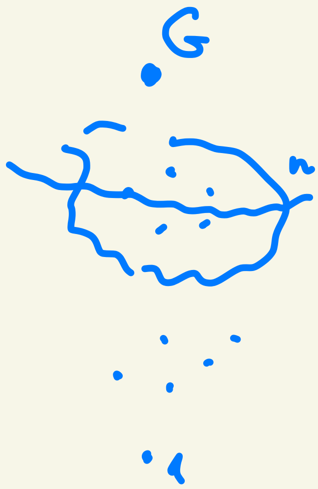
GRK mini lecture series 3.12.2020

Subgroup Growth (1 of 3)



G fin gen group

$$a_n(G) = \#\{ H \leq G \mid |G:H| = n \}$$

$$s_n(G) = \sum_{k=1}^{n} a_k(G) < \infty$$

$\rightsquigarrow$ non-decreasing fct $\mathbb{N} \to \mathbb{N}$

$n \mapsto s_n(G)$

"subgp growth"

WHY?

- finiteness conditions

$\sim$ 1970/80s Olsh, Rips

Tarski monsters $p \in \mathbb{P}$, $p > 10^{75}$

G inf

$\forall H,\ 1 \neq H \lneq G : H \cong C_p$

to avoid TM:

G residually finite

- $\bigcap_{N \trianglelefteq_f G} N = 1$
- $G \hookrightarrow \hat{G}$

Prop G fg & linear $\Rightarrow$ G resid finite

- formal Dirichlet series,

  subgp (growth) zeta fct,

$$\zeta_G^{\leq}(s) = \sum_{n=1}^{\infty} a_n(G)\, n^{-s} = \sum_{H \leq_f G} |G:H|^{-s}$$

in analogy to Dedekind zeta fct

$\rightsquigarrow$ non-comm analytic # theory

typical tools

- CFSG, post-80s version
- Lie methods, linearisation techniques
- p-adic integration

Source: Lubotzky, Segal 2003

§ 1 Warm-up: ideas ...

- G finite $\quad \underline{a} = (a_n(G))_{n\in\mathbb{N}}$

ex: $\underline{a}$ knows
- $|G|$
- G cyclic

- G nilpotent
- G elem ab
- $G \cong \mathrm{Alt}(5)$
- $d(G)$ for G nilp
  ($\uparrow$ min # gens of G)

what else ?

Caveat : $\exists\ G_1 \not\cong G_2$ of order $3^5 = 243$

and bij $\beta : \mathcal{L}(G_1) \to \mathcal{L}(G_2)$ betw subgp lattices st

$\forall\ H, K \leq G_1$:

- $H \not\cong H\beta$ unless $H = G_1$
- $H \underset{\trianglelefteq}{\leq} K \leftrightarrow H\beta \underset{\trianglelefteq}{\leq} K\beta$
- $G_1/H \cong G_2/H\beta$ for $H \underset{\neq 1}{\trianglelefteq} G_1$

- $G = C_\infty \ (\cong \mathbb{Z})$

  $$\underline{a} = (a_n(G))_n = (1, 1, \dots)$$

  $$\zeta_G^{\leq}(s) = \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \text{Riemann zeta fct}$$

- $G \cong \mathbb{Z}^d$ free ab

$$\rightsquigarrow \quad \boxed{\zeta_G^{\leq}(s) = \zeta(s)\,\zeta(s-1)\cdots\zeta(s-d+1)}$$

$$s_n(\mathbb{Z}\times\mathbb{Z}) \sim \underline{\frac{\pi^2}{12}}\, n^2$$ (via Tauberian thm)

§2 Free groups

M Hall 1949

M Newman 1976

$$m_n(G) = \#\{ M \leq_{\max} G \mid |G:M| = n\} \leq a_n(G)$$

Thm $F = F_d$ free, $d \geq 2$

(1) $$a_n(F) = \underline{n\,(n!)}^{d-1} - \sum_{k=1}^{n-1} \big(\underline{(n-k)!}\big)^{d-1} a_k(F)$$

(2) $$a_n(F) \sim m_n(F) \sim n\,(n!)^{d-1}$$

Rem: $n^{n/2} \leq n! \leq n^n \rightsquigarrow$

F has 'strict subgp growth type'

$n^n = 2^{n \log(n)}$

Sketch proof:

$$t_n(G) = \#\{\varphi : G \to \mathrm{Sym}_n \mid G\varphi \text{ transitive}\}$$

$m_n(G)$ primitive

$$\boxed{a_n(G) = \frac{t_n(G)}{(n-1)!}}$$

$H\backslash G \curvearrowleft G$

$$t_n(G) \leq h_n(G) := |\mathrm{Hom}(G, S_n)|$$

$$\leq (n!)^{d(G)} \qquad \leadsto \quad a_n(G) \leq n\,(n!)^{d(G)-1}$$

$$h_n(G) = \sum_{k=1}^{n} \underbrace{h_{n,k}(G)}_{\substack{\#\text{ homom } G \to S_n \text{ st} \\ \text{orbit of } 1 \text{ has length } k}}$$

$$h_{n,k}(G) = \binom{n-1}{k-1} t_k(G)\, h_{n-k}(G)$$

$$h_{n,n}(G) = t_n(G)$$

$$\leadsto \quad (*) \quad a_n(G) = \frac{1}{(n-1)!} h_n(G) - \sum_{k=1}^{n-1} \frac{1}{(n-k)!} h_{n-k}(G) \cdot a_k(G)$$

for $G = F$ free on d gens:

$$h_n(G) = (n!)^d$$

substitute in $(*)$ $\leadsto$ Hall's formula

elementary treatment of $h_n(F) - t_n(F)$

similar to above $\leadsto$

$$1 - t_n(F)/h_n(F) \leq \frac{4}{n} \to 0 \quad \text{as } n \to \infty$$

thus $t_n(F) \sim h_n(F)$ Newman

$$\Rightarrow \quad a_n(F) = \frac{t_n(F)}{(n-1)!} \sim \frac{h_n(F)}{(n-1)!} = n\,(n!)^{d-1}$$

this also follows from **Dixon's thm:** (1969)

$$\mathrm{Prob}(\sigma_1, \sigma_2 \in S_n \text{ gen } A_n \text{ or } S_n) \to 1 \quad \text{as } n \to \infty$$

Thm (Lubotzky 2001)

$$a_n^{\trianglelefteq}(F) < n^{2(d+1)(1+\lambda(n))}$$

$n = \prod p_i^{e_i}$

$\lambda(n) = \sum e_i \leq \log(n)$

- subexponential normal subgp growth
- relies on 'short' presentations for finite non-ab simple gps (C$\overset{2}{\mathrm{F}}$SG) in category of profinite groups