Triangulated Categories and Localization

"Last week":

more generally of an abelian category $\mathcal{A}$

$Ch(Ab)$ = category of chain complexes of abelian groups and chain maps

$K(Ab)$ = category of chain complexes of abelian groups and homotopy classes of chain maps

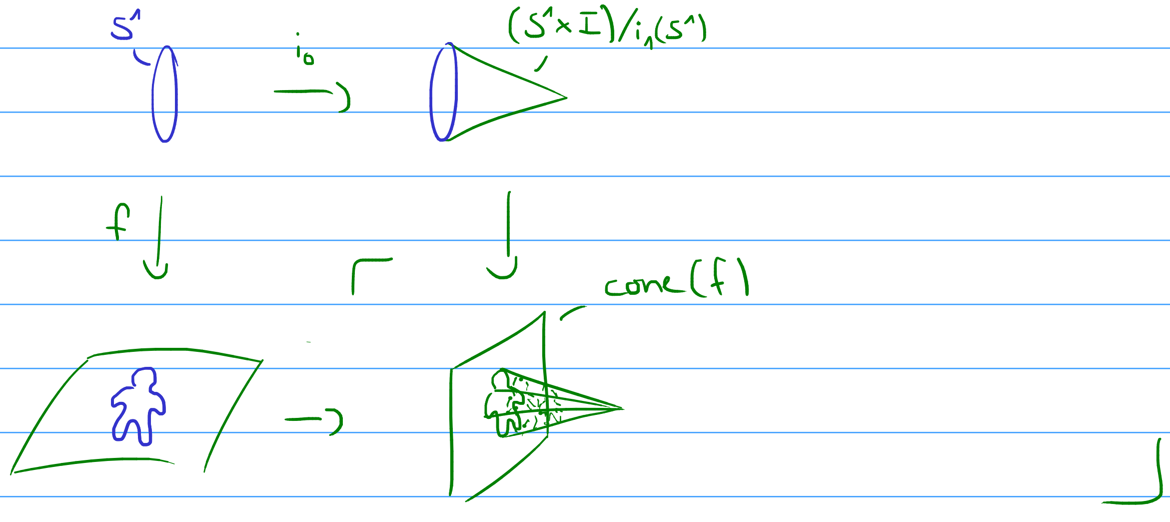
We will now introduce mapping cones of chain maps.

Quick recollection from algebraic topology:

$$\mathrm{cone}(f) = \mathrm{colim}\left(\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d, "f"'] & (X \times I)/i_1(X) \\ Y & \end{tikzcd} \right)$$

$$= \left(Y \amalg (X \times I)/i_1(X)\right)/_{f(x) \sim i_0(x)}$$

Ex.:



Def.:

The mapping cone of a chain map $f_\bullet : (X_\bullet, d_\bullet^X) \to (Y_\bullet, d_\bullet^Y)$ is

$$\mathrm{cone}(f)_\bullet = X_{\bullet-1} \oplus Y_\bullet$$

together with the differential

$$d_\bullet^{\mathrm{cone}(f)} = \begin{pmatrix} -d^X_{\bullet-1} & 0 \\ -f_{\bullet-1} & d^Y_\bullet \end{pmatrix}.$$

Rem.:

Why should this be a mapping cone?

Using the "unit interval chain complex" $I_\bullet$, we can mimic the topological definition of mapping cones. Turns out to result in our definition of mapping cones for chain maps.

The mapping cone of a chain map $f_\bullet : X_\bullet \to Y_\bullet$ fits into the following short exact sequence in $\mathrm{Ch}(\mathrm{Ab})$:

$$0_\bullet \to Y_\bullet \xrightarrow{(i_2)_\bullet} \mathrm{cone}(f)_\bullet \xrightarrow{(-pr_1)_\bullet} X_\bullet[-1] \to 0_\bullet$$

$$\Vert$$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r, "0"] \arrow[d, "0"] & Y_n \arrow[r, "i_2"] \arrow[d, "d_n^Y"] \arrow[dr, phantom, "1"] & X_{n-1} \oplus Y_n \arrow[r, "-pr_1"] \arrow[d, "\begin{pmatrix} -d^X_{n-1} & 0 \\ -f_{n-1} & d^Y_n \end{pmatrix}"] \arrow[dr, phantom, "2"] & X_{n-1} \arrow[r, "0"] \arrow[d, "-d^X_{n-1}"] & 0 \arrow[d, "0"] & \text{degree } n \\
0 \arrow[r, "0"] \arrow[d] & Y_{n-1} \arrow[r, "i_2"] \arrow[d] & X_{n-2} \oplus Y_{n-1} \arrow[r, "-pr_1"] \arrow[d] & X_{n-2} \arrow[r, "0"] \arrow[d] & 0 \arrow[d] & \text{degree } n-1 \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$$\begin{tikzcd} y \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "(1)"] & \begin{pmatrix} 0 \\ y \end{pmatrix} \arrow[d, mapsto] \\ d_n^Y(y) \arrow[r, mapsto] & \begin{pmatrix} 0 \\ d_n^Y(y) \end{pmatrix} \end{tikzcd} \qquad \begin{tikzcd} \begin{pmatrix} x \\ y \end{pmatrix} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "(2)"] & -x \arrow[d, mapsto] \\ \begin{pmatrix} -d_{n-1}(x) \\ -f_{n-1}(x) + d_n^Y(y) \end{pmatrix} \arrow[r, mapsto] & d_{n-1}(x) \end{tikzcd}$$

and

clearly exact.

This data, considered in $K(Ab)$, is usually written in the form

will use such arrows to indicate that the morphism actually maps to $X_\bullet[-1]$

$$\begin{tikzcd} & \mathrm{cone}(f)_\bullet \arrow[dl, dashed, "(-pr_1)_\bullet"'] & \\ X_\bullet \arrow[rr, "f_\bullet"] & & Y_\bullet \arrow[ul, "(i_2)_\bullet"'] \end{tikzcd}$$

and called the strict triangle on $f_\bullet : X_\bullet \to Y_\bullet$.

