Around Ax-Kochen/Ershov transfer principle

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Long story short...

Definition

Given integers $i \geqslant 0$ and $d \geqslant 1$, a field K is called

- $C_i(d)$ if every homogeneous polynomial of degree d with coefficients in K and $n > d^i$ variables has a non-trivial solution in K.
- C_i if it is $C_i(d)$ for every $d \geqslant 1$.

So far in this workshop we have proved:

- K is algebraically closed if and only if K is C_0.
- If K is C_1, then K has trivial Brauer group.
- Finite fields are C_1.
- If K is C_i and $L|K$ is a finite extension, then L is C_i.
- If K is C_i, then $K(t)$ is C_{i+1}.
- If K is C_i, then $K(\!(t)\!)$ is C_{i+1}.
- $\mathbb{F}_p(t)$ and $\mathbb{F}_p(\!(t)\!)$ are C_2, and $\mathbb{C}(\!(t)\!)$ is C_1 (so in particular, it has trivial Brauer group).
- $\mathbb{Q}_p$ is $C_2(3)$.

Long story short...

Knowing that the fields $\mathbb{F}_p((t))$ and $\mathbb{Q}_p$ share many properties, and that $\mathbb{F}_p((t))$ was C_2, Artin conjectured that $\mathbb{Q}_p$ was also C_2. However, (as for $\mathbb{Q}$ and $\mathbb{R}$) $\mathbb{Q}_p$ is not C_i for any i.
However, Ax and Kochen showed that, to a certain extend, Artin had the right intuition.

Theorem (Ax-Kochen)

Fix $d > 0$. Then, there is a finite set E_d of prime numbers such that for every $p \notin X_d$, $\mathbb{Q}_p$ is $C_2(d)$.

This is the situation:

- Since $\mathbb{Q}_p$ is not C_i for every $i \geqslant 0$: for every $i \geqslant 0$ and every prime p, there is $d = d(i, p)$ such that $\mathbb{Q}_p$ is not $C_i(d)$.
- By the previous theorem, for every $d \geqslant 1$, there is $N = N(d) \geqslant 1$ such that if $p > N$ then $\mathbb{Q}_p$ is $C_2(d)$ (and hence $C_i(d)$ for every $i \geqslant 2$).

How similar are $\mathbb{F}_p((t))$ and $\mathbb{Q}_p$?

Clearly we have that

$$\mathbb{F}_p((t)) \ncong \mathbb{Q}_p$$

but what kind of relation could one establish between these two fields?

$$\mathbb{F}_p((t)) \overbrace{\sim}^{?} \mathbb{Q}_p$$

- both fields are complete (and hence henselian)
- both fields have residue field $\mathbb{F}_p$
- both fields have value group $\mathbb{Z}$

The key is to forget the previous question and look at the classes $(\mathbb{F}_p((t))_p)_{p>0}$ and $(\mathbb{Q}_p)_{p>0}$ asymptotically!

The transfer principle

Theorem (Ax-Kochen/Ershov)

Let φ *be a first order sentence in the language of valued fields. Then there is a finite set of prime numbers* E_φ *such that for all* $p \notin E_\varphi$

$$\varphi \text{ holds in } \mathbb{Q}_p \text{ if and only if } \varphi \text{ holds in } \mathbb{F}_p((t)).$$

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$$\mathbb{Q}_p \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \models \varphi$$

What is a first order formula in the language of... rings?

Informally, a first-order formula in the language of rings $\mathcal{L}_{\text{ring}}$ is a formal expression build up using

- variables x_1, x_2, x, y, z etc.
- boolean connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
- quantifiers $\forall, \exists$
- the equality symbol $=$
- the formal symbols of the language of rings $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 1, 0\}$
- and parenthesis symbols (for convenience),

following "natural" rules of construction.

The slogan: "given a finite sequence of symbols φ build up from the symbols above, if after replacing (the free occurrences of) variables by elements in any ring A, we obtain a statement which is true or false in A, then φ is an $\mathcal{L}_{\text{ring}}$-formula".

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$1 + 1 + 1 + 1 + 1 = 0$	$(\forall n \in \mathbb{N})(\exists x)(n \cdot x = y)$
	$\neg(1 + 1 + 1 = \pi)$

What is a first order formula in the language of... rings?

