Advanced Seminar on Group Theory - WISE 2020-2021

Topics on Subgroup Growth

Let G be a finitely generated residually finite group and denote $a_n(G)$ the number of subgroups of G of index n. The subgroup growth is the study of the asymptotic behaviour of the sequence $(a_n(G))_{n\in\mathbb{N}}$ and it turned out that this holds a wealth of information about G. Analogously one can define the sequence for normal subgroups that is denoted by $(a_n^{\triangleleft}(G))_{n\in\mathbb{N}}$.

The aim of the seminar is to present 6 self-contained talks around this topic. We will see some results that were not covered during the GRK lectures and some explicit computations. You can find the suggested topics below.

Notice that the order of the talks could be changed, since they are independent from one to another. Please let me know if you have any question about the preparation of your talk or if you have any suggestion for some other topic and we can try to include that in the programme.

Plan of the seminar:

Talk 1. Normal subgroup growth of free groups. *(12 January 2020)*

In the GRK lectures in December, we have seen the subgroup growth of d-generated free groups (see Section 2.1 of [4]). The aim of this talk is to show the following result of Lubotzky on normal subgroup growth of such groups.

Theorem. *Let F be a free group on d generators. Then for each n,*

$$a_n^{\triangleleft}(F) < n^{(2d+2)(1+\lambda(n))}$$

where $\lambda(n) = \sum l_i$ when $n = \prod p_i^{l_i}$ with distinct primes p_i.

References: Section 2.3 of [4] and [3].

Talk 2. Gap Theorem for Pro-p Groups. *(19 January 2020)*

We have seen in the GRK lectures last semester that the finitely generated pro-p groups with polynomial subgroup growth are precisely the pro-p groups of finite rank (if you missed them you can look at Section 4.1 of [4]). This result is a forerunner of the 'PSG Theorem'.

The aim of this talk is to prove that there is a gap in the 'subgroup growth spectrum' of pro-p groups. That means that any finitely generated pro-p group either has growth type $\leqslant n$ or has growth type at least $n^{\log n}$.

References: Section 4.2 and Window: Pro-p groups of [4]. See also Theorem 11.4 of [1].

Talk 3. The PSG Theorem. *(26 January 2020)*

Theorem. *Let G be a finitely generated residually finite group. Then G has PSG if and only if G is virtually soluble of finite rank.*

The 'if' direction is easy to prove, however the other direction involves several different techniques. The idea of this talk is to give a guideline of the proof giving some details when possible.

Reference: Chapter 5 of [4].

Talk 4. Subgroup growth of Baumslag-Solitar Groups. *(2 February 2020)*

There are very few classes of groups for which explicit formulas for the sequence $a_n(G)$ are known. Some of them could be found in Chapert 14 of [4]. The idea for this talk is to show the explicit formulas for the Baumslag-Solitar group defined by $G(p,q) = \langle a, b | a^p = ba^q b^{-1} \rangle$ for two non-zero coprime integers p and q. The result is the following.

Theorem. *Let p and q be non-zero integers such that $(p,q) = 1$. Then*

$$a_n(G(p,q)) = \sum_{l|n,(l,pq)=1} l.$$

Reference: See [2].

Talk 5. Zeta functions of plane crystallographic groups. *(9 February 2020)*

There are seventeen isomorphism types of plane crystallographic groups. Each of them is given by an extension of a free abelian group of rank 2 by a small finite solvable group. It is possible to calculate explicitly the subgroup zeta function of these groups. The aim of this talk is to show some explicit calculations of one of these crystallographic groups.

Reference: See [5] Sections 4 and 5.

Talk 6. Normal subgroup growth of the groups $\mathbf{SL}_d^1(\mathbb{F}_p[[t]])$ for $d \in \{2,3\}$. *(16 February 2020) tbc*

In [6] Ilir Snopce computed the normal subgroup zeta function for the group $\mathrm{SL}_2^1(\mathbb{F}_p[[t]])$. The aim of this talk is to show how this computation was done and what changes when one considers the group $\mathrm{SL}_3^1(\mathbb{F}_p[[t]])$.

References

[1] J. D. Dixon, M. P. F. du Sautoy and A. Mann, *Analytic Pro-p Groups*, Cambridge Univ. Pess, 1999.

[2] E. Gelman, *Subgroup Growth of Baumslag-Solitar Groups*, J. Group Theory 8, no. 6, 801-806, 2005.

[3] A. Lubotzky, *Enumerating Boundedly Generated Finite Groups*, J. Algebra **238**, 194-199, 2001.

[4] A. Lubotzky and D. Segal, *Subgroup Growth*, Progress in Mathematics, 212, Birkhäuser Basel, 2003.

[5] M. P. F. du Sautoy, J. J. McDermott and G. C. Smith, *Zeta Functions of Crystallographic Groups and Analytic Continuation*, Proc. London Math. Soc. (3) 79, no. 3, 511-534, 1999.

[6] I. Snopce, *Normal Zeta Functions of Some Pro-p Groups*, Comm. Algebra 39, no. 11, 3969-3980, 2011.