Introduction to subgp growth 17.12.20

Part 3

(The Sem from 12 I '21 :
5-6 lectures on
SGG ...)

'puzzle' : 6, 10, 16, 26, 44, 78, ...

recap : lec 1 : a_n, s_n, ζ, free gps → fast growth

lec 2 : PSG, Heisenberg gp, $\alpha(G)$

today :
- continue with fg (torsion-free) nilp gp
- connect to p-adic formalism / integration

[G fg tf nilp gp ($\mathcal{T}$-gps)
(→ resid finite)

∃ Malcev basis ... Malcev compl / radicable hull

$\mathcal{G}$ h-dim'l unipotent affine alg gp
(h Hirsch length of G)

$G^R = \mathcal{G}(R)$ gp of R-valued pts
← binomial ring , ie $\frac{r(r-1)\cdots(r-n+1)}{n!} \in R$ ($r \in R$, $n \in \mathbb{N}$)

$G^{\mathbb{Z}} = G$, $G^{\mathbb{Q}}$ Malcev compl for G, H $\mathbb{Z}$-grps, write

$$G \cong_{\mathbb{R}} H \iff_{\text{def}} G^{\mathbb{R}} \cong H^{\mathbb{R}}$$

for instance,

$$\mathbb{A}_\infty = \prod_p \mathbb{Z}_p \quad \text{finite integral adèles}$$

$$\mathbb{A}_\infty \subset \mathbb{A}$$

$$G^{\mathbb{A}_\infty} = \prod_p \underbrace{G^{\mathbb{Z}_p}}_{\text{pro-}p\text{ compl}} \cong \hat{G} \quad \text{profinite compl}$$

$$\begin{tikzcd} G & & H \\ K_1 \arrow[u, "fi"] \arrow[rr, phantom, "\cong"] & & K_2 \arrow[u, "fi"'] \end{tikzcd}$$

<u>facts</u>:

$$G \cong_{\mathbb{Q}} H \iff G, H \text{ commensurable}$$

$$\forall p: G \cong_{\mathbb{Q}_p} H \iff G \cong_{\mathbb{A}} H \iff \hat{G}, \hat{H} \text{ commen.}$$

$$\boxed{G \cong_{\mathbb{A}_\infty} H \iff \hat{G} \cong \hat{H}}$$

[Pickel: each $\mathbb{A}_\infty$-class splits into fin many $\cong$-classes]

$$\exists p: G \cong_{\mathbb{Q}_p} H \iff \exists p: G \cong_{\mathbb{Z}_p} H$$

$$\iff G \cong_{\mathbb{C}} H$$

- <u>back to SGG & ζ fcts</u>

[Grunewald, Segal, Smith '88]

G, H $\mathbb{Z}$-grps

$\#\{H \leq G \mid |G:H|, \hat{G} \cong \hat{H}\}$
pro-isomorphic subgrps

$$\zeta_G^{\hat{(\leq)}}(s) = \sum_{n=1}^{\infty} a_n^{\hat{}}(G)\, n^{-s} \qquad (s \in \mathbb{C})$$

$$\alpha^{\hat{(\leq)}}(G) = \overline{\lim} \frac{\log s_n^{\hat{}}(G)}{\log n} < \infty$$

<u>Prop</u> $\alpha^{\wedge}(G) \leq d(G) \leq \alpha^{\leq}(G) \leq h(G)$

<u>Prop</u> G, H comm $\to \alpha^{\leq}(G) = \alpha^{\leq}(H)$

<u>Prop</u> $\zeta_G^{\hat{(\leq)}}(s) = \prod_p \zeta_{G,p}^{\hat{(\leq)}}(s)$ (Euler product)

<u>Thm</u> $$\boxed{\zeta_{G,p}^{\hat{(\leq)}}(s) = \frac{\Phi_p(p^{-s})}{\Psi_p(p^{-s})}}$$

$\Phi_p, \Psi_p \in \mathbb{Z}[X]$

$\deg \Phi_p, \deg \Psi_p$ unif bounded

proof relies on p-adic formalism (Denef, Macintyre)

$\prod_i (1 - p^{a_i - b_i s})$

poles on $\mathbb{R}$ are in fact in $\mathbb{Q}$

gen for p-adic analytic grps by du Sautoy ('93)

<u>Thm</u> (Grunewald, du Sautoy '00)

$\zeta_G^{\leq}(s)$ can be expressed in terms of p-adic 'cone integrals'

$\to$ $\alpha^{\leq}(G) \in \mathbb{Q}$ — what rat come up?

merom continuation — how far?

$s_n^{\leq}(G) \sim c(G) \cdot n^{\alpha^{\leq}(G)} (\log n)^{b(G)}$

(they ten: can do same for $\wedge$ ...)

ex $G \cong \mathbb{Z}^d$ free ab, $\zeta^{\leq} = \zeta^{\wedge}$

$G = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{pmatrix}$ $H \leq_f G$

$$\begin{array}{cc|c} x^a & y^b & z^r \\ & y^c & z^s \\ \hline & & z^{t} \end{array}$$

$a \geq 1$, $c \geq 1$

$0 \leq b < c$

$t = ac$ ... change

$0 \leq r, s < t$

$\rightsquigarrow$ $\zeta^{\wedge}_{G,p}(s) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} p^{\ell} p^{-2(k+\ell)s}$

$= (\quad)(\quad) = \underbrace{\zeta_p(2s)}_{\text{from } k} \underbrace{\zeta_p(2s-1)}_{\text{from } \ell}$

$\zeta_G^{\wedge}(s) = \zeta(2s)\zeta(2s-1)$

- transition to p-adic formalism

Γ $\mathcal{T}$-gp $\rightsquigarrow$ $\mathbb{Z}$-Lie lattice L

st $\forall p$: $\zeta^{\wedge}_{\Gamma, p}(s) = \zeta^{\wedge}_{L,p}(s)$

($\forall p$ for nilp cl 2) $= \zeta^{iso}_{\mathbb{Z}_p \otimes L}(s)$

$L_p = \mathbb{Z}_p \otimes L$ $\qquad d = \dim_{\mathbb{Z}}(L)$

$\underline{\mathrm{Aut}}(L) = \underline{G} \leq GL_d$ $\quad$ $\mathbb{Z}$-gp scheme

$\underline{G}(R) = \mathrm{Aut}_R(R \otimes L)$

Prop (GSS '88)

$G_p = \underline{G}(\mathbb{Q}_p)$ lc p-adic gp, with Haar measure μ

$G_p^+ = G_p \cap \mathrm{Mat}_d(\mathbb{Z}_p)$

$\cong \mathrm{Aut}(\mathbb{Q}_p \otimes L) \cap \mathrm{End}(\mathbb{Z}_p \otimes L)$

2) $\forall p$: $\zeta^{iso}_{L_p}(s) = \int_{G_p^+} |\det(g)|_p^s \, d\mu$

du Sautoy, Lubotzky '96 :

reduction to $\int$ over reductive part of $\underline{G}$

Berman, Onn '18

connects to classical studies of Hey, Weil, ...

$\boxed{6, 10, 16, \ldots}$