GRK Workshop - Summer Semester 2022
The McKay Correspondence

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The special unitary group SU(2) is the group of 2×2 unitary matrices with determinant 1. Its elements can be seen as rotations of the 3-dimensional space, up to a sign. Furthermore, most of the finite subgroups of SU(2) are related to groups of rotational symmetries of Platonic solids (the tetrahedron, octahedron, cube, icosahedron, and dodecahedron).

In 1980, McKay [McK] noticed that there are correspondences between non-trivial finite subgroups of SU(2), up to conjugation, and certain Euclidean graphs on the one hand, and between Kleinian singularities and Dynkin graphs on the other hand.

In a series of four talks, we will unravel some of the 'mysteries' behind these correspondences. Our main references are [K] and [L].

TALK 1: INTRODUCTION, FINITE SUBGROUPS OF SU(2), AND QUIVERS

Give a brief overview of the contents and aims of the workshop (see the beginnings of Sections 8.2, 8.3, and Theorems 8.9, 8.15 of [K]). Mention that the correspondences illustrated in Theorems 8.9 and 8.15 are related (see [K, §12.4]), even if we will not investigate this connection.

Explain Section 8.1 of [K] and outline the proof of the classification of finite subgroups of SU(2), following Section 10 of [M]. Recall the necessary definitions and results of the previous sections of the same notes. For further background information see [L].

In the last part of the talk, introduce the basics of quivers by giving Definitions 1.1 and 1.2 and at least one example illustrating the relevant concepts from [K, §1.1].

TALK 2: DYNKIN AND EUCLIDEAN GRAPHS, *ADE* CLASSIFICATION

Prove the classification of Dynkin and Euclidean graphs. In particular, define the Euler and Tits forms as in Section 1.5 of [K] and state and prove Theorems 1.28 and 1.30, [K, §1.6].

Explain Section 8.2 of [K] up to Theorem 8.9. If time permits, give a sketch of the proof of Theorem 8.6 [K] that can be found in Section 11.7 of [C].

TALK 3: THE McKay CORRESPONDENCE

The aim of this talk is to illustrate the McKay correspondence, following Section 8.3 of [K] (up to the end of page 144). Recall the necessary definitions and results from representation theory (see [I], [H], and [L, Chapter 2]). Illustrate the correspondence by at least one example (see [L, Chapter 2] and [K, §8.3]).

TALK 4: KLEINIAN SINGULARITIES

Let G be a finite subgroup of SU(2) and consider the quotient space $\mathbb{C}^2/G$. The singularities of this algebraic variety are called *Kleinian singularities*. In this talk we will see how certain resolutions of these singularities are related to Dynkin graphs.

Follow Section 12.1 of [K] to give an overview of the talk. Give an idea of how to find the equations of Table 12.1 by looking at Section 4.1 of [L] (see in particular Theorem 4.3, without proof, and Definition 4.4). Show Example 12.2 of [K]. Illustrate briefly the general theory of blow-ups and give at least one example (see [L, §3.3], [S, §2.4], and [G, Chapter 9]). Explain how to draw diagrammatically the irreducible components of the exceptional divisors (see [L, §3.3] and [B, Chapter 4]). Finally show that, in the case of Kleinian singularities, the resulting diagrams are the Dynkin graphs (see [L, Chapter 5] and [B, Chapter 4]).

Main references

[B] R. Bocklandt, *Kleinian Singularities*, 2006. https://algebra.datanose.nl/website/notes/kleinian.pdf

[C] H. S. M. Coxeter, *Regular complex polytopes*, Cambridge University Press, London-New York, 1974.

[G] A. Gathmann, *Algebraic Geometry*, Version 21/22. https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2021/alggeom-2021.pdf

[H] B. Huppert, *Character theory of finite groups*, De Gruyter Expositions in Mathematics, 25. Walter de Gruyter & Co., Berlin, 1998.

[I] M. Isaacs, *Character theory of finite groups*, Pure and Applied Mathematics, no. 69, Academic Press, New York-London, 1976.

[K] A. Kirillov, *Quiver representations and quiver varieties*, Graduate Studies in Mathematics, 174, American Mathematical Society, Providence, RI, 2016.

[L] M. Lindh, *An introduction to the McKay correspondence*, Master thesis in Physics, 2018. http://www.diva-portal.org/smash/get/diva2:1184051/FULLTEXT01.pdf

[M] H. Mark, *Classifying finite subgroups of* SO(3), 2011. http://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/MarkH.pdf

[S] I. R. Shafarevich, *Basic Algebraic Geometry 1. Varieties in projective space*, Third edition, Translated from the 2007 third Russian edition, Springer, Heidelberg, 2013.

Further readings

[BKR] T. Bridgeland, A. King, and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer.MAth. Soc. 14, no. 3, 535–554, 2001.

[D] I. Dolgachev, *McKay correspondence*, (Winter 2006/07), 2009. http://www.math.lsa.umich.edu/~idolga/McKaybook.pdf

[GSV] G. Gonzalez-Sprinberg and J.-L. Verdier, *Contruction géométrique de la correspondance de McKay* (French), Ann. Sci. École Norm. Sup. (4) 16, no. 3, 409–449, 1983.

[McK] J. McKay, *Graphs, singularities, and finite groups*, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), pp. 183–186, Proc. Sympos. Pure Math., 37, Amer. Math. Soc., Providence, R.I., 1980.

[S] P. Slodowy, *Platonic solids, Kleinian singularities, and Lie groups*, in Algebraic geometry (Ann Arbor, Mich., 1981), 102–138, Lecture Notes in Math., 1008, Springer, Berlin, 1983.

If you have problems in getting access to any of the references above, or if you require further details regarding the content or the organization of the talks, please let us know.

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