Probabilistic Methods in Finite and Profinite Group Theory

Spring 2016

This is a proposal for next semester "Research Seminar on Group Theory".

21/04/16 **TALK 1: Introduction.**

03/05/16 **TALK 3: Probability theory.** Refreshing of probabilistic concepts. The numbering follows the one in [7]. σ-algebras (1.1, 1.3), (1.4) without proofs, measures (1.5, 1.6 without proof of Theorem 1.6.1), (1.7) Borel sets (1.8) and Probability measure (1.9)[1]. Independence (1.12 without Theorem 1.12.1), Borel-Cantelli Lemmas (1.13) with proofs.

BLOCK A: Finite groups

24/05/16 **TALK 4.** Dixon's proof of the following theorem: the probability that two random elements in the symmetric group S_n generate A_n or S_n tends to 1 as n tends to infinity. Reference [2]. Mention that this result holds for general finite non-abelian simple groups (see for example [5]). See also **TALK 9***.

BLOCK B: Profinite groups

09/06/16 **TALK 5: Haar measure.** Definition of Haar measure for profinite groups and first properties (Sections 16.1 and 16.2 from [3]). Observe that the Haar measure makes any profinite group into a probability space.

16/06/16 **TALK 6: Subgroup growth.** This talk should introduce the notions of "subgroup growth" and "maximal subgroup growth" (Beginning of Section 1.1, Proposition 1.1.1 and Corollary 1.1.2 of [6]). Then present Section 2.1 of [6] (note that instead of Proposition 2.1.1 there one can use the results of **TALK 4**, making everthing much shorter). Finally present the proof of Theorem 3.5(i) in [6] (present only the results in Section 3.1 regarding maximal subgroup growth).

30/06/16 **TALK 7: Probability and maximal subgroups.** The goal of this talk is the following theorem: a profinite group G has "polynomial maximal subgroup growth" if and only if G is "positively finitely generated". Present the beginning of Chapter 11 of [6] (Theorem 11.1, Corollary 11.2, 11.3 and 11.4; maybe mention Proposition 11.6 and Corollary 11.7). Present Section 11.2 and Section 11.3 (without the results on $s_n^{\triangleleft}$).

[1] Here it would be good to give some examples of probability spaces.

07/07/16 **TALK 8: Further applications: generation of open subgroups and zeta functions.** Present the second half of the beginning of Chapter 11 of [6] (Theorem 11.8, 11.9, 11.10 and Corollary .11). Present Section 11.4 of [6]: here one can find complementary results about growth of certain families of open subgroups in a profinite group. Present 11.5 of [6]: this section studies probabilistic generation in pro-p groups and it ends with the explicit calculation of the "subgroup zeta function" of the group $\mathbb{Z}^d$.

BLOCK C: Complements The following talks are all independent of each other, so it will be up to the participarts to decide which we are going to cover.

21/07/16 **TALK 9*.** "A sneak peek" of the aforementioned generalization of [2]: present Section 2 of [4] (this is the proof the following: the probability that two random elements in $\mathrm{PSL}(V)$ generate the whole group tends to 1 as the size of the group tends to infinity).

28/07/16 **TALK 10*.** Present the following result: the probability of generating the inverse limit of wreath products of alternating groups of degree ≥ 5 by two elements is strictly positive and tends to 1 as the degree of the first factor tends to infinity. Reference [1], give an overview of the paper and only mention the results of Section 3 there.

04/08/16 **TALK 11*.** Section 5.3 of [8]. Here there is a "local field generalization" of [2]. Moreover, it gives a nice overview about the "lower rank" of a profinite group and some open problems.

References

[1] M. Bhattacharjee. The probability of generating certain profinite groups by two elements. *Israel J. Math.*, 86(1-3): 311–329, 1994.

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[3] M. D. Fried and M. Jarden. *Field arithmetic*, volume 11 of *Results in Mathematics and Related Areas. 3rd Series.* Springer-Verlag, Berlin, 2008.

[4] W. M. Kantor and A. Lubotzky. The probability of generating a finite classical group. *Geom. Dedicata*, 36(1):67–87, 1990.

[5] M. W. Liebeck and A. Shalev. The probability of generating a finite simple group. *Geom. Dedicata*, 56(1):103–113, 1995.

[6] A. Lubotzky and D. Segal. *Subgroup Growth*, volume 212 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2003.

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[8] A. Shalev. *Lie methods in the theory of pro-p groups* in New horizons in pro-p groups, volume 184 of *Progr. Math.* Birkhäuser Boston, Boston, MA, 2000.