We will from now on abuse of notation, and write expressions like

$$
\begin{array}{rcr}
x^n & \text{for} & x \cdot \overbrace{\cdots}^{n-\text{times}} \cdot x \\
xy & \text{for} & x \cdot y \\
x \neq y & \text{for} & \neg(x = y) \\
x - y & \text{for} & x + (-y) \\
\bigwedge_{i=1}^{n} \varphi_i & \text{for} & \varphi_1 \wedge \cdots \wedge \varphi_n \text{ (where each } \varphi_i \text{ is a formula)}
\end{array}
$$

For example,

$$(\forall x)(x^2 + y^2 = 1 \to xy \neq 0)$$

is an abbreviation for the $\mathcal{L}_{\text{ring}}$-formula

$$(\forall x)((x \cdot x + y \cdot y = 1) \to \neg(x \cdot y = 0))$$

What is a first order formula in the language of... valued fields?

We basically play the same game as for $\mathcal{L}_{\text{ring}}$ but we add one new formal symbol: a binary relation $\text{VF}(x, y)$ which we interpret in any valued field (K, v) as

$$\text{VF}(x, y) \Leftrightarrow v(x) \leqslant v(y).$$

We denote this language $\mathcal{L}_{\text{VF}}$.
The following are examples of $\mathcal{L}_{\text{VF}}$-formulas

- $\text{VF}(1, x)$
- $\text{VF}(x^2 + 1, xy - 1) \to x \neq 1$
- $(\forall x)(\forall y)(\text{VF}(1, x) \wedge \text{VF}(1, y) \to \text{VF}(1, x + y))$
- every $\mathcal{L}_{\text{ring}}$-formula!

Sentences

An **$\mathcal{L}$-sentence** (where $\mathcal{L}$ is either $\mathcal{L}_{\text{ring}}$ or $\mathcal{L}_{\text{VF}}$) is an $\mathcal{L}$-formula which has no free variables.

In particular, if φ is an $\mathcal{L}_{\text{ring}}$-sentence, then for any ring A it either holds in A or not. Similarly, if φ is an $\mathcal{L}_{\text{VF}}$-sentence and (K, v) is a valued field then φ either holds in K or not.

Sentences

Examples:

- $(\exists x)(x^2 = -1)$
- $(\forall x)(\forall y)(xy = yx)$
- $\chi_p := 1 + \overbrace{\cdots}^{p} + 1 = 0$
- $(\forall y_0) \cdots (\forall y_m)(\forall x)(\forall z)(\neg(\bigwedge_{i=0}^{m} y_i = 0) \to (\sum_{i=0} y_i x^i = \sum_{i=0} y_i x^i \to x = z)).$

A trivial instance of the transfer principle (in order to get used to it)

Theorem (Ax-Kochen/Ershov)

Let φ be a $\mathcal{L}_{\mathrm{VF}}$-sentence in the language of valued fields. Then there is a finite set of prime numbers E_φ such that for all $p \notin E_\varphi$

$$\mathbb{Q}_p \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \models \varphi$$

Given a prime number q, consider the sentence $\chi_q := 1 + \overbrace{\cdots}^{q} + 1 = 0$.

Clearly, if p is a prime number bigger than q, we have both $\mathbb{F}_p((t)) \not\models \chi_q$ and $\mathbb{Q}_p \not\models \chi_q$ so setting $E_{\chi_q} = \{q' \in \mathbb{P} : q' \leqslant q\}$ we have that for all $p \notin E_{\chi_q}$

$$\mathbb{Q}_p \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \models \varphi$$

A less trivial application

Theorem (Ax-Kochen/Ershov)

Let φ *be a* $\mathcal{L}_{\mathrm{VF}}$*-sentence in the language of valued fields. Then there is a finite set of prime numbers* E_φ *such that for all* $p \notin E_\varphi$

$$\mathbb{Q}_p \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \models \varphi$$

Proposition

The property "having characteristic 0" is not expressible by a $\mathcal{L}_{\mathrm{VF}}$*-sentence.*

How to express being $C_i(d)$ in the language of rings?

For integers $d > 0$, $i \geqslant 0$ and $n > d^i$, say that a field K satisfies the property $C_i(d, n)$ if every homogeneous polynomial of degree d in n variables with coefficients in K has a non-trivial root in K.
Clearly, K is $C_i(d)$ if it satisfies $C_i(d, n)$ for every $n > d^i$.

Being $C_i(d, n)$ is expressible by an $\mathcal{L}_{\text{ring}}$-sentence!

How to express being $C_i(d)$ in the language of rings?