Def.:

A diagram in $K(Ab)$ of the form

$$\begin{tikzcd} & Z_\bullet \arrow[dl, dashed] & \\ X_\bullet \arrow[rr] & & Y_\bullet \arrow[ul] \end{tikzcd}$$

is an exact triangle on $(X_\bullet, Y_\bullet, Z_\bullet)$, if it is isomorphic to a strict triangle.

commutative diagram

$$\begin{tikzcd}
X_\bullet \arrow[r, "f_\bullet"] \arrow[d, "\alpha_\bullet"] & Y_\bullet \arrow[r, "g_\bullet"] \arrow[d, "\beta_\bullet"] & Z_\bullet \arrow[r, "h_\bullet"] \arrow[d, "\gamma_\bullet"] & X[-1] \arrow[d, "\alpha_\bullet[-1]"] \\
X'_\bullet \arrow[r, "f'_\bullet"] & Y'_\bullet \arrow[r, "(i_2)_\bullet"] & \operatorname{cone}(f')_\bullet \arrow[r, "(-pr_1)_\bullet"] & X'_\bullet[-1]
\end{tikzcd}$$

in $K(Ab)$ such that the vertical morphisms are isomorphisms

Ex.:

- $$\begin{tikzcd}
 & 0_\bullet \arrow[dl, dotted, "0"'] & \\
X_\bullet \arrow[rr, equal] & & X_\bullet \arrow[ul, "0"']
\end{tikzcd}$$ is an exact triangle:

consider $\operatorname{cone}(id_{X_\bullet})_* = X_{*-1} \oplus X_*$ with differential

$$d_* = \begin{pmatrix} -d^X_{*-1} & 0 \\ -id_{X_{*-1}} & d^X_* \end{pmatrix}.$$

and the diagram

$$\begin{tikzcd}
\ldots \arrow[r, "d_{n+2}"] & X_n \oplus X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, equal, shift right] \arrow[d, "0", shift left] & X_{n-1} \oplus X_n \arrow[r, "d_n"] \arrow[d, equal, shift right] \arrow[d, "0", shift left] \arrow[dl, "\eta_n"'] & X_{n-2} \oplus X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, equal, shift right] \arrow[d, "0", shift left] \arrow[dl, "\eta_{n-1}"'] & \ldots \\
\ldots \arrow[r, "d_{n+2}"] & X_n \oplus X_{n+1} \arrow[r, "d_{n+1}"] & X_{n-1} \oplus X_n \arrow[r, "d_n"] & X_{n-2} \oplus X_{n-1} \arrow[r, "d_{n-1}"] & \ldots
\end{tikzcd}$$

with $\eta_n = \begin{pmatrix} 0 & -id_{X_n} \\ 0 & 0 \end{pmatrix}$.

Since

$$\underbrace{d_{n+1} \circ \eta_n}_{= \begin{pmatrix} -d_n^X & 0 \\ -id_{X_n} & d_{n+1}^X \end{pmatrix} \begin{pmatrix} 0 & -id_{X_n} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d_n^X \\ 0 & id_{X_n} \end{pmatrix}} + \underbrace{\eta_{n-1} \circ d_n}_{= \begin{pmatrix} 0 & -id_{X_{n-1}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_{n-1}^X & 0 \\ -id_{X_{n-1}} & d_n^X \end{pmatrix} = \begin{pmatrix} id_{X_{n-1}} & -d_n^X \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} id_{X_{n-1}} & 0 \\ 0 & id_{X_n} \end{pmatrix} = id_{conc(id_{X_\bullet})_n}$$

$$= id_{conc(id_{X_\bullet})_n} - 0,$$

we have $conc(id_{X_\bullet})_* \simeq 0_*$, which yields an isomorphism

$$\begin{tikzcd} & 0_\bullet \arrow[dl, dashed] & \\ X_\bullet \arrow[rr, "="] & & X_\bullet \arrow[ul] \end{tikzcd} \qquad \text{exact triangle}$$

$$\wr\!\wr$$

$$\begin{tikzcd} & conc(id_{X_\bullet})_\bullet \arrow[dl, dashed] & \\ X_\bullet \arrow[rr, "="] & & X_\bullet \arrow[ul] \end{tikzcd} \qquad \text{strict triangle}$$

of diagrams in $K(Ab)$.

- If

$$\begin{tikzcd} & Z_\bullet \arrow[dl, dashed, "h_\bullet"'] & \\ X_\bullet \arrow[rr, "f_\bullet"] & & Y_\bullet \arrow[ul, "g_\bullet"'] \end{tikzcd}$$

is an exact triangle, then so are its rotates

$$\begin{tikzcd} & X_\bullet[-1] \arrow[dl, dashed, "-f_\bullet[-1]"'] & \\ Y_\bullet \arrow[rr, "g_\bullet"] & & Z_\bullet \arrow[ul, "h_\bullet"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y_\bullet \arrow[dl, dashed, "g_\bullet"'] & \\ Z_\bullet[1] \arrow[rr, "-h_\bullet[1]"] & & X_\bullet \arrow[ul, "f_\bullet"'] \end{tikzcd}$$

We will show that

$$\begin{tikzcd}
 & X_\bullet[-1] \arrow[dl, dashed, "-f_\bullet[-1]"'] & \\
Y_\bullet \arrow[rr, "g_\bullet"] & & Z_\bullet \arrow[ul, "h_\bullet"']
\end{tikzcd}$$

is exact. The argument for the other rotate is similar.

