GRK 2240 WORKSHOP: SCHUBERT CALCULUS ON GRASSMANNIANS WINTER SEMESTER 2021/22

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1. Introduction

Named after Hermann Schubert for his pioneering work on enumerative geometry at the center of Hilbert's 15th problem, Schubert calculus lies at the crossroad of representation theory, algebraic geometry, algebraic combinatorics and algebra. The goal of this seminar is to illustrate some aspects of this theory by considering its simplest and probably most celebrated example, that of grassmannians. More specifically, we will illustrate the relationship existing between the algebra of symmetric functions and the cohomology ring of grassmannians by relating different families of polynomials to an additive basis, the importance of which is due to its links with enumerative geometry.

Talk 1: Symmetric Functions

The aim of this talk is to make the audience familiar with the notions of symmetric functions and Young tableaux. The talk should provide several examples in order to clarify the mentioned definitions.

The lecturer should

(i) define the ring of symmetric functions Λ_n [2, p.17];
(ii) define partitions [4, Definition 3.3] (preferably mention Young diagrams) and monomial symmetric polynomials;
(iii) define elementary symmetric functions [2, p.19-20] and give examples [4, Section 3.1];
(iv) show that the ring of symmetric functions Λ_n is generated by elementary symmetric functions [2, p.20-21];
(v) define complete symmetric functions [2, p.21] and give examples [4, Section 3.1];
(vi) describe the relation between the $H(t)$, the generating function of the complete symmetric functions, and that of the elementary ones [2, p.21];
(vii) conclude that the ring of symmetric functions is also generated by complete symmetric functions Λ_n [2, p.22];
(viii) explain that one can obtain a symmetric polynomial from any partition by counting the Young tableaux [1, p.3] and give several examples (e.g. for partitions (3), (1,1,1) and (2,1) with three variables).

For the purpose of this workshop, it is not necessary to talk about the inverse limit described in [2, p.18]. In (viii) the focus should be on the procedure and not on the fact that the polynomial is indeed symmetric. Further, the lecturer should preferably not refer to these polynomials as Schur polynomials.

Talk 2: Schur functions

The second talk introduces the Schur functions and their relations to elementary symmetric functions and complete symmetric functions. The main reference for this talk is [2, p.40-42].

The lecturer should

(i) briefly recall the last construction (viii) from the previous talk, express the polynomial obtained from the partition $(2,1)$ in terms of elementary symmetric functions and motivate that a formula for this procedure would be very helpful by stating that even for e.g. the partition (3) this computation is not obvious;
(ii) define Schur functions;
(iii) remark without a proof that the Schur functions are precisely the polynomials obtained by the construction (viii) of talk 1;
(iv) show that the polynomials $a_{\lambda+\delta}$ in [2, p.40] form a basis of the $\mathbb{Z}$-module A_n of skew-symmetric polynomials (short computation needed);
(v) deduce that the Schur functions form a $\mathbb{Z}$-basis of the ring of symmetric functions [2, p.41] and mention that there is an isomorphism between the free $\mathbb{Z}$-module R^n generated by the irreducible characters of the symmetric group S_n and the ring of symmetric functions Λ_n [3, Proposition 1.6.3] (the irreducible characters map to the Schur functions);
(vi) prove the first of the Jacobi-Trudi formulae [2, p.41 (3.4)] and state that one can deduce the second formula [2, p.41 (3.5)];
(vii) provide some examples using these formulae including at least the basic examples [2, p.42 (3.9)] and also express $s_{(3)}$ in terms of elementary symmetric functions in order to come back to the motivation from (i);
(viii) formulate the motivating question how one could multiply two additive generators which will be answered during the workshop. More explicitly: "How can one compute the structure constants $c^{\nu}_{\lambda,\mu}$ coming from the expansion of the product $s_\lambda \cdot s_\mu = \sum_\nu c^{\nu}_{\lambda,\mu} s_\nu$?"

Again, it is not necessary to provide details concerning the ring Λ. If there are questions concerning point (iv), do not hesitate to contact us. Preferably, the formulae [2, p.41 (3.4), (3.5)] should be declared as the Jacobi-Trudi formulae, although it is not mentioned in the main source for this talk. For more details on the topic, see also [3, 4].

Talk 3: Grassmannians

The first part of this talk focuses on Grassmannians and their geometry which will be essential in the sequel of the workshop. The second half is dedicated to Chern classes and the cohomology of Grassmannians. The main reference will be [3, Chapter 3].

