Wuppertal, 16.9.2020
ring lecture
16:50–17:50

Deformation theory I
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thank Jens for organizing
Jens asked me to take over, regretfully because the other speaker could not be found

Welcome to this first part of my ring lecture.

Very sorry

Topic was suggested by Karen Martinez Acosta and Fabian Korthauer – excellent proposal, because it appears over and over in various areas of pure math.

Deformation theory has reputation of being „difficult". This is actually true. No way to deny. There are many reasons for this – one of them is that „deformation" refers both to a precise notion and an intuitive idea, furthermore, there are many almost synonymous terms (perturbations, variations, general fibers, etc)

Today I want to use rare physical powers to spread more on the intuitive side. Apologize for those who know this. Will proceed more formally in second part, and touch some research topics in the third.

Reference. Monographs of Hartshorne and Sernesi.

Roughly intuition. If you want to study solutions of

$$f_i(T_1, \dots, T_n) = 0, \qquad 1 \le i \le m$$

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Then look at solutions to $f_i(T_1,\dots,T_n)=\varepsilon$ for ε small. Immediate seen over $k=\mathbb{C}$, but one has to attach meaning to it, say over $K=\mathbb{F}_p$.

A more invariant point of view: For certain [just one] morphism $f: X\to Y$, regard fibers $X_t=f^{-1}(t)$ [or better, the ...] as deformations of X_a, for t close to a.

One difficulty in deformation theory is that you hardly see it in differential topology.

Lemma (Ehresmann) Let $f: X\to Y$ diff. map between diff. mfd. with Y connected. Suppose f is proper, surjective, and all tangent maps

$$T_a(X) \longrightarrow T_{f(a)}(Y)$$

[tangent spaces]

are surjective. Then $X\xrightarrow{f} Y$ is twisted form of [very important to every way of thinking] $F\times Y\xrightarrow{pr} Y$ for some connected diff. mfd F.

Locally it looks like product, $X_t=X_a$.

Related with fundamental group / univ. cover.

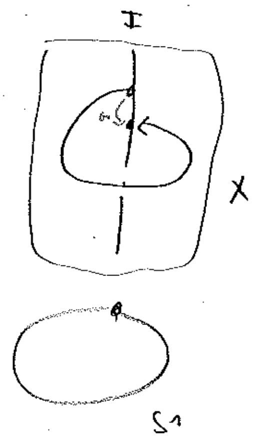
$$X\times_Y \tilde{Y} = F\times\tilde{Y} \quad\text{and}\quad X=(F\times\tilde{Y})/\pi_1(Y,b_0)$$

[univ. cov]

17:05

<u>Example 1</u> Let $\sigma \in \mathrm{Aut}(F)$. Consider diagonal action of $G = \mathbb{Z}$ on $F \times \mathbb{R}$:

$$X = (F \times \mathbb{R})/G \xrightarrow{\pi} \mathbb{R}/G = S^1$$



Monodromy action

<u>Example 2</u> Consider Weierstraß equation

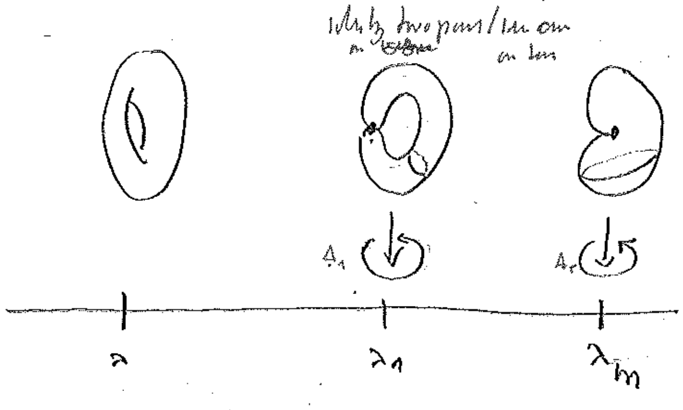
$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

with $a_i \in \mathbb{C}[T]$, with discriminant $\Delta \neq 0$. Homogenous gives family of cubic curves

not always ... - dimension

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^2_{\mathbb{C}[T]} \\ & \searrow & \downarrow \\ & & \mathbb{A}^1 \end{array}$$

Discriminant $\Delta \in \mathbb{C}[T]$ has finitely many zeros $\lambda_1, \dots, \lambda_m \in \mathbb{C}$. Ehresmann Lemma applies to $\mathbb{A}^1(\mathbb{C}) \setminus \{\lambda_1, \dots, \lambda_m\}$, with $\pi_1 = \mathop{\ast}\limits_{1}^{m} \mathbb{Z}$.



other degenerations do not occur, thanks to the special form of Weierstraß equation.

17.12.

Gives monodromy matrices $A_1, \ldots, A_m \in GL_2(\mathbb{Z})$.

Isomorphism class of elliptic curves X_t, $\Delta(t) \neq 0$

are given by "invariant" $j \in \mathbb{C}[T]$, a rational function
as complex manifold, not as deformation

$j: \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$. Usually $X_t \not\cong X_a$ for almost all t.