It suffices to treat the case of a strict triangle

$$\begin{tikzcd}
 & \mathrm{cone}(f_*)_\bullet \arrow[dl, dashed, "(-pr_1)_\bullet"'] & \\
X_\bullet \arrow[rr, "f_\bullet"] & & Y_\bullet \arrow[ul, "(i_2)_\bullet"']
\end{tikzcd}$$

with its rotate

$$\begin{tikzcd}
 & X_\bullet[-1] \arrow[dl, dashed, "-f_\bullet[-1]"'] & \\
Y_\bullet \arrow[rr, "(i_2)_\bullet"] & & \mathrm{cone}(f_*)_\bullet \arrow[ul, "(-pr_1)_\bullet"']
\end{tikzcd}.$$

Consider the morphisms φ and ψ

$$\begin{tikzcd}
Y_\bullet \arrow[r, "(i_2)_\bullet"] \arrow[d, equal] & \mathrm{cone}(f_*)_\bullet \arrow[r, "(-pr_1)_\bullet"] \arrow[d, equal] & X_\bullet[-1] \arrow[r, "-f_\bullet[-1]"] \arrow[d, shift right, "\begin{pmatrix} -f \\ id \\ 0 \end{pmatrix}"'] & Y_\bullet[-1] \arrow[d, equal] \\
Y_\bullet \arrow[r, "(i_2)_\bullet"] & \mathrm{cone}(f_*)_\bullet \arrow[r, "(i_2)_\bullet"] & \mathrm{cone}((i_2)_*)_\bullet \arrow[r, "(-pr_1)_\bullet"] \arrow[u, shift right, "(0, id, 0)"'] & Y_\bullet[-1]
\end{tikzcd}$$

$$\mathrm{cone}((i_2)_*)_\bullet = Y_{\bullet-1} \oplus \mathrm{cone}(f_*)_\bullet = Y_{\bullet-1} \oplus X_{\bullet-1} \oplus Y_\bullet$$

of triangles. Now we have

$$\psi \circ \varphi = (0,\ id,\ 0) \circ \begin{pmatrix} -f \\ id \\ 0 \end{pmatrix} = id$$

and

$$\varphi \circ \psi = \begin{pmatrix} -f \\ id \\ 0 \end{pmatrix} \circ (0,\ id,\ 0) = \begin{pmatrix} 0 & -f & 0 \\ 0 & id & 0 \\ 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} id & 0 & 0 \\ 0 & id & 0 \\ 0 & 0 & id \end{pmatrix}$$

$$\begin{pmatrix} id & 0 & 0 \\ 0 & id & 0 \\ 0 & 0 & id \end{pmatrix} - \begin{pmatrix} 0 & -f & 0 \\ 0 & id & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} id & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & id \end{pmatrix}$$

$$= \begin{pmatrix} id & f & -d^Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & d^Y \\ 0 & 0 & 0 \\ 0 & 0 & id \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & -id \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -d^Y & 0 & 0 \\ 0 & -d^X & 0 \\ -i_2 & -f & d^Y \end{pmatrix} + \begin{pmatrix} -d^Y & 0 & 0 \\ 0 & -d^X & 0 \\ -i_2 & -f & d^Y \end{pmatrix} \begin{pmatrix} 0 & 0 & -id \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{homotopy}} ,$$

so that the rotate is an exact triangle.

By definition of exact triangles, the functoriality of homology yields:

Prop.:
For every exact triangle

$$\begin{tikzcd} & Z_\bullet \arrow[dl, dashed, "h_\bullet"'] & \\ X_\bullet \arrow[rr, "f_\bullet"] & & Y_\bullet \arrow[ul, "g_\bullet"'] \end{tikzcd}$$

the induced sequence

$$\ldots \xrightarrow{H_{n+1}(g)} H_{n+1}(Z) \xrightarrow{H_{n+1}(h)} H_n(X) \xrightarrow{H_n(f)} H_n(Y) \xrightarrow{H_n(g)} H_n(Z) \xrightarrow{H_n(h)} H_{n-1}(X) \xrightarrow{H_{n-1}(f)} \ldots$$

on the homology groups is exact.

Triangulated categories

Let $\mathcal{C}$ be a category and let $T \in \mathrm{Aut}(\mathcal{C})$. A diagram in $\mathcal{C}$ of the form

here now $Z \to T(X)$

$$\begin{tikzcd} & Z \arrow[dl, dashed] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

will be called a triangle in $\mathcal{C}$ (with respect to T).

This notion allows us to generalize the structure exact triangles give to $K(Ab)$:

Def.:
An additive category $\mathcal{C}$ together with an additive automorphism $T \in \mathrm{Aut}(\mathcal{C})$ (called translation/shift functor) and a collection Δ of distinguished triangles in $\mathcal{C}$ (with respect to T) (also called exact triangles in $\mathcal{C}$) is called a triangulated category, if

existence axiom

(TC1) For every morphism $f: X \to Y$ in $\mathcal{C}$ there exists an object $Z \in \mathcal{C}$ together with morphisms $g: Y \to Z$ and $h: Z \to T(X)$ such that

$$\begin{tikzcd} & Z \arrow[dl, dashed, "h"'] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd} \in \Delta.$$

(TC2) • $$\begin{tikzcd} & 0 \arrow[dl, dashed, "\text{can.}"'] & \\ X \arrow[rr, equal] & & X \arrow[ul, "\text{can.}"'] \end{tikzcd} \in \triangle$$

bookkeeping axiom

for all objects $X \in \mathcal{C}$

• $\triangle$ is closed under rotations.

