Introduction to Subgp Growth — Part 2

10.12.20

recap: G f.g., resid-fin

$G \hookrightarrow \hat{G}$

$$a_n(G) = \#\{H \leq G \mid |G:H| = n\}$$

$$s_n(G) = \sum_{k=1}^{n} a_n(G)$$

free grs ... upper end

today: lower end : $\boxed{\text{PSG groups}}$

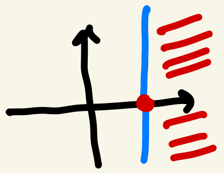
polynomial subgp gr

$$\exists c\ \forall n : s_n(G) \leq n^{c}$$

$c_1 n^{c_2}$

Rem/def

$$G \text{ PSG} \rightsquigarrow \zeta_G^{\leq}(s) = \sum_{n=1}^{\infty} a_n(G)\, n^{-s} = \sum_{H \leq_f G} |G:H|^{-s}$$



$Re(s) = \alpha(G)$

converges (abs) to an analytic fct on $\{s \in \mathbb{C} \mid Re(s) > \alpha(G)\}$

$$\boxed{\alpha(G) = \overline{\lim_{n\to\infty}} \frac{\log s_n(G)}{\log n}}$$

degree of PSG

Questions : • What grp have PSG ? ($\alpha(G) < \infty$)

• What is $\alpha(G)$?

• merom cont (⇝ pole at $s = \alpha(G)$) ?

(• analogues of classical class number formula)

examples : (0) $G \cong C_\infty \cong \mathbb{Z}$

$$\zeta_G^{\leq}(s) = \zeta(s) = \sum n^{-s}$$

(1) $G = \mathbb{Z} \times \mathbb{Z}$

every $H \leq_f G$ has a standard basis of the form (a, b), $(0, c)$

where $a \geq 1$, $c \geq 1$, $0 \leq b < c$

$$\zeta_G^{\leq}(s) = \sum_{a=1}^{\infty} \sum_{c=1}^{\infty} c \cdot (ac)^{-s} \qquad |G:H| = ac$$

$$= \sum_{a=1}^{\infty} a^{-s} \cdot \sum_{c=1}^{\infty} c^{1-s} = \zeta(s) \cdot \underline{\zeta(s-1)}$$

$\alpha(G) = 2$

pole of order 1

$\cancel{(\log n)^{1-1}} = 1$

⇝ $\boxed{s_n(\mathbb{Z} \times \mathbb{Z}) \sim \frac{\pi^2}{12} \cdot n^2}$

$\frac{\pi^2}{12} = \frac{\zeta(2)}{2\,\Gamma(1)}$

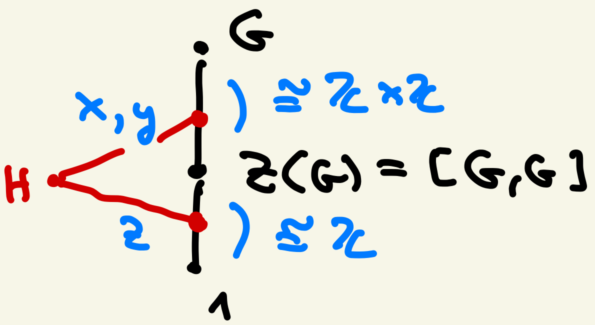
(2) generalisation:

$$\zeta^{\le}_{\mathbb{Z}^d}(s) = \zeta(s)\,\zeta(s-1)\cdots\zeta(s-d+1)$$

(3) $G = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$ discrete Heisenberg gp

free class-2 nilp gp on 2 gens

$$x = \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix},$$



$$z = [x,y] = x^{-1}y^{-1}xy = \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \text{ central}$$

every $H \le_f G$ has stand gen set

$$\boxed{\begin{matrix} x^a & y^b \\ & y^c \end{matrix}} \begin{matrix} z^r \\ z^s \\ z^t \end{matrix}$$

$a \ge 1$, $c \ge 1$,

$0 \le b < c$,

$1 \le t \le ac$ (*)

$0 \le r,s < t$

$|G:H| = act$

ad (*):

$$[x^a \cancel{y^b} \cancel{z^r}, y^c \cancel{z^s}] = [x^a, y^c] = [x,y]^{ac} = z^{ac}$$

- trick: if G is … then

$$\zeta_G^{\le}(s) = \prod_{p\in\mathbb{P}} \underbrace{\zeta_{G,p}^{\le}(s)}_{=\sum_{k=0}^{\infty} a_{p^k}(G)\cdot p^{-ks}}$$

hence focus on

$a=p^k$, $c=p^\ell$, $t=p^m$ where

$k,\ell\ge 0$; $0\le m\le k+\ell$

$$\leadsto \quad \zeta_{G,p}^{\le}(s) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{k+\ell} p^{\ell+2m}\cdot p^{-(k+\ell+m)s}$$

$$= \ldots = \frac{1+p^{1-s}+p^{2-2s}}{(1-p^{-s})(1-p^{2-2s})(1-p^{3-2s})}$$

$$\leadsto \quad \zeta_G^{\le}(s) = \frac{\zeta(s)\,\zeta(s-1)\,\zeta(2s-2)\,\zeta(2s-3)}{\zeta(3s-3)}$$

