A-H-bimodules and equivalences

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Abstract

In [6, Theorem 2.2] Doi gave a Hopf-algebraic proof of a generalization of Oberst's theorem on affine quotients of affine schemes. He considered a commutative Hopf algebra H over a field, coacting on a commutative H-comodule algebra A. If A^{coH} denotes the subalgebra of coinvariant elements of A and $\beta : A \otimes_{A^{coH}} A \longrightarrow A \otimes H$ the canonical map, he proved that the following are equivalent:

(a) $A^{coH} \subset A$ is a faithfully flat Hopf Galois extension;

(b) the functor $(-)^{coH} : \mathcal{M}_A^H \longrightarrow A^{coH}$-Mod is an equivalence;

(c) A is coflat as a right H-comodule and β is surjective.

Schneider generalized this result in [14, Theorem 1] to the non-commutative situation imposing as a condition the bijectivity of the antipode of the underlying Hopf algebra. Interpreting the functor of coinvariants as a Hom-functor, Menini and Zuccoli gave in [10] a module-theoretic presentation of parts of the theory. Refining the techniques involved we are able to generalize Schneiders result to H-comodule-algebras A for a Hopf algebra H (with bijective antipode) over a commutative ring R under fairly weak assumptions.

Introduction

Let H denote a Hopf algebra over a commutative ring R with ${}_RH$ projective, and A a right H-comodule algebra. This setup generalizes such different situations as group scheme actions on affine schemes or R-algebras graded by a group. Using this setup with commutativity-conditions for the algebras involved, Doi gave in [6, Theorem 2.2] a Hopf algebraic proof of Oberst's theorem on affine quotients of affine schemes (conf. [12]). Schneider generalized this result to the non-commutative situation, i.e., he showed in [14, Theorem 1] that for a Hopf algebra H with bijective antipode over a field k and an H-comodule-Algebra A, the functor of coinvariants $(-)^{coH} : \mathcal{M}_A^H \to A^{coH}$-Mod is an equivalence of categories if and only if A is injective as a right H-comodule and the canonical map $\beta : A \otimes_{A^{coH}} A \to A \otimes H$ is surjective.

It was observed by Menini and Zuccoli in [10] that parts of the theory can be described by general module-theoretic methods. Refining these techniques we are able to generalize the main part of Schneiders paper to H-comodule-algebras A for a Hopf algebra H over a ground ring R.

In Section 1 we study general properties of $(A$-$H)$-bimodules for a Hopf algebra H over an arbitrary commutative ring R, provided ${}_RH$ is projective. In particular we show in 1.6 that $\mathcal{M}_A^H$ is subgenerated by $A \otimes_R H$ and hence can be identified with $\sigma_{A^{op}\#H^*}[A \otimes_R H]$ where $A^{op}\#H^*$ is a suitable smash-product.

Using results from [10] it is shown in 1.10 that for H a semiperfect Hopf algebra over a QF-ring R the category $\mathcal{M}_A^H$ can be identified with $A^{op}\#T$-Mod, where T is the left rational part of H^*.

The second section is devoted to the question when A is a (projective) generator in $\mathcal{M}_A^H$. This part presents the module-theoretic background which makes it possible to describe (faithfully) flat Hopf-Galois extensions $A^{coH} \subset A$ as (projective) generators in the category of $(A$-$H)$-bimodules (conf. 2.5 and 2.6).

In Section 3 we assume that the antipode of the Hopf algebra H is bijective. We proof in 3.1 that $A \otimes_R H$ is a generator in the category $\mathcal{M}_A^H$ if and only if A is H-generated as a right H-comodule. In 3.3 we give a refinement of Corollary 3.2 in [14]: For ${}_RA$ flat over R we characterise coflatness of A as an object in $\mathcal{M}^H$ as projectivity of A as an object in $\mathcal{M}_A^H$. The main result in Section 3 is a generalisation of Schneiders theorem on faithfully flat Hopf-Galois extensions ([14]. Theorem 1) under fairly weak assumptions from ground-fields to ground-rings (see 3.5). The same theorem is extended to Hopf-Galois extensions over ground-QF-rings in 3.7.

1 Right $(A$-$H)$-bimodules

Let H be a Hopf R-algebra with multiplication $\mu : H \otimes_R H \to H$, unit 1_H, comultiplication $\Delta : H \to H \otimes_R H$ and counit $\varepsilon : H \to R$. We will always assume H to be projective as an R-module.

The dual module $H^* = \mathrm{Hom}_R(H, R)$ endowed with the convolution product is an R-algebra. For the canonical structures on H and H^* we use the notation (for $h, x \in H, f \in H^*$)

$$\begin{array}{ll} f \rightharpoonup h = (1 \otimes f)\Delta(h) & \text{for } H \text{ as left } H^*\text{-module and} \\ h \rightharpoondown f = [x \mapsto f(xh)] & \text{for } H^* \text{ as left } H\text{-module.} \end{array}$$

Let A be a *right H-comodule algebra*, i.e., an R-algebra $\mu_A : A \otimes_R A \to A$ with unit 1_A and a right H-comodule structure $\varrho_A : A \to A \otimes_R H$ which is an algebra morphism.