• $\triangle$ is closed under isomorphisms.

$$\begin{tikzcd} & Y \arrow[dl, dashed, "g"'] & \\ T^{-1}(Z) \arrow[rr, "-T^{-1}(h)"] & & X \arrow[ul, "f"'] \end{tikzcd} \in \triangle$$

$\Uparrow$

$$\begin{tikzcd} & Z \arrow[dl, dashed, "h"'] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd} \in \triangle$$

$\Downarrow$

$$\begin{tikzcd} & T(X) \arrow[dl, dashed, "-T(f)"'] & \\ Y \arrow[rr, "g"] & & Z \arrow[ul, "h"'] \end{tikzcd} \in \triangle$$

(TC3) For each two distinguished triangles

morphism axiom

$$\begin{tikzcd} & Z \arrow[dl, dashed, "h"'] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd} , \quad \begin{tikzcd} & Z' \arrow[dl, dashed, "h'"'] & \\ X' \arrow[rr, "f'"] & & Y' \arrow[ul, "g'"'] \end{tikzcd} \in \triangle$$

and morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commutes, there exists a morphism $\gamma : Z \to Z'$ such that

$$\begin{tikzcd} & Z \arrow[dl, dashed, "h"'] \arrow[dd, "\gamma"] & \\ X \arrow[rr, "f"] \arrow[dd, "\alpha"'] & & Y \arrow[ul, "g"'] \arrow[dd, "\beta"] \\ & Z' \arrow[dl, dashed, "h'"'] & \\ X' \arrow[rr, "f'"] & & Y' \arrow[ul, "g'"'] \end{tikzcd}$$

$$= \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, "\gamma"] & T(X) \arrow[d, "T(\alpha)"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X') \end{tikzcd}$$

is a morphism a triangles.

(TC4) Whenever

octahedron axiom

$$
\begin{tikzcd}[column sep=small]
 & Z' \arrow[dl, dashed, "\beta"'] & \\
X \arrow[rr, "f"'] & \arrow[u, phantom, "1"] & Y \arrow[ul, "\alpha"']
\end{tikzcd},
\quad
\begin{tikzcd}[column sep=small]
 & X' \arrow[dl, dashed, "\delta"'] & \\
Y \arrow[rr, "g"'] & \arrow[u, phantom, "2"] & Z \arrow[ul, "\gamma"']
\end{tikzcd},
\quad
\begin{tikzcd}[column sep=small]
 & Y' \arrow[dl, dashed, "\xi"'] & \\
X \arrow[rr, "g\circ f"'] & \arrow[u, phantom, "3"] & Z \arrow[ul, "\varepsilon"']
\end{tikzcd}
\in \triangle,
$$

there exist morphisms $\eta : Y' \to X'$ and $\vartheta : X' \to Z'$ such that

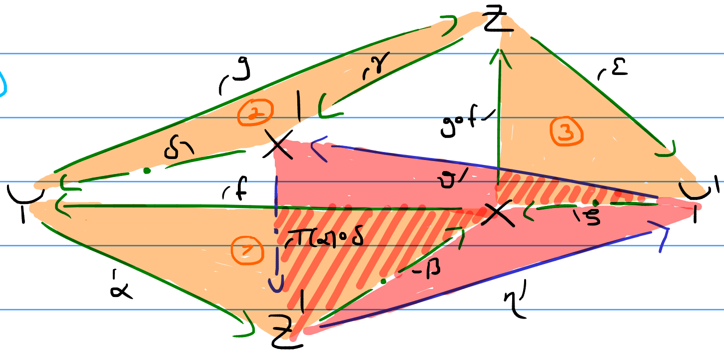
- $$
\begin{tikzcd}[column sep=small]
 & X' \arrow[dl, dashed, "T(\alpha)\circ\delta"'] & \\
Z' \arrow[rr, "\eta"'] & \arrow[u, phantom, "4"] & Y' \arrow[ul, "\vartheta"']
\end{tikzcd}
\in \triangle
$$

- the four diagrams

$$
\begin{tikzcd}[column sep=small]
 & T(X) & \\
Z' \arrow[ur, "\beta"] \arrow[rr, "\eta"'] & & Y' \arrow[ul, "\xi"']
\end{tikzcd},
\quad
\begin{tikzcd}[column sep=small]
 & X' & \\
Z \arrow[ur, "\gamma"] \arrow[rr, "\varepsilon"'] & & Y' \arrow[ul, "\vartheta"']
\end{tikzcd},
\quad
\begin{tikzcd}
Y \arrow[r, "\alpha"] \arrow[d, "g"] & Z' \arrow[d, "\eta"] \\
Z \arrow[r, "\varepsilon"] & Y'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Y' \arrow[r, "\vartheta"] \arrow[d, "\xi"] & X' \arrow[d, "\delta"] \\
T(X) \arrow[r, "T(f)"] & T(Y)
\end{tikzcd}
$$

commute.

the four remaining diagrams commute



We can think of the third object in a dist. triangle

is unique up to (non!)-unique isom.
$\rightsquigarrow$ cone(f)

$$\begin{tikzcd} & Z \arrow[dl, dashed, "h"'] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

as a cone of f. Using this, we can explain the meaning of the octahedron axiom:
The octahedron axiom demands the existence of a well-behaved distinguished triangle relating the cones of two morphisms and the cone of their composition.

Thinking about it as the cokernel of f leads to the following explanation:

Given $Z' = Y/X$, $X' = Z/Y$ and $Y' = Z/X$, we also have

$$Z/Y = X' = Y'/Z' = (Z/X)/(Y/X),$$

the third isomorphism theorem.

Rem.:
Distinguished triangles behave very similarly to exact sequences in abelian categories:

- composition $g \circ f$ in

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow T(X)$$

is 0.

$$\begin{tikzcd} X \arrow[d, equal] & = & X \arrow[r, "0"] \arrow[d, "f"] \arrow[dl, phantom, "\circlearrowleft"] & 0 \arrow[r, "0"] \arrow[d, dashed] & T(X) \arrow[d, equal] \\ X \arrow[rr, "f"] & & Y \arrow[r, "g"] & Z \arrow[r] & T(X) \end{tikzcd}$$

$\rightsquigarrow \quad g \circ f = 0$

- 2/3 property (includes 5-lemma):

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, "\gamma"] & T(X) \arrow[d, "T(\alpha)"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X') \end{tikzcd}$$

morph. of triangles. If two of α, β, γ are isomorphisms, the so is the third.