$\leadsto \quad \alpha(G)=2$

$$s_n(G)\sim \frac{\pi^4}{72\,\zeta(3)}\cdot n^2(\log n)$$

expl for : for G (pro) finite :

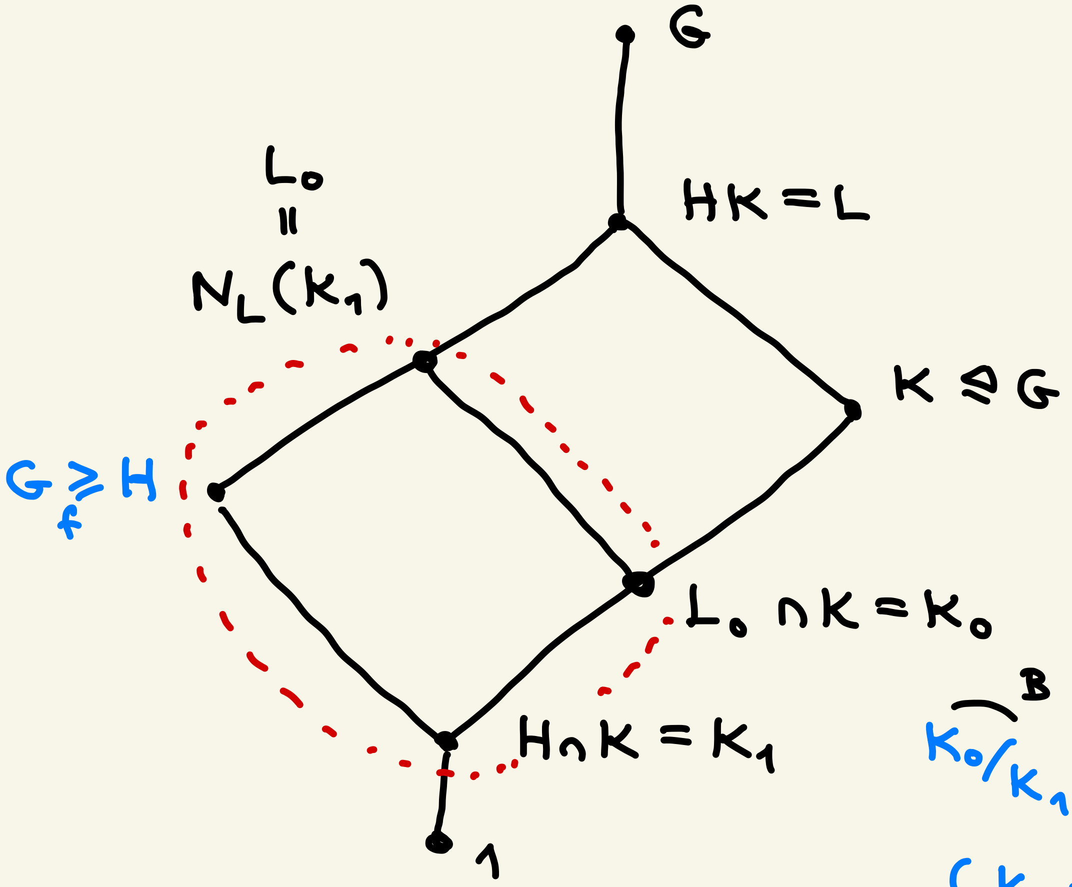
G (pro) nilp $\Leftrightarrow$ G cartes product of its Sylow-(pro-) p-subgp

$\to \quad a_n(G) = \prod a_{p_i^{r_i}}(G) \qquad n = \prod_i p_i^{r_i}$

back to quest : Which gps are PSG ?

What about $\alpha(G)$?

key idea : gp extensions



$\overbrace{K_0/K_1}^{B} \circlearrowleft \overbrace{L_0/K_0}^{A}$

$(K_1 \trianglelefteq L_0)$

complements H

$= \# \mathrm{Der}(A, B)$

$\delta : A \to B$

$(xy)\delta = (x\delta)^y (y\delta)$

special case : B abelian , central ...

in general : $\underline{\# \mathrm{Der}(A, B) \le |B|^{d(A)}}$

2_p

$$a_n(G) \leq \sum_{t \mid n} a_{n/t}(G/K) \cdot a_t(K) \cdot t^{rk(G/K)}$$

$$s_n(G) \leq s_n(G/K) \cdot s_n(K) \cdot n^{rk(G/K)}$$

<u>Thm</u> (Lubotzky, Mann, Segal 1993)

G f g, resid-fin $\quad \infty > (\; G \;\; G_0 \text{ sol}$

Then: $\quad G$ has PSG $\overset{\rightarrow}{\longleftrightarrow}$

G virtually soluble of $rk(G) < \infty$

$\leftarrow$ see above $\qquad$ ($\leftrightarrow$ G virt sol linear over $\mathbb{Q}$

$\leftrightarrow$ G built by extn of subgps of $(\mathbb{Q},+)$ and finite gps)

<u>Thm</u> (improves KR '98)

G inf virt sol minimax gp
($\leftarrow$ inf fg sol gps of finite rk)

$\Rightarrow$

$$\tfrac{1}{6}\, h(G) \leq \alpha(G) \leq h(G) + 1$$

can prob be improved

($h(G)$ Hirsch length $\equiv$ dimension)