For any two right H-comodules $\varrho_M : M \to M \otimes_R H$ and $\varrho_N : N \to N \otimes_R H$ the tensor product $M \otimes_R N$ can be either endowed with the trivial comodule structure (e.g. $id \otimes \varrho_N$) or with the twisted one - intertwining the two comodule structures involved - i.e.

$$\varrho_{M \otimes_R N} := (id \otimes id \otimes \mu_H) \circ (id \otimes \tau \otimes id) \circ (\varrho_M \otimes \varrho_N).$$

The resulting comodule with this *crossed* comodule strucure we denote by $M \otimes_R^c N$.

For any right A-module N, we consider $N \otimes_R H$ as a right $A \otimes_R H$-module and a right A-module by

$$(n \otimes c)(a \otimes h) := na \otimes ch, \text{ and } (n \otimes c) \cdot a := (n \otimes c)\varrho_A(a)\,.$$

1.1 $(A$-$H)$-bimodules. An R-module M is called a *right $(A$-$H)$-bimodule* if M is a right A-module $\psi_M : M \otimes_R A \to M$, and a right H-comodule $\varrho_M : M \to M \otimes_R H$, such that ϱ_M is A-linear, i.e., for $m \in M$, $a \in A$,

$$\varrho_M(ma) = \varrho_M(m) \cdot a \; (= \varrho_M(m)\varrho_A(a))\,,$$

or - equivalently - ψ_M is a right comodule morphism, where $M \otimes_R^c A$ has the right comodule structure defined above, i.e.,

$$(\psi_M \otimes id) \circ \varrho_{MA}(m \otimes a) = \varrho_M(\psi_M(m \otimes a)) = \varrho_M(ma).$$

We denote by $\mathcal{M}_A^H$ the category which has as objects all $(A$-$H)$-bimodules and as set of morphisms between $(A$-$H)$-bimodules M and N the mappings which are both A-module and H-comodule maps (denoted by $\mathrm{Bim}_A^H(M, N)$). This is obviously an additive category which is closed under infinite direct sums and has kernels and cokernels.

1.2 Basic properties of $(A\text{-}H)$-bimodules.

(1) For every right A-module N, $N \otimes_R H$ is an $(A\text{-}H)$-bimodule by

$$\begin{array}{rcl} \varrho_{N\otimes_R H} & : & N \otimes_R H \to (N \otimes_R H) \otimes_R H, \quad n \otimes c \mapsto n \otimes \Delta c\,, \\ \psi_{N\otimes_R H} & : & (N \otimes_R H) \otimes_R A \to N \otimes_R H, \quad n \otimes c \otimes a \mapsto (n \otimes c)\varrho_A(a)\,. \end{array}$$

For any $(A\text{-}H)$-bimodule M, the structure map $\varrho_M : M \to M \otimes_R H$ is an $(A\text{-}H)$-bimodule map.

(2) If $\alpha : N_1 \to N_2$ is an (epi) morphism in Mod-A*, then*

$$\alpha \otimes id_H : N_1 \otimes_R H \to N_2 \otimes_R H$$

is an (epi) morphism of $(A\text{-}H)$-bimodules.

(3) For every right H-comodule L, $L \otimes_R^c A$ is an $(A\text{-}H)$-bimodule by

$$\begin{array}{rcl} \psi_{L\otimes_R^c A} & : & (L \otimes_R^c A) \otimes_R A \to L \otimes_R^c A, \quad l \otimes a \otimes b \mapsto l \otimes ab\,, \\ \varrho_{L\otimes_R^c A} & : & L \otimes_R^c A \to (L \otimes_R^c A) \otimes_R H, \quad l \otimes a \mapsto \sum_{i,j} l_j \otimes a_i \otimes \tilde{l}_j \tilde{a}_i\,. \end{array}$$

For any $(A\text{-}H)$-bimodule M, the structure map $\psi_M : M \otimes_R^c A \to M$ is an $(A\text{-}H)$-bimodule map.

(4) If $\beta : L_1 \to L_2$ is an (epi) morphism of H-comodules, then

$$\beta \otimes id_A : L_1 \otimes_R^c A \to L_2 \otimes_R^c A$$

is an (epi) morphism of $(A\text{-}H)$-bimodules.

Proof. This can be immediately verified from the definitions (see [5, Example 1.1, 1.2]). □

As a first interesting application we observe:

1.3 Corollary. *Assume the right H-comodule G is a generator in $\mathcal{M}^H$. Then (with the structure from (3)) $G \otimes_R^c A$ is a generator in $\mathcal{M}_A^H$.*

Proof. Let M be any $(A\text{-}H)$-bimodule. Then there exists an H-comodule epimorphism $G^{(\Lambda)} \to M$ which yields the $(A\text{-}H)$-epimorphisms

$$G^{(\Lambda)} \otimes^c A \to M \otimes^c A \to M.$$

□

1.4 Remark. *By 1.2, the tensor products $H\otimes^c_R A$ and $A\otimes_R H$ both are $(A$-$H)$-bimodules. If the antipode S has a composition inverse $\bar S$, then the two bimodules are isomorphic by the maps*

$$\begin{array}{rclcrcl} H\otimes^c_R A & \to & A\otimes_R H, & & h\otimes a & \mapsto & \sum_i a_i\otimes h\tilde a_i, \\ A\otimes_R H & \to & H\otimes^c_R A, & & a\otimes h & \mapsto & \sum_i h\bar S(\tilde a_i)\otimes a_i, \end{array}$$

where $\varrho(a)=\sum_i a_i\otimes\tilde a_i\in A\otimes H$.