WLOG α, β isom (otherwise rotate)

Applying $\mathrm{Hom}(A,-)$ (to Ab) keeps exactness (see below), then apply 5-lemma.

homological functor

- f in

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$$

is an isomorphism if and only if $Z \cong 0$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed] & T(X) \arrow[d, "T(f)"] \\ Y \arrow[r, equal] & Y \arrow[r] & 0 \arrow[r] & T(Y) \end{tikzcd}$$

$\rightsquigarrow$ f isom $\overset{2/3}{(\Leftarrow)}$ $\downarrow$ isom

- ...

Ex.:

The category $K(Ab)$ is triangulated with distinguished triangles given by the exact triangles:

(TC1): ✓ by definition

(TC2): ✓ by definition / seen already

(TC3): Can clearly assume that we are given two strict triangles

$$\begin{tikzcd} & \mathrm{cone}(f_\bullet) \arrow[dl, "(-pr_1)_\bullet"'] & \\ X_\bullet \arrow[rr, "f_\bullet"] & & Y_\bullet \arrow[ul, "(i_2)_\bullet"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathrm{cone}(f'_\bullet) \arrow[dl, "(-pr_1)_\bullet"'] & \\ X'_\bullet \arrow[rr, "f'_\bullet"] & & Y'_\bullet \arrow[ul, "(i_2)_\bullet"'] \end{tikzcd}$$

and morphisms $\alpha_\bullet : X \to X'$ and $\beta_\bullet : Y \to Y'$ such that the diagram

$$\begin{tikzcd} X_\bullet \arrow[r, "f_\bullet"] \arrow[d, "\alpha_\bullet"] & Y_\bullet \arrow[d, "\beta_\bullet"] \\ X'_\bullet \arrow[r, "f'_\bullet"] & Y'_\bullet \end{tikzcd}$$

commutes up to chain homotopy, i.e. there exists a chain homotopy $\varphi_\bullet$ such that

$$\beta_\bullet \circ f_\bullet - f'_\bullet \circ \alpha_\bullet = d^{Y'}_{\bullet+1} \circ \varphi_\bullet + \varphi_{\bullet-1} \circ d^X_\bullet \qquad (*)$$

The map

$$\gamma_\bullet = \begin{pmatrix} \alpha_{\bullet-1} & 0 \\ -\varphi_{\bullet-1} & \beta_\bullet \end{pmatrix} : \operatorname{cone}(f_*)_\bullet \longrightarrow \operatorname{cone}(f'_*)_\bullet$$

now does the job:

$\gamma_\bullet$ is morphism of chain complexes

- $$\begin{pmatrix} \alpha & 0 \\ -\varphi & \beta \end{pmatrix}\begin{pmatrix} -d^X & 0 \\ -f & d^Y \end{pmatrix} = \begin{pmatrix} -\alpha \circ d^X & 0 \\ \varphi \circ d^X - \beta \circ f & \beta \circ d^Y \end{pmatrix}$$

α $\shortparallel$ β morphism $+$ $(*)$

$$\begin{pmatrix} -d^{X'} \circ \alpha & 0 \\ -f' \circ \alpha - d^{Y'} \circ \varphi & d^{Y'} \circ \beta \end{pmatrix} = \begin{pmatrix} -d^{X'} & 0 \\ -f' & d^{Y'} \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ -\varphi & \beta \end{pmatrix}$$

$\gamma_\bullet$ yields morphism of triangles

- $$(i_2)_\bullet \circ \beta_\bullet = \begin{pmatrix} 0 \\ \beta_\bullet \end{pmatrix} = \begin{pmatrix} \alpha_{\bullet-1} & 0 \\ -\varphi_{\bullet-1} & \beta_\bullet \end{pmatrix} \circ (i_2)_\bullet = \gamma_\bullet \circ (i_2)_\bullet$$
- $$(-pr_1)_\bullet \circ \gamma_\bullet = (-pr_1)_\bullet \circ \begin{pmatrix} \alpha_{\bullet-1} & 0 \\ -\varphi_{\bullet-1} & \beta_\bullet \end{pmatrix} = (-\alpha_{\bullet-1}, 0) = \alpha_\bullet[-1] \circ (-pr_1)_\bullet$$

(TC4): Once again it suffices to treat strict triangles. So consider strict triangles

$$\begin{tikzcd} & \operatorname{cone}(f_*)_\bullet \arrow[dl, "(-pr_1)_\bullet"'] & \\ X_\bullet \arrow[rr, "f_\bullet"] & & Y_\bullet \arrow[ul, "(i_2)_\bullet"'] \end{tikzcd}, \quad \begin{tikzcd} & \operatorname{cone}(g_\bullet)_\bullet \arrow[dl, "(-pr_1)_\bullet"'] & \\ Y_\bullet \arrow[rr, "g_\bullet"] & & Z_\bullet \arrow[ul, "(i_2)_\bullet"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \operatorname{cone}(g_* \circ f_*)_\bullet \arrow[dl, "(-pr_1)_\bullet"'] & \\ X_\bullet \arrow[rr, "g_\bullet \circ f_\bullet"] & & Z_\bullet \arrow[ul, "(i_2)_\bullet"'] \end{tikzcd}$$

and the morphisms

$$\eta = \begin{pmatrix} id_{X_{\bullet-1}} & 0 \\ 0 & g_\bullet \end{pmatrix} : \underbrace{\operatorname{cone}(f_*)_\bullet}_{= X_{\bullet-1} \oplus Y_\bullet} \longrightarrow \underbrace{\operatorname{cone}(g_* \circ f_*)_\bullet}_{= X_{\bullet-1} \oplus Z_\bullet},$$

