

Introduction to subgroup growth 17.12.20

Part 3

(The Sem from 12 I '21 :

5-6 lectures on
SGG ...)

'muzzle' : 6, 10, 16, 26, 44, 78, ...

recap: lec 1 : a_n, s_n, ξ , free gr → fast growth

lec 2 : PSG, Heisenberg gr, $\alpha(G)$

today:

- continue with fg (torsion-free) nilp gp
- connect to p-adic formalism / integration

[G fg tf nilp gr (\mathbb{Z} -grs)
(→ resid finite)

\exists Malcev basis ... Malcev compl / radical hull

of h -divisible unipotent affine alg gr
(h Hirsch length of G)

$G^R = \text{Og}(R)$ gr of R -valued pts

↑ binomial ring, ie $\frac{r(r-1)\dots(r-u+1)}{u!}$

$G^{\mathbb{Z}} = G$, $G^{\mathbb{Q}}$ Malcev compl

$r \in R$ ($r \in R, m \in \mathbb{N}$)

for G, H \mathbb{Z} -grps, write

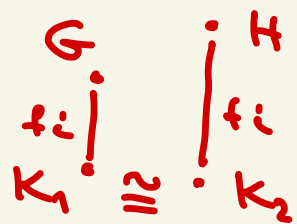
$$G \cong_{\mathbb{Z}} H \xrightarrow{\text{def}} G^{\mathbb{R}} \cong H^{\mathbb{R}}$$

for instance,

$$\mathbb{A}_{\mathbb{Z}} = \prod_{\mathbb{P}} \mathbb{Z}_{\mathbb{P}} \quad \text{pick integral orders}$$

$$G^{\mathbb{A}_{\mathbb{Z}}} = \prod_{\mathbb{P}} G^{\mathbb{Z}_{\mathbb{P}}} \cong \widehat{G} \quad \text{profinite compl}$$

$\underbrace{\hspace{10em}}_{\text{pro-}\mathbb{P} \text{ compl}}$



facts:

$$G \cong_{\mathbb{Q}} H \iff G, H \text{ commensurable}$$

$$\mathbb{A}_{\mathbb{P}}: \iff G \cong_{\mathbb{A}} H \iff \widehat{G}, \widehat{H} \text{ commens}$$

$$G \cong_{\mathbb{Q}} H$$

$$G \cong_{\mathbb{A}_{\mathbb{Z}}} H \iff \widehat{G} \cong \widehat{H}$$

[Pickel: each $\mathbb{A}_{\mathbb{Z}}$ -class

splits into fin many \cong -classes]

$$\exists \mathbb{P}: G \cong_{\mathbb{Q}_{\mathbb{P}}} H \iff \exists \mathbb{P}: G \cong_{\mathbb{Z}_{\mathbb{P}}} H$$

$$\iff G \cong_{\mathbb{Z}} H$$

• Back to SGG & S Pts

[Gorenwald, Segal, Smith '88]

G, H \mathbb{F} -grps

$\# \{H \leq G \mid |G:H|, \hat{G} \cong \hat{H}\}$
 pro-isomorphic subgrps

$$\zeta_G^{\hat{}}(s) = \sum_{n=1}^{\infty} a_n(G) n^{-s} \quad (s \in \mathbb{C})$$

$$\alpha^{\hat{}}(G) = \liminf \frac{\log S_n(G)}{\log n} < \infty$$

Prop $\alpha^{\hat{}}(G) \leq d(G) \leq \alpha^{\leq}(G) \leq h(G)$

Prop G, H comm $\rightarrow \alpha^{\leq}(G) = \alpha^{\leq}(H)$

Prop $\zeta_G^{\hat{}}(s) = \prod_p \zeta_{G,p}^{\hat{}}(s)$ (Euler product)

Thm $\zeta_{G,p}^{\hat{}}(s) = \frac{\Phi_p(p^{-s})}{\Psi_p(p^{-s})}$

$\Phi_p, \Psi_p \in \mathbb{Z}[x]$
 $\deg \Phi_p, \deg \Psi_p$
 unif bounded

proof relies on p -adic formalism

(Dedekind, Macintyre)

$$\prod_i (1 - p^{a_i - b_i s})$$

poles on \mathbb{R} are in fact in \mathbb{Q}

gen for p -adic analytic grps by du Sautoy ('93)

Thm (Greenwood, du Sautoy '00)

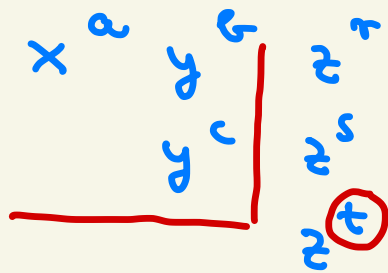
$\zeta_G^{\leq}(s)$ can be expressed in terms

of p -adic 'core integrals'

→ $\alpha^{\leq}(G) \in \mathbb{Q}$ what not come up?
 map continuation how far?
 $S_n^{\leq}(G) \sim c(G) \cdot n^{\alpha^{\leq}(G)} (\log n)^{b(G)}$
(they rem: can do same for $\wedge \dots$)

ex $G \cong \mathbb{Z}^d$ free ab, $\zeta^{\leq} = \zeta^{\wedge}$

$$G = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{pmatrix} \quad H \leq_f G$$



$$a \geq 1, c \geq 1$$

$$0 \leq G < c$$

$$\underline{t = ac} \dots \text{change}$$

$$0 \leq \tau, s < t$$

$$\begin{aligned} \zeta_{G, p}^{\wedge}(s) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} p^k p^{-2(k+l)s} \\ &= (\quad) (\quad) = \underbrace{\zeta_p(2s)}_{\text{from } k} \underbrace{\zeta_p(2s-1)}_{\text{from } l} \\ \zeta_G^{\wedge}(s) &= \zeta(2s) \zeta(2s-1) \end{aligned}$$

• transition to p -adic formalism

$\Gamma \quad \mathbb{Z}$ -gn $\mapsto \mathbb{Z}$ -lie lattice L

$$\text{st } \forall p : \sum_{\Gamma, p} \hat{\Gamma}(s) = \sum_{L, p} \hat{L}(s)$$

$$\left(\forall p \text{ for nice } d \right) = \sum_{\mathbb{Z}_p \otimes L}^{\text{iso}}(s)$$

$$L_p = \mathbb{Z}_p \otimes L \quad d = \dim_{\mathbb{Z}}(L)$$

$$\underline{\text{Aut}}(L) = \underline{G} \leq \text{GL}_d \quad \mathbb{Z}\text{-gr scheme}$$

$$\underline{G}(R) = \text{Aut}_R(R \otimes L)$$

Propn (GLS '88)

$$G_p = \underline{G}(\mathbb{Q}_p) \quad \mathbb{R} \subset p\text{-adic gr, with Haar measure } \mu$$

$$G_p^+ = G_p \cap \text{Mat}_d(\mathbb{Z}_p)$$

$$\cong \text{Aut}(\mathbb{Q}_p \otimes L) \cap \text{End}(\mathbb{Z}_p \otimes L)$$

$$\Rightarrow \forall p : \sum_{L_p}^{\text{iso}}(s) = \int_{G_p^+} |\det(g)|_p^s d\mu$$

du Sautoy, Lubotzky '96:

reduction to \int over reductive part of \underline{G}

Berman, Tam '18

6, 10, 16, ...

connects to classical studies of Hej, Weil, ...