The $(A$-$H)$-bimodules may be considered as modules over an algebra which is defined by a suitable multiplication on $A^{op}\otimes_R H^*$.

1.5 The smash product $A^{op}\#H^*$. Any module $M\in\mathcal{M}^H_A$ is a left H^*-module and we have in fact a left action

$$(A^{op}\otimes_R H^*)\otimes_R M\to M,\quad (a\otimes k)\otimes m\mapsto (a\otimes k)\varrho_M(m).$$

Notice that this does not make M an $A^{op}\otimes_R H^*$-module with respect to the usual ring structure on $A^{op}\otimes_R H^*$. We define a new multiplication on $A^{op}\otimes_R H^*$ by

$$(a\otimes k)(b\otimes h)=\sum\nolimits_j b_j a\otimes(\tilde b_j\rightharpoonup k)*h,$$

where $a,b\in A$, $k,h\in H^*$ and $\varrho_A(b)=\sum_j b_j\otimes\tilde b_j$.

The resulting algebra is called the *(left) smash product* of A and H^*. We denote it by $A^{op}\#H^*$ and for $a\otimes f$ we write $a\#f$. $1_A\#\varepsilon_H$ is the unit of $A^{op}\#H^*$ and it is an exercise in handling the definitions to show that every $(A$-$H)$-bimodule is a left $A^{op}\#H^*$-module and $(A$-$H)$-bimodules morphisms are precisely the $A^{op}\#H^*$-module morphisms.

The maps $A^{op}\to A^{op}\#H^*, a\mapsto a\#\varepsilon_H$, and $H^*\to A^{op}\#H^*, k\mapsto 1_A\#k$, are algebra embeddings. In particular every left $A^{op}\#H^*$-module is a right A-module and a left H^*-module.

1.6 The category $\mathcal{M}^H_A$. *Let A be a right H-comodule algebra.*

(1) The category $\mathcal{M}^H_A$ is equal to $\sigma_{A^{op}\#H^*}[A\otimes_R H]=\sigma_{A^{op}\#H^*}[H\otimes^c_R A]$, the subcategory of left $A^{op}\#H^*$-modules subgenerated by $A\otimes_R H$ or $H\otimes^c_R A$.

(2) For any $M\in\mathcal{M}^H_A$ and $N\in$ Mod-A,

$$\mathrm{Bim}^H_A(M,N\otimes_R H)\to\mathrm{Hom}_A(M,N),\ f\mapsto(id\otimes\varepsilon)\circ f,$$

is an R-module isomorphism with inverse map $h\mapsto(h\otimes id)\circ\varrho$.

(3) For $N \in \mathcal{M}^H$ and $M \in \mathcal{M}_A^H$,

$$\mathrm{Bim}_A^H(N \otimes_R^c A, M) \to \mathrm{Com}^H(N, M), \ g \mapsto g(- \otimes 1_A),$$

is an R-isomomorphism (functorial in M) with inverse map $f \mapsto \psi_M \circ (f \otimes id_A)$.

Proof. (1) By 1.2, $A \otimes_R H$ is an $(A$-$H)$-bimodule hence an $A^{op}\#H^*$-module and so are all objects in $\sigma_{A^{op}\#H^*}[A \otimes_R H]$.

Let $M \in \mathcal{M}_A^H$ and $\alpha : A^{(\Lambda)} \to M$ an epimorphism in Mod-A. Then the maps

$$\alpha \otimes id : \ A^{(\Lambda)} \otimes_R H \to M \otimes_R H \text{ and } \varrho_M : M \to M \otimes_R H$$

are morphisms in $\mathcal{M}_A^H$, proving that M is subgenerated by $A \otimes_R H$.

To show that $H \otimes_R^c A$ is also a subgenerator recall that for every $(A$-$H)$-bimodule M, $M \otimes_R H$ is an H-generated right H-comodule. A comodule epimorphism $H^{(\Lambda)} \to M \otimes_R H$ yields a bimodule epimorphism

$$(H \otimes^c A)^{(\Lambda)} \simeq H^{(\Lambda)} \otimes^c A \to (M \otimes H) \otimes_R^c A.$$

Since $\varrho_M : M \to M \otimes_R H$ splits in R-Mod, we have $M \otimes_R^c A \subset (M \otimes_R H) \otimes_R^c A$ in $\mathcal{M}_A^H$. But $\mu_M : M \otimes_R^c A \to M$ is an epimorphism in $\mathcal{M}_A^H$ and we are done.

(2) This fact comes from the adjunction of the functors $U_H : \mathcal{M}_A^H \to$ Mod-A (forgetting the H-comodule structure) and $- \otimes_R H$: Mod-$A \to \mathcal{M}_A^H$ ([22, 3.12]).