$$\varsigma_\bullet = \begin{pmatrix} f_{\bullet-1} & 0 \\ 0 & id_{Z_\bullet} \end{pmatrix} : \underset{\substack{\| \\ X_{\bullet-1}\oplus Z_\bullet}}{cone(g_* \circ f_*)_\bullet} \longrightarrow \underset{\substack{\| \\ Y_{\bullet-1}\oplus Z_\bullet}}{cone(g_*)_\bullet}$$

and

$$\beta_\bullet = (i_2)_\bullet[-1] \circ (-pr_1)_\bullet : \underset{\substack{\| \\ Y_{\bullet-1}\oplus Z_\bullet}}{cone(g_*)_\bullet} \longrightarrow \underset{\substack{\| \\ X_{\bullet-2}\oplus Y_{\bullet-1}}}{cone(f_*)_\bullet[-1]}.$$

$$\begin{tikzcd}
Y_{\bullet-1}\oplus Z_\bullet \arrow[d, "(-pr_1)_\bullet"] & X_{\bullet-2}\oplus Y_{\bullet-1} \\
Y_{\bullet-1} \arrow[ur, bend right, "(i_2)_\bullet[-1]"'] &
\end{tikzcd}$$

We want to show that

$$\begin{tikzcd}
& cone(g_*)_\bullet \arrow[dl, dashed, "\beta_\bullet"'] & \\
cone(f_*)_\bullet \arrow[rr, "\eta_\bullet"] & & cone(g_* \circ f_*) \arrow[ul, "\varsigma_\bullet"']
\end{tikzcd}$$

is an exact triangle by showing that it is isomorphic to the strict triangle

$$\begin{tikzcd}
& cone(\eta_*)_\bullet \arrow[dl, dashed, "(-pr_1)_\bullet"'] & \\
cone(f_*)_\bullet \arrow[rr, "\eta_\bullet"] & & cone(g_* \circ f_*)_\bullet \arrow[ul, "(i_2)_\bullet"']
\end{tikzcd}.$$

Consider the morphisms

$$\underbrace{\begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y_{\bullet-1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{Z_\bullet} \end{pmatrix}}_{\varphi_\bullet} : \begin{array}{ccc} \mathrm{cone}(g_*)_\bullet & \longrightarrow & \mathrm{cone}(\eta_*)_\bullet \\ \| & & \| \\ Y_{\bullet-1} \oplus Z_\bullet & & \mathrm{cone}(f_*)_{\bullet-1} \oplus \mathrm{cone}(g_* \circ f_*)_\bullet \\ & & \| \\ & & X_{\bullet-2} \oplus Y_{\bullet-1} \oplus X_{\bullet-1} \oplus Z_\bullet \end{array}$$

$$\underbrace{\begin{pmatrix} 0 & \mathrm{id}_{Y_{\bullet-1}} & f_{\bullet-1} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Z_\bullet} \end{pmatrix}}_{\psi_\bullet} : \mathrm{cone}(\eta_*)_\bullet \longrightarrow \mathrm{cone}(g_*)_\bullet .$$

We get two morphisms of diagrams:

$$\begin{tikzcd}
\mathrm{cone}(f_*)_\bullet \arrow[r, "\eta_\bullet"] \arrow[d, equal] & \mathrm{cone}(g_* \circ f_*)_\bullet \arrow[r, "\xi_\bullet"] \arrow[d, equal] & \mathrm{cone}(g_*)_\bullet \arrow[r, "\beta_\bullet"] \arrow[d, shift right, "\varphi_\bullet"'] & \mathrm{cone}(f_*)_\bullet[-1] \arrow[d, equal] \\
\mathrm{cone}(f_*)_\bullet \arrow[r, "\eta_\bullet"] & \mathrm{cone}(g_* \circ f_*)_\bullet \arrow[r, "(i_2)_\bullet"] & \mathrm{cone}(\eta_*)_\bullet \arrow[r, "(-pr_1)_\bullet"] \arrow[u, shift right, "\psi_\bullet"'] & \mathrm{cone}(f_*)_\bullet[-1]
\end{tikzcd}$$

$\beta_\bullet \circ \psi_\bullet \simeq (-pr_1)_\bullet$ via $\begin{pmatrix} 0 & 0 & \mathrm{id}_{X_{\bullet-1}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\varphi_\bullet \circ \xi_\bullet \simeq (i_2)_\bullet$ via $\begin{pmatrix} \mathrm{id}_{X_{\bullet-1}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

Now we have

$\psi_\bullet \circ \varphi_\bullet = \ldots = \mathrm{id}$

and

$$\varphi_\bullet \circ \psi_\bullet \simeq \mathrm{id} \text{ via } \begin{pmatrix} 0 & 0 & -\mathrm{id}_{X_{\bullet-1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover one shows, that the remaining four diagrams of the octahedron commute.

Def.:
An abelian category $\mathcal{A}$ is called split, if every short exact sequence in $\mathcal{A}$ splits.