Being $C_i(d,n)$ is expressible by an $\mathcal{L}_{\text{ring}}$-sentence! Indeed, set

- $x = (x_1, \ldots, x_n)$,
- let $I \subseteq \mathbb{N}^d$ be the set of tuples such that the sum of its coordinates is equal to d, so for $i = (i_1, \ldots, i_d) \in I$
$$\sum_{j=1}^{d} i_j = d,$$
- for $i \in I$, let $x^i = \prod_{j=1}^{n} x_j^{i_j}$
- let N be the cardinality of I and $s \colon I \to \{1, \ldots, N\}$ be a bijection.

Then, let $\varphi(d,i,n)$ be the $\mathcal{L}_{\text{ring}}$-sentence

$$(\forall y_1)\cdots(\forall y_N)(\exists x_1)\cdots(\exists x_n)(\neg \bigwedge_{j=1}^{N} y_i = 0 \to (\neg \bigwedge_{j=1}^{n} x_i = 0 \wedge \sum_{i \in I} y_{s(i)} x^i = 0))$$

Applying the transfer principle

We have that for every (K, v)

$$(K, v) \models \varphi(d, i, n) \Leftrightarrow (K, v) \text{ is } C_i(d, n)\ .$$

We can apply the transfer principle to the $\mathcal{L}_{\text{ring}}$-sentence $\varphi(d, 2, n)$ and obtain that there is a finite set of primes $E = E(d, 2, n)$ such that for all $p \notin E$

$$\mathbb{Q}_p \text{ is } C_2(d, n)\ \Leftrightarrow \mathbb{Q}_p \models \varphi(d, 2, n) \Leftrightarrow \mathbb{F}_p((t)) \models \varphi(d, 2, n) \Leftrightarrow \mathbb{F}_p((t)) \text{ is } C_2(d, n)\ .$$

Since $\mathbb{F}_p((t))$ is C_2, we have in particular that $\mathbb{F}_p((t))$ is $C_2(d, n)$, and therefore, $\mathbb{Q}_p \models C_2(d, n)$ for all primes $p \notin E$.

But how to show that $\mathbb{Q}_p$ is actually $C_2(d)$ for all but finite many primes? Here we use simple trick:

$$K \text{ is } C_2(d, n) \text{ for all } n > d^2 \Leftrightarrow K \text{ is } C_2(d, d^2 + 1).$$

Applying the transfer principle

$$K \text{ is } C_2(d,n) \text{ for all } n > d^2 \Leftrightarrow K \text{ is } C_2(d, d^2+1).$$

Applying the transfer principle

We apply the transfer principle to the $\mathcal{L}_{\text{ring}}$-sentence $\varphi = \varphi(d, 2, d^2 + 1)$ and obtain that there is a finite set of primes $E = E(d)$ such that for all $p \notin E$

$$\mathbb{Q}_p \text{ is } C_2(d) \Leftrightarrow \mathbb{Q}_p \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \text{ is } C_2(d) .$$

Since $\mathbb{F}_p((t))$ is C_2, we have in particular that $\mathbb{F}_p((t))$ is $C_2(d)$, and therefore, $\mathbb{Q}_p \models C_2(d)$ for all primes $p \notin E$.

Another application

Is there perhaps a similar trick in order to express the property C_2 (resp. C_i for $i \geqslant 2$) as a first order sentence in $\mathcal{L}_{\text{ring}}$ or $\mathcal{L}_{\text{VF}}$?

No. Suppose for a contradiction it was an let ψ be an $\mathcal{L}_{\text{VF}}$-sentence such that for K either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$

$$K \text{ is } C_2 \Leftrightarrow K \models \psi.$$

Then by the transfer principle where would be a finite set of primes E_ψ such that for every $p \notin E_\psi$

$$\mathbb{Q}_p \models \psi \Leftrightarrow \mathbb{F}_p((t)) \models \psi$$

But we know that $\mathbb{Q}_p$ is not C_2 (resp. not C_i for every $i \geqslant 0$), so $\mathbb{Q}_p \not\models \psi$. But then this implies that there are primes p for which $\mathbb{F}_p((t)) \not\models \psi$, and hence $\mathbb{F}_p((t))$ is not C_2, a contradiction. Hence the property C_i is not expressible by an $\mathcal{L}_{\text{VF}}$-sentence. Note: the property C_i is of course an infinite conjunction of $\mathcal{L}_{\text{ring}}$-sentences, namely the sentences $C_i(d, d^i + 1)$.

Many thanks for your attention.