(3) This is dual two (2). The relation stems from the adjunction of the functors $U_A : \mathcal{M}_A^H \to \mathcal{M}^H$ (the functor forgetting the A-module structure) and $- \otimes_R^c A : \mathcal{M}^H \to \mathcal{M}_A^H$. □

Remark: It is worth noting that properties which can be characterised via morphism functors like generation or projectivity are closley related through the adjunctions. For example - if N is a projective object (a generator) in $\mathcal{M}^H$, than $N \otimes_R^c A$ becomes projective (a generator) in $\mathcal{M}_A^H$ using (3) of 1.6 - thus giving an alternative proof of 1.3.

We denote by $\mathcal{T}^H : H^*$-Mod $\to \mathcal{M}^H$ the *rational functor*, assigning to any left H^*-module M the largest right H-subcomodule of M, i.e. the trace of $\mathcal{M}^H$ in M (the *rational part* of M). Similarly for any left $A^{op}\#H^*$-module M the trace of $\mathcal{M}_A^H$ in M (as a bimodule) is the largest R-submodule of M belonging to $\mathcal{M}_A^H$, which we denote by $\mathcal{T}_A^H(M)$.

For a left $A^{op}\#H^*$-module M the trace $\mathcal{T}^H(M)$ is an H-subcomodule and the trace $\mathcal{T}_A^H(M)$ is a subbimodule of M. The next proposition connects these two concepts.

1.7 Proposition. *Let* $M \in A^{op}\#H^*$-Mod*. Then*

(1) $\mathcal{T}^H(M) \in \mathcal{M}_A^H$.

(2) $M \in \mathcal{M}_A^H$ *if and only if* $M = \mathcal{T}^H(M)$.

Proof. (1) First of all note that the H^*-module structure of $\mathcal{T}^H(M)$ coincides with the one inherited from M so that for every $m \in \mathcal{T}^H(M)$ and for $f \in H^*$ we have

$$f \cdot m = (1\#f)m = (1 \otimes f)\varrho(m),$$

where $\varrho : \mathcal{T}^H(M) \to \mathcal{T}^H(M) \otimes H$ is the structure map. Moreover for every $m \in \mathcal{T}^H(M)$ and $a \in A$, putting $\varrho_A(a) = \sum_j a_j \otimes \tilde{a}_j$ and $\varrho(m) = \sum_i m_i \otimes \tilde{m}_i$ we have:

$$\begin{aligned} f \cdot ma &= (1\#f)(a\#\varepsilon)m = ((1\#f)(a\#\varepsilon))m \\ &= \textstyle\sum_j (a_j\#(\tilde{a}_j \rightharpoondown f))m \\ &= \textstyle\sum_{i,j} (a_j\#\varepsilon)m_i(\tilde{a}_j \rightharpoondown f)(\tilde{m}_i) \\ &= \textstyle\sum_{i,j} (a_j\#\varepsilon)m_i f(\tilde{m}_i\tilde{a}_j) \\ &= \textstyle\sum_{i,j} m_i a_j f(\tilde{m}_i\tilde{a}_j). \end{aligned}$$

The map

$$\varphi : \mathcal{T}^H(M) \otimes^c A \longrightarrow \mathcal{T}^H(M)A, \quad m \otimes a \longmapsto ma \ (= (a\#\varepsilon)m),$$

is an H^*-morphism, since

$$f \cdot (m \otimes a) = (1\#f)\sum_{i,j}(m_i \otimes a_j) \otimes \tilde{m}_i\tilde{a}_j \longmapsto \sum_{i,j} m_i a_j f(\tilde{m}_i\tilde{a}_j) = f \cdot ma,$$

and therefore the image of the right H-comodule $\mathcal{T}^H(M) \otimes^c A$ under φ is also an H-comodule, i.e. $\mathcal{T}^H(M)A \subset \mathcal{T}^H(M)$.

(2) ($\Rightarrow$) clear by definition. ($\Leftarrow$) follows by (1). □

As a consequence of the last proposition we get

1.8 Corollary. *If* $\mathcal{M}^H$ *is closed under extensions in* H^*-Mod*, then* $\mathcal{M}_A^H$ *is closed under extensions in* $A^{op}\#H^*$-Mod*.*

Proof. Consider an exact sequence in $A^{op}\#H^*$-Mod,

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0,$$

where $K, M \in \mathcal{M}_A^H$. Then $L = \mathcal{T}^H(L) \in \mathcal{M}^H$ and so $L \in \mathcal{M}_A^H$ by 1.7. □

The following observations are inspired by ideas from Cai-Chen [4].

1.9 Dense subalgebras of $A^{op}\#H^*$. *Let $T \subset H^*$ be a subalgebra and assume ${}_RA$ to be flat.*

(1) If T is dense in H^*, then $A^{op}\#T$ is an $A \otimes_R H$-dense subalgebra of $A^{op}\#H^*$ and we have

$$\sigma_{A^{op}\#T}[A \otimes_R H] = \mathcal{M}_A^H.$$

(2) If T is a ring with enough idempotents then $A^{op}\#T$ also has enough idempotents.