Prop.:
Every triangulated abelian category $\mathcal{C}$ is split.

proof.:
Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence in $\mathcal{C}$. By (TC1), there is a distinguished triangle of the form

$$\begin{tikzcd} & Z \arrow[dl, dashed, "\beta"'] & \\ A \arrow[rr, "f"] & & B \arrow[ul, "\alpha"'] \end{tikzcd}$$

and by (TC2),

$$\begin{tikzcd} & 0 \arrow[dl, dashed] & \\ B \arrow[rr, equal] & & B \arrow[ul] \end{tikzcd}$$

is a distinguished triangle. Therefore we have a diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "\alpha"] \arrow[d, equal] & Z \arrow[r, "\beta"] & T(A) \arrow[d, "T(f)"] \\
B \arrow[r, equal] & B \arrow[r] & 0 \arrow[r] & T(B)
\end{tikzcd}
\qquad (*)
$$

whose rows are given by distinguished triangles and whose left square is commutative. By (TC3), this allows us to find a morphism $\varphi : Z \to 0$ that makes $(*)$ commutative. In particular, we have

$$T(f \circ T^{-1}(\beta)) = T(f) \circ \beta = 0,$$

so that $f \circ T^{-1}(\beta) = 0$. Since the sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact, f is monic, which yields $T^{-1}(\beta) = 0$ and hence $\beta = 0$ by applying T. Using (TC2), we have a commutative diagram of the form

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "\alpha"] & Z \arrow[r, "0"] \arrow[d, "0"] & T(A) \arrow[d, equal] \\
A \arrow[r, equal] & A \arrow[r] & 0 \arrow[r] & T(A)
\end{tikzcd}
$$

By (TC2), also

$$
\begin{tikzcd}
T^{-1}(Z) \arrow[r, "0"] \arrow[d, "0"'] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "\alpha"] & Z \arrow[d, "0"] \\
0 \arrow[r, "0"] & A \arrow[r, equal] & A \arrow[r] & 0
\end{tikzcd}
$$

is a diagram whose rows are distinguished triangles. Since the left square obviously commutes, (TC3) yields a morphism $\psi : B \to A$ such that

$$\begin{array}{ccccccc} T^{-1}(Z) & \xrightarrow{0} & A & \xrightarrow{f} & B & \xrightarrow{\alpha} & Z \\ {\scriptstyle 0}\downarrow & & \| & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle 0} \\ 0 & \xrightarrow{0} & A & = & A & \to & 0 \end{array}$$

(and hence also

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \to & Z & \xrightarrow{0} & T(A) \\ \| & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle 0} & & \| \\ A & = & A & \to & 0 & \xrightarrow{0} & T(A) \end{array}$$
)

commutes. In particular, f is a retraction, so that the short exact sequence

$$0 \to A \overset{\psi}{\underset{f}{\leftrightarrows}} B \xrightarrow{g} C \to 0$$

splits. $\square$

Cor.:

The category Ch(Ab) does not have the structure of a triangulated category.

proof:

Since

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\text{can.}} \mathbb{Z}/2\mathbb{Z} \to 0$$

is a non-split short exact sequence of abelian groups, the same holds for the associated chain complexes concentrated in degree 0, so that the category Ch(Ab) of chain complexes is not split

Def.:

An additive functor $F: \mathcal{C} \to \mathcal{C}'$ between two triangulated categories $\mathcal{C}$ and $\mathcal{C}'$ with translation functors T and T' respectively is called exact (or triangulated), if

F commutes with translations { (i) There exists an isomorphism $F \circ T \cong T' \circ F$.

F behaves like "normal" exact functors { (ii) F maps distinguished triangles in $\mathcal{C}$ to distinguished triangles in $\mathcal{C}'$.

This allows us to define:

Def.:

Let $\mathcal{D}$ be a (full) subcategory of a triangulated category $\mathcal{C}$. We call $\mathcal{D}$ a (full) triangulated subcategory of $\mathcal{C}$, if $\mathcal{D}$ is triangulated and the inclusion functor $\mathcal{D} \to \mathcal{C}$ is exact.

Ex.:

The categories $K^b(Ab)$, $K^+(Ab)$ and $K^-(Ab)$ are full triangulated subcategories of $K(Ab)$.

épaisse actually "only" equivalent to thick

Def.:

A full triangulated subcategory $\mathcal{D}$ of a triangulated category is called thick (saturated or épaisse), if whenever $X \oplus Y \in \mathcal{D}$, also $X, Y \in \mathcal{D}$.

Ex.:

If $F: \mathcal{C} \to \mathcal{C}'$ is an exact functor between two triangulated categories $\mathcal{C}$ and $\mathcal{C}'$, then

$$\ker(F) := \{X \in \mathcal{C} \mid F(X) \cong 0\}$$

every thick subcategory is a kernel (see Verdier quotient)

is a thick subcategory of $\mathcal{C}$.

opposite category of a triangulated category is triangulated with translation $(T^{-1})^{op}$

Def.:

An additive functor $H: \mathcal{C}^{op} \to \mathcal{A}$ from a triangulated category $\mathcal{C}$ to an abelian category $\mathcal{A}$ is called (co-)homological, if for every distinguished triangle

$$\begin{tikzcd} & Z \arrow[dl, dashed, "h"'] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

in $\mathcal{C}$, the sequence

$$H(X) \overset{H(f)}{\rightleftarrows} H(Y) \overset{H(g)}{\rightleftarrows} H(Z)$$

is exact.

Ex.:

- $\mathrm{Hom}(X,-)$ is homological
- $\mathrm{Hom}(-,X)$ is cohomological
- H_n is homological for every $n \geq 0$

homology of chain complexes

Localization

Def.:
Let S be a collection of morphisms of a category $\mathcal{C}$. A localization of $\mathcal{C}$ with respect to S consists of a category $S^{-1}\mathcal{C}$ together with a functor $Q_S : \mathcal{C} \to S^{-1}\mathcal{C}$ such that

(i) $Q_S(s)$ is an isomorphism for all $s \in S$.