<u>Example 3</u> Let $G = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$ be

a finitely presented group. Often studies representations

$\rho: G \longrightarrow GL_d(\mathbb{C})$. Nothing but
degree

$$A_1, \ldots, A_m \in GL_d(\mathbb{C}) = \mathbb{C}^{d^2} \setminus \det^{-1}(0)$$

subject to $r_j(A_1, \ldots, A_m) = E$. Can move these

matrices, e.g. via conjugation $S A_i S^{-1}$. Are there

other ways? This gives "infinitesimal" deformations of ρ.

Recall Ehresmann condition about $f: X \to Y$

that are proper, (smooth, curve, surj. on tangent space.

Seek to translate this to alg. geometry:

Given a field k, set $k[\varepsilon] = k[T]/(T^2)$

and regard

$$\mathrm{Spec}(k[\varepsilon]) = \{ \bullet \!\to \}$$

as k-valued point with tangent vector.

could try to ... tangent vectors ...

Let X noetherian scheme. Each $a \in X$ gives

local ring $\mathcal{O}_{X,a}$ with residue field $\kappa(a) = \mathcal{O}_{X,a}/m_a$

and first-order vector space m_a/m_a^2. NAK: dim = min number of gen. of m_a ≥ dim.

Each non-zero form $\varphi : m_a/m_a^2 \to \kappa(a)$ gives

surjection

choose ...

$$\mathcal{O}_{X,a}/m_a^2 \longrightarrow \kappa(a)[\varepsilon], \quad f \longmapsto f \bmod \ker(\varphi)$$

monomorphism of schemes.

Then $\mathrm{Spec}(\kappa(a)[\varepsilon]) \subset X$. So we call

$$\mathrm{Hom}_{\kappa(a)}(m_a/m_a^2, \kappa(a))$$

the tangent space of X at $a \in X$.

... ✓ ...

Let $f : X \to Y$ be a morphism of schemes

between noetherian schemes. Smoothness on tangent space

means the ...

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(k[\varepsilon]) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

can be verified ...

can be completed. This leads to:

<u>Def</u> The morphism $f: X \to Y$ is smooth if

each diagram

$$\begin{tikzcd} \mathrm{Spec}(A/I) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Same map on top, [illegible] [illegible] $\to$ [illegible]

Grothendieck!
Note that it was smooth [illegible] for $X(R)$ and $Y(R)$

can be completed, when A is a ring and I is an ideal with $I^2 = 0$.

(same top. space, [illegible] small

So if $f: X \to Y$ is proper, smooth, then we have the analogue of the Ehresmann lemma.

But in alg. geom. it would be [illegible] to think that $X_5 = M(3)$ are smooth.

Now the lifting is unique [illegible]

is

17:35

<u>Thm</u> The morphism $f: X \to Y$ is smooth if and only if the following holds.

(i) The fibers $f^{-1}(b)$, $b \in Y$ are smooth

(ii) For each $A = \mathcal{O}_{X,a}$ and $R = \mathcal{O}_{Y,f(a)}$, [illegible] lin sys of eqns

every A-valued solution of some $R^m \to R^n$

is linear combination of R-valued solutions.

If (ii) holds one says that the R-algebra A is flat. (Could be rephrased in terms of tensor products.) Each short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of R-modules gives exact sequence of A-modules

$$\ldots \to \mathrm{Tor}_1^R(A, M') \to M' \otimes_R A \to M \otimes_R A \to M'' \otimes_R A \to 0$$

Turning things upside down:

<u>Prop</u> The R-algebra A is flat if and only if $M \mapsto M \otimes_R A$ is exact.

17.1.42

A morphism $f: X \to Y$ of schemes is flat at a
well-defined if $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$
if $O_{X,a}$ is flat over $O_{Y,f(a)}$

We now regard the proper flat morphism

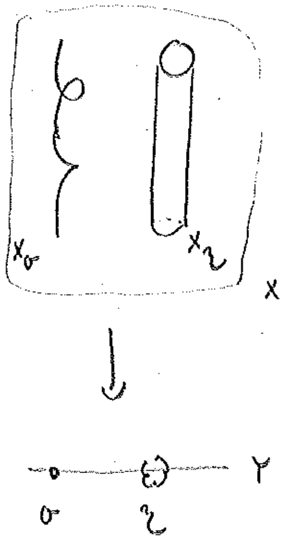
$f: X \to Y$ as families of proper schemes $X_b = f^{-1}(b)$,

parametrized by Y. Particularly instructive is

$Y = \mathrm{Spec}(R) = \{0, \eta\} =$ 

is the spectrum of a discrete valuation ring R, e.g.

$R = k[[t]]$ or $R = \mathbb{Z}_p$ or $R = O_{C,a}$ point of a curve / closed point



Regard X_0 as degeneration of X_η, and X_η as deformation of X_0.

reduced: no nilpotents

Lemma: A reduced Noetherian R-scheme X is flat if and only if all its generic points $\eta_i \in X$ map to the generic point $\eta \in \mathrm{Spec}(R)$.

So for each irreducible homogeneous $P \in R[T_0, \dots, T_n]$,

the zero-scheme $X = V_+(P) \subset \mathbb{P}^n_R$ defines

a family of hypersurfaces

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "f"'] & & \mathbb{P}^n_R \arrow[dl] \\ & \operatorname{Spec}(R) & \end{tikzcd}$$

parametrized by $Y = \operatorname{Spec}(R)$.

Note that the fibers X_σ may be reducible or non-reduced