(3) Now let T be $\mathcal{T}^H({}_{H^*}H^*)$. If $T \subset H^*$ is H-dense, then we have

$$A^{op}\#T = \mathcal{T}^H(A^{op}\#H^*) = \mathcal{T}_A^H(A^{op}\#H^*).$$

Moreover for any $M \in \mathcal{M}_A^H$ we have $(A^{op}\#T)M = M$.

Proof. (1) Consider any $(A$-$H)$-bimodule M. Let $a\#f \in A^{op}\#T$ and take any $m_1, \ldots, m_n \in M$. Putting $\varrho_M(m_l) = \sum_i m_{li} \otimes \tilde{m}_{li}$, we have for each $l \leq n$,

$$(a\#f)(m_l) = \sum\nolimits_i f(\tilde{m}_{li}) m_{li} a.$$

By assumption there exists $t \in T$ such that $t(\tilde{m}_{li}) = f(\tilde{m}_{li})$, for all (finitely many) i and $l \leq n$. So $(a\#f)(m_l) = (a\#t)(m_l)$ for all $l \leq n$, showing that $A^{op}\#T$ is M-dense in $A^{op}\#H^*$ (modulo the annihilator of M).

In particular $A^{op}\#T$ is $A \otimes_R H$-dense in $A^{op}\#H^*$ and this implies

$$\sigma_{A^{op}\#T}[A \otimes_R H] = \sigma_{A^{op}\#H^*}[A \otimes_R H] = \mathcal{M}_A^H$$

(e.g., [1, Proposition 3.1], [20, 15.7]).

(2) Assume $\{e_\lambda\}_\Lambda$ is a set of enough orthogonal idempotents of T. Then $\{1_A\#e_\lambda\}_\Lambda$ is a set of enough orthogonal idempotents of $A^{op}\#T$.

(3) Since T is dense in H^*, we know that $\mathcal{T}^H(N) = TN$ for any $N \in H^*$-Mod(see [22, 2.6] for details). Now by 1.7,(1)

$$\mathcal{T}^H(A^{op}\#H^*) = T(A^{op}\#H^*) = (1\#T)(A^{op}\#H^*) = A^{op}\#T \in \mathcal{M}_A^H.$$

Using (1) above we get density of $A^{op}\#T$ in $A^{op}\#H^*$. Applying the formalism of [22, 2.6] to $\sigma_{A^{op}\#T}[A \otimes_R H]$ we get $\mathcal{T}_A^H(M) = (A^{op}\#T)M$, for any $M \in A^{op}\#H^*$-Mod. □

As a special case of the situation described above we recall the properties of semiperfect coalgebras.

1.10 Proposition. *Let H be a (left) semiperfect Hopf algebra over a QF ring R with trace ideal $T := \mathrm{Rat}({}_{H^*}H^*)$. Then*

$$\sigma_{A^{op}\#T}[A \otimes_R H] = \mathcal{M}_A^H = A^{op}\#T - Mod,$$

and $T \otimes_R^c A$ and $H \otimes_R^c A$ are generators in $\mathcal{M}_A^H$.

Moreover $A^{op}\#T$ is an algebra with enough idempotents and $A \otimes_R H$ and $H \otimes_R^c A$ are isomorphic as $A^{op}\#H^*$-modules.

Proof. By [9, 3.9] and [22, 6.4] H and T are generators in $\mathcal{M}^H$ and T is dense in H^*. So the first assertions follow from 1.9 and 1.3.

Since the antipode of H is bijective (by [9, 3.8]) $A \otimes_R H$ and $H \otimes_R^c A$ are isomorphic bimodules (by 1.4). □

2 A as an A-H-bimodule

In this section we study the structure of A as an A-H-bimodule. It turns out, that this leads to the equivalence-theorems for bimodule-categories studied in [14] or [10].

2.1 Coinvariants. For any $M \in \mathcal{M}_A^H$ put

$$\begin{aligned} M^{coH} &= \{m \in M \mid \varrho_M(m) = m \otimes 1_H\}, \text{ in particular} \\ A^{coH} &= \{a \in A \mid \varrho_A(a) = a \otimes 1_H\}. \end{aligned}$$

(1) There is an R-module isomorphism

$$\nu_M : \mathrm{Bim}_A^H(A, M) \to M^{coH}, \; f \mapsto f(1_A),$$

with inverse map $\omega_M : m \mapsto [a \mapsto (a \otimes \varepsilon_H)m]$.

(2) In particular we have a ring isomorphism

$$\nu_A : \; \mathrm{End}_A^H(A) = \mathrm{Bim}_A^H(A, A) \to A^{coH},$$

M^{coH} is a right A^{coH}-module and ν_M is an A^{coH}-module morphism.