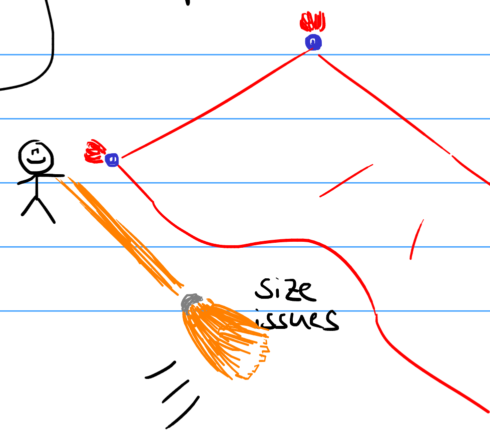
(ii) Any functor $F: \mathcal{C} \to \mathcal{D}$ mapping all elements of S to isomorphisms factors uniquely through Q_S, i.e. there exists a unique functor $G: S^{-1}\mathcal{C} \to \mathcal{D}$ with $F \cong G \circ Q_S$.

$\rightsquigarrow$ $S^{-1}\mathcal{C}$ unique up to equivalence

Ex.:

- The localization of $Ch(Ab)$ with respect to homotopy equivalences is $K(Ab)$
- The localization of rings (as category with one object)

for experts:



Thm.:
Localizations of categories exist.

- contains id's
- closed under composition
- can extend $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \\ Z & \end{tikzcd}$ to $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, "s'"] \\ Z \arrow[r, dashed] & W \end{tikzcd}$
- given $Z \overset{s}{\underset{s'}{\rightrightarrows}} X \xrightarrow{f} Y$, can find $X \xrightarrow{f} Y \overset{t}{\underset{t'}{\rightrightarrows}} W$

idea of proof:

suffices for us $\Big\{$ If S is a left multiplicative system:

$\rightsquigarrow$ focus on this case

$S^{-1}\mathcal{C}$ is given by:

results in left calculus of fractions; for right turn arrows (and need right mult. system)

- objects = objects of $\mathcal{C}$
- morphisms: $\mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y) = \Big\{ \underbrace{\big(X \xrightarrow{f} W \xleftarrow{s} Y\big)}_{\text{roofs / hats}} \;\Big|\; \begin{matrix} f \in \mathrm{Hom}(\mathcal{C}) \\ s \in S \end{matrix} \Big\} \Big/ \sim$

$(X \to W \leftarrow Y) \sim (X \to W' \leftarrow Y)$ iff ex. $(X \to Z \leftarrow Y)$ and morphisms $W \to Z$, $W' \to Z$ st.

$$\begin{tikzcd} & W \arrow[d] & \\ X \arrow[ur] \arrow[r] \arrow[dr] & Z & Y \arrow[ul] \arrow[l] \arrow[dl] \\ & W' \arrow[u] & \end{tikzcd}$$

commutes; think of $[X \xrightarrow{f} W \xleftarrow{s} Y]$ as $s^{-1} \circ f$

- composition: $(Y \to W' \leftarrow Z) \circ (X \to W \leftarrow Y)$

$$\|$$

$$\left(\begin{tikzcd} & & W \amalg_Y W' & & \\ & W \arrow[ur] & & W' \arrow[ul] & \\ X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z \arrow[ul] \end{tikzcd} \right)$$

composition of roofs only defined up to isomorphism

- identities: $(X \overset{=}{\to} X \overset{=}{\leftarrow} X)$

$\rightsquigarrow Q_S : \mathcal{C} \to S^{-1}\mathcal{C}$, $X \xrightarrow{f} Y \mapsto (X \xrightarrow{f} Y \overset{=}{\leftarrow} Y)$

more generally:

hat-pilling: replace hats

$$X \rightarrow W \leftarrow Y$$

by zigzags

$$X_1 \rightarrow W_1 \xleftarrow{\in S} X_2 \rightarrow W_2 \xleftarrow{\in S} X_3 \rightarrow \cdots \xleftarrow{\in S} X_{n-1} \rightarrow W_{n-1} \xleftarrow{\in S} X_n$$

and adapt $\sim$ to this situation...

"$\square$"

Actually creates inverse morphisms for $s \in S$:

$$(Y \xrightarrow{s} W \xleftarrow{t} Z) \circ (X \xrightarrow{f} W \xleftarrow{s} Y)$$

$$\shortparallel$$

$$\left(\begin{tikzcd}[column sep=small] & & & W \sqcup_Y W \arrow[d, phantom, "\cong"] & & & \\ & & & W & & & \\ & W \arrow[uurr] & & & & W \arrow[uull] & \\ X \arrow[ur, "f"] & & & Y \arrow[ull, "s"'] \arrow[urr, "s"] & & & Z \arrow[ul, "t"'] \end{tikzcd} \right)$$

$$\shortparallel$$

$$(X \xrightarrow{f} W \xleftarrow{t} Z)$$

write $\left(X \xrightarrow{f} W \xleftarrow{s} Y\right) = s^{-1} \circ f$

Rem.:

see rem. after Def. of triangulated cat.

If $\mathcal{D}$ is a full triangulated subcategory of a triangulated category $\mathcal{C}$, the localization of $\mathcal{C}$ with respect to $S = \{ s \mid \mathrm{cone}(s) \in \mathcal{D} \}$ is a triangulated category (called the Verdier quotient) and is denoted by $\mathcal{C}/\mathcal{D}$.

- translation: extended from $\mathcal{D}$ by setting

$$\left(X \xrightarrow{f} W \xleftarrow{s} Y\right) \overset{T}{\longmapsto} \left(T(X) \xrightarrow{T(f)} T(W) \xleftarrow{T(s)} T(Y)\right)$$

- distinguished triangles: triangles isomorphic to images of distinguished triangles of $\mathcal{C}$ under quotient morphism Q_S

Ex.:

$\mathcal{C} = K(Ab)$, $\mathcal{D}$ = acyclic cochain complexes

here cochain complexes

$\mathrm{Hom}(\mathcal{D})$ = quasi-isoms.

$\leadsto \mathcal{C}/\mathcal{D} = D(\mathcal{C}) = \underbrace{D(Ab)}$

derived category of abelian groups

$\leadsto$ next talk!