The lecturer should

(i) define Grassmannians [4, Definition 4.1+Remark];
(ii) shortly recall the tautological bundle T and the quotient bundle Q over the Grassmannian [3, Section 3.5.3] which will be relevant in the sequel;
(iii) define Schubert cells, Schubert varieties and special Schubert varieties [3, Section 3.2.1];

(iv) give at least one explicit example of a Schubert cell (e.g. for the partition $(2,1,1)$ and the corresponding Schubert cell in $\mathrm{Gr}(3,5)$) and the most natural examples of Schubert varieties ($X_\emptyset, X_{m\times n}$ and X_k in Manivel's notation);
(v) state [3, Proposition 3.2.3] and give some intuitive examples or remarks;
(vi) recall that the module structure of the cohomology of cellular varieties can be described using their cells, deduce the module structure of the cohomology of Grassmannians (cf. [3, Corollary 3.2.4]) and remark that for Grassmannians there is an isomorphism between the Chow ring and cohomology;
(vii) recall the existence of Chern classes, exemplify the general behaviour by stating $c_1(L) = [X_s]$ and state the Whitney-sum formula for Chern classes $c_k(E)$ [3, Section 3.5.1];
(viii) define the Chern polynomial $c_t(E) = \sum_{i\geq 0} c_i(E)t^i$, conclude $c_t(T)c_t(Q) = 1$ and make a connection to the equation $H(t)E(-t) = 1$ from talk 1.

For the purpose of this workshop it is not necessary to discuss more properties of Chern classes and cohomology/Chow rings. Furthermore, the connection in (viii) should rather be formulated as an observation.

Talk 4: Cohomology ring of the Grassmannian

The last talk of the program should conclude with the main result which is the computation of the ring structure of the cohomology of the Grassmannian $\mathrm{Gr}(m, m+n)$.

The lecturer should

(i) define skew diagrams, m-horizontal (vertical) strips and provide examples [2, p.4-5];
(ii) recall the involution ω from Talk 1 [2, p.21-22] and then prove Pieri's formula [2, p.73 (5.16), (5.17)];
(iii) conclude that (together with the Jacobi-Trudi formulae) this determines the multiplicative structure of Λ_m using Schur functions as additive generators;
(iv) give examples of Pieri's formula in Λ_m by computing the product of two Schur functions and remark that this procedure gives an answer to the question raised in Talk 2 (viii);
(v) state Pieri's formula for $H^*(\mathrm{Gr}(m, m+n))$ (e.g. [4, Theorem 5.3]);
(vi) deduce Giambelli's formula [3, Corollary 3.2.10] using the module homomorphism $\phi_{m,n} : \Lambda_m \to H^*(\mathrm{Gr}(m, m+n))$ in [3, Corollary 3.2.9] (hint: use both Pieri formulae, Jacobi-Trudi formula and compare the coefficients), and finally conclude that $\phi_{m,n}$ is a surjective map of rings;
(vii) show that the kernel of $\phi_{m,n}$ is given by $\langle s_\lambda \mid \lambda \not\subseteq m \times n\rangle = \langle h_j \mid j > n\rangle$, deduce that it is enough to consider the generators $h_{n+1}, ..., h_{n+m}$ in order to determine the kernel and deduce $H^*(\mathrm{Gr}(m, m+n)) \cong \mathbb{Z}[e_1, ..., e_m]/(h_{n+1}, ..., h_{n+m})$ where $h_{n+i} = P_i(e_1, ..., e_m)$ is a polynomial in the variables $e_1, ..., e_m$;
(viii) state as a fact that $\phi_{m,n}(e_i) = (-1)^i c_i(T)$ and $\phi_{m,n}(h_i) = c_i(Q)$ hold for all $i \geq 0$, the tautological bundle T and the quotient bundle Q and conclude that the cohomology can be completely described by the Chern classes of the quotient bundle, i.e.

$$H^*(\mathrm{Gr}(m, m+n)) \cong \frac{\mathbb{Z}[-c_1(T), ..., (-1)^m c_m(T)]}{(P_i(-c_1(T), ..., (-1)^m c_m(T)))_{1\leq i\leq m}}.$$

References

[1] Fulton, W. *Young tableaux*, vol. 35 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[2] Macdonald, I. G. *Symmetric functions and Hall polynomials*, second ed. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, 2015.

[3] Manivel, L. *Symmetric functions, Schubert polynomials and degeneracy loci*, vol. 6 of *SMF/AMS Texts and Monographs.* American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001.

[4] Pech, C. *Schubert calculus on Grassmannians.* Lecture Notes King's College London. 2013, `http://math.univ-lyon1.fr/~pech/Publications/LTCC-course.pdf`.