Proof. The proof of [10, 3.15] also applies for coalgebras over rings. □

2.2 A^{coH}-modules and $(A$-$H)$-bimodules. *Let $V \in \mathrm{Mod}\text{-}A^{coH}$. Then $V \otimes_{A^{coH}} A$ is a right A-module and has a right H-comodule structure induced by the right comodule structure of A,*

$$V \otimes_{A^{coH}} A \to V \otimes_{A^{coH}} A \otimes_R H, \quad v \otimes a \mapsto v \otimes \varrho_A(a).$$

For any $M \in \mathcal{M}_A^H$, there are $(A$-$H)$-bimodule morphisms

$$\begin{array}{rrcl} \Psi_M : & \mathrm{Bim}_A^H(A, M) \otimes_{A^{coH}} A & \to & M, \quad f \otimes a \mapsto f(a), \\ \Phi_M : & M^{coH} \otimes_{A^{coH}} A & \to & M, \quad m \otimes a \mapsto ma, \end{array}$$

which are connected by the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Bim}_A^H(A, M) \otimes_{A^{coH}} A & \xrightarrow{\Psi_M} & M & & f \otimes a & \mapsto & f(a) \\ \downarrow_{\nu_M \otimes id} & & \| & & \downarrow & & \| \\ M^{coH} \otimes_{A^{coH}} A & \xrightarrow{\Phi_M} & M & & f(1) \otimes a & \mapsto & f(1_A)a\,, \end{array}$$

where $\nu_M \otimes id$ is an isomorphism (by 2.1) and hence Ψ_M is injective (surjective) if and only if Φ_M is.

The next proposition provides some more technical relationships between coinvariants and constructions related to A-modules.

2.3 A-modules and coinvariants. *For every $N \in$ Mod-A,*

$$\Lambda_N : N \to (N \otimes_R H)^{coH}, \quad n \mapsto n \otimes 1_H,$$

is an isomorphism of right A^{coH}-modules with inverse map $\sum_i n_i \otimes h_i \mapsto \sum_i n_i \varepsilon_H(h_i)$.

Combined with the isomorphism $\nu_{N\otimes H} : \mathrm{Bim}_A^H(A, N \otimes_R H) \to (N \otimes_R H)^{coH}$ (see 2.1) this yields an isomorphism

$$\Theta_N : \mathrm{Bim}_A^H(A, N \otimes_R H) \to N, \; f \mapsto (1 \otimes \varepsilon) f(1_A).$$

We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Bim}_A^H(A, N \otimes_R H) & \xrightarrow{\Theta_N} & N \\ \downarrow_{\nu_{N \otimes_R H}} & & \| \\ (N \otimes_R H)^{coH} & \xrightarrow{id \otimes \varepsilon} & N \otimes R\,, \end{array}$$

From this we derive an isomorphism

$$\begin{array}{rrcl} \Theta_N \otimes id : & \mathrm{Bim}_A^H(A, N \otimes_R H) \otimes_{A^{coH}} A & \to & N \otimes_{A^{coH}} A, \\ & f \otimes a & \mapsto & (1 \otimes \varepsilon) f(1_A) \otimes a\,. \end{array}$$

Moreover we obtain a map

$$\begin{array}{rccccc} \beta^N : & N \otimes_{A^{coH}} A & \xrightarrow{\Lambda_N \otimes id} & (N \otimes_R H)^{coH} \otimes_{A^{coH}} A & \xrightarrow{\Phi_{N \otimes_R H}} & N \otimes_R H, \\ & n \otimes a & & \mapsto & & (n \otimes 1)\varrho(a)\,, \end{array}$$

which yields the commutative diagram

$$\begin{array}{ccc} \mathrm{Bim}_A^H(A, N\otimes_R H)\otimes_{A^{coH}} A & \stackrel{\Psi_{N\otimes_R H}}{\longrightarrow} & N\otimes_R H \\ \downarrow \Theta_N\otimes id & & \| \\ N\otimes_{A^{coH}} A & \stackrel{\beta^N}{\longrightarrow} & N\otimes_R H\,. \end{array}$$

Proof. All these assertions are straightforward to verify (e.g., [10, Lemma 3.18, ff]). □

We will use the mappings introduced above to characterize A as a generator in $\mathcal{M}_A^H$. Hereby it is helpful to observe that an isomorphism for some single module implies isomorphisms for a whole class of modules. Such a situation is considered in our next proposition.

2.4 $\Psi_{A\otimes_R H}$ **as isomorphism**. *With the previous notation assume that* ${}_{A^{coH}}A$ *is flat and*

$$\Psi_{A\otimes_R H}: \mathrm{Bim}_A^H(A, A\otimes_R H)\otimes_{A^{coH}} A\to A\otimes_R H, \quad f\otimes a\mapsto f(a)\,,$$

is an isomorphism. Then

(1) A is a subgenerator in $\mathcal{M}_A^H$;

(2) for each $M\in\mathcal{M}_A^H$, Ψ_M is a monomorphism;

(3) for every A-generated $M\in\mathcal{M}_A^H$, Ψ_M is an isomorphism.

Proof. (1) Since $\Psi_{A\otimes_R H}$ is an isomorphism, $A\otimes_R H$ is A-generated as a bimodule and hence A is a subgenerator in $\sigma_{A^{op}\#H^*}[A\otimes_R H]=\mathcal{M}_A^H$.

(2) With slight modifications the proof of [10, Lemma 3.22] applies.

(3) If M is A-generated as an $(A$-$H)$-bimodule, then Ψ_M is surjective. □

We are now prepared to give a number of interesting properties which make A a generator in $\sigma_{A^{op}\#H^*}[A\otimes_R H]=\mathcal{M}_A^H$. This essentially extends [10, Theorem 3.27] from base fields to base rings.

2.5 A **as generator in** $\mathcal{M}_A^H$. *Let A be a right H-comodule algebra. The following are equivalent:*

(a) A is a generator in $\mathcal{M}_A^H$;

(b) the functor $\mathrm{Bim}_A^H(A,-):\mathcal{M}_A^H\to$ Mod-A^{coH} is faithful;

(c) for every $M \in \mathcal{M}_A^H$, we have an $(A$-$H)$-bimodule isomorphism

$$\Phi_M : M^{coH} \otimes_{A^{coH}} A \to M,\ m \otimes a \mapsto ma;$$

(d) ${}_{A^{coH}}A$ is flat and for every $A \otimes_R H$-generated $(A$-$H)$-bimodule M,

$$\Psi_M : \mathrm{Bim}_A^H(A, M) \otimes_{A^{coH}} A \to M,\ f \otimes a \mapsto f(a)\,,$$

is an isomorphism;

(e) ${}_{A^{coH}}A$ is flat and we have an isomorphism

$$\Psi_{A \otimes_R H} : \mathrm{Bim}_A^H(A, A \otimes_R H) \otimes_{A^{coH}} A \to A \otimes_R H,\ f \otimes a \mapsto f(a)\,;$$

(f) ${}_{A^{coH}}A$ is flat and we have an isomorphism

$$\beta^A :\ A \otimes_{A^{coH}} A \to A \otimes_R H,\ a \otimes b \mapsto (b \otimes 1_H)\varrho_A(a)\,.$$

Proof. $(a) \Leftrightarrow (b)$ This holds for any category (e.g., [20, 13.2]).

$(a) \Leftrightarrow (c) \Leftrightarrow (d)$ are proved in [10, Theorems 2.2, 2.3].

$(d) \Leftrightarrow (e)$ This is shown in 2.4.

$(e) \Leftrightarrow (f)$ follows from the diagrams in 2.3. □

In any Grothendieck category, a finitely generated projective generator determines a category equivalence and we have this if we impose on A slightly stronger conditions than in 2.5. Our result extends [10, Theorem 3.29] (for the case $D = H$) from base fields to base rings.

2.6 A as projective generator in $\mathcal{M}_A^H$. *Let A be a right H-comodule algebra. Then the follwing are equivalent:*

(a) A is a projective generator in $\mathcal{M}_A^H$;

(b) we have a category equivalence

$$\begin{array}{rcl} \mathrm{Bim}_A^H(A,-) & : & \mathcal{M}_A^H \to \text{Mod-}A^{coH}, \\ (\ (\ )^{coH} & : & \mathcal{M}_A^H \to \mathrm{Mod} - A^{coH},\ M \mapsto M^{coH}\ ); \end{array}$$

(c) ${}_{A^{coH}}A$ is faithfully flat and we have isomorphisms

$$\begin{array}{rcl} \Psi_{A \otimes_R H} & : & \mathrm{Bim}_A^H(A, A \otimes_R H) \otimes_{A^{coH}} A \to A \otimes_R H; \\ (\ \beta^A & : & A \otimes_{A^{coH}} A \to A \otimes_R H,\ a \otimes b \mapsto (b \otimes x)\varrho_A(a)\ ); \end{array}$$

(d) *for any* $M \in \mathcal{M}_A^H$ *and* $N \in Mod\text{-}A^{coH}$, *we have isomorphisms*

$$\begin{aligned} \Psi_M &: \operatorname{Bim}_A^H(A, M) \otimes_{A^{coH}} A \to M, \quad f \otimes a \mapsto f(a), \\ \Omega_N &: N \to \operatorname{Bim}_A^H(A, N \otimes_{A^{coH}} A), \quad n \mapsto [a \mapsto n \otimes a] \; . \end{aligned}$$

Proof. Since A is finitely generated as an $(A\text{-}H)$-bimodule the assertions follow from characterizations of progenerators in $\sigma[M]$ (see [20, 18.5 and 46.2]) and 2.5. □

2.7 Remarks. (1) For $A = H$, $H^{coH} = R \cdot 1_H$ and the map

$$\beta : H \otimes_R H \to H \otimes_R H, \; h \otimes g \mapsto (h \otimes 1_H)\Delta(g),$$

is an isomorphism. So H is a generator in Bimod-H and we obtain the Fundamental Theorem of Hopf modules.

(2) Let A be a right H-comodule algebra. If

$$\beta^A : A \otimes_{A^{coH}} A \to A \otimes_R H, \; a \otimes b \mapsto (a \otimes 1_H)\varrho_A(b)\,,$$

is an isomorphism then $A^{coH} \subset A$ is called a *right Hopf-Galois extension* (see [10, 3.23]). 2.5 and 2.6 characterize such extensions.

Combining our observations we obtain an extension of Beattie-Dăscălescu-Raianu [2, Theorem 3.1] to Hopf algebras over QF rings. Recall that a left modules M over any ring S is said to be a *weak generator* if for any right S-module L, $L \otimes_S M = 0$ implies $L = 0$.

2.8 Comodule algebras over semiperfect Hopf algebras. *Let* H *be a semiperfect Hopf algebra over a QF ring* R *and* A *be a right* H*-comodule algebra.*

(1) The following are equivalent:

(a) A *is a generator in* $\mathcal{M}_A^H$;

(b) A *generates* $H \otimes_R^c A$ *(or* $T \otimes_R^c A$*) as bimodules;*

(c) *the map* $\Psi_{H\otimes_R^c A} : \operatorname{Bim}_A^H(A, H \otimes_R^c A) \otimes_{A^{coH}} A \to H \otimes_R^c A$ *is surjective (bijective);*

(d) *the map* $\Psi_{T\otimes_R^c A} : \operatorname{Bim}_A^H(A, T \otimes_R^c A) \otimes_{A^{coH}} A \to T \otimes_R^c A$ *is surjective.*

(2) The following are equivalent:

(a) A *is a projective generator in* $\mathcal{M}_A^H$;

(b) *the map* $\Psi_{A\otimes^c_R H} : \mathrm{Bim}^H_A(A, H\otimes^c_R A)\otimes_{A^{coH}} A \to H\otimes^c_R A$ *is surjective and* ${}_{A^{coH}}A$ *is a weak generator;*

(c) A *is injective in* $\mathcal{M}^H_A$ *and the map* $\Psi_{A\otimes^c_R H} : \mathrm{Bim}^H_A(A, H\otimes^c_R A)\otimes_{A^{coH}} A \to H\otimes^c_R A$ *is surjective.*

Proof. (1) By [9, 3.9], H and T are (projective) generators in $\mathcal{M}^H$ and hence $H\otimes^c_R A$ and $T\otimes^c_R A$ are generators in $\mathcal{M}^H_A$ (see 1.3). Now the assertions follow from 2.5.

(2) $(a) \Rightarrow (b)$ follows from 2.6.

$(b) \Rightarrow (a)$ As in the proof of (1), $H\otimes^c A$ is a generator in $\mathcal{M}^H_A$. The surjectivity of $\Psi_{A\otimes^c_R H}$ implies, that A is a generator in $\mathcal{M}^H_A$ as well. So by 2.5 A is flat over A^{coH}. Now the weak generator property makes A faithfully flat over A^{coH} and we are done using 2.6.

$(a) \Leftrightarrow (c)$ will be proved in the next section 3.6,(4). Note that under the assumptions on H the antipode is bijective. □

3 Schneiders theorem revisited

From now on we assume, that the antipode S of the Hopf algebra H is bijective with composition inverse $\bar{S}$. In [14] Schneider generalizes Oberst's result on affine quotients [12] to the non-commutative situation. His proof is a Hopf algebraic one. But the result is module theoretic in nature - relating equivalences between categories of modules to the exactness of functors between them - so we will give a module theoretic proof of the theorem in this section.

The key observation is the following :

3.1 $A\otimes_R H$ **as generator in** $\mathcal{M}^H_A$**.**

Let H *be a Hopf algebra with bijective antipode,* A *a right* H*-comodule algebra. Then the following are equivalent:*

(a) $A\otimes_R H$ *is a generator in* $\mathcal{M}^H_A$*;*

(b) $A\otimes_R H$ *generates* A *in* $\mathcal{M}^H_A$*;*

(c) A *is* H*-generated as right* H*-comodule.*

Proof. $(a) \Rightarrow (b)$ is trivial.

$(b) \Rightarrow (c)$ By assumption $A\otimes_R H$ generates A in $\mathcal{M}^H_A$and therefore in $\mathcal{M}^H$. Recall that $A\otimes_R H$ has trivial right comodule structure and therefore is H-generated in $\mathcal{M}^H$. Combining this two facts we see that A is H-generated as right H-comodule.

$(c) \Rightarrow (a)$ Let $\phi : H^{(\Lambda)} \to A$ be an epimorphism in $\mathcal{M}^H$ and $M \in \mathcal{M}^H_A$ with right A-module structure given by $\mu_M : M \otimes^c_R A \to M$. Since M is a right H-comodule the module $M \otimes^c_R H$ with the structure given 1.2(2) becomes a Hopf module (i.e. an object in $\mathcal{M}^H_H$). By the Fundamental theorem of Hopf modules $M \otimes^c_R H$ is H-generated as a Hopf module and therefore it is H-generated as right H-comodule, say by $\psi : H^{(\Delta)} \to M \otimes^c_R H$ in $\mathcal{M}^H$. This gives rise to a surjective map in $\mathcal{M}^H$

$$\left(H^{(\Delta)}\right)^{(\Lambda)} \longrightarrow (M \otimes^c_R H)^{(\Lambda)} \simeq M \otimes^c_R H^{(\Lambda)}.$$

Combining this map with the surjection

$$M \otimes^c_R H^{(\Lambda)} \xrightarrow{id \otimes \phi} M \otimes^c_R A \xrightarrow{\mu_M} M$$

yields that each $(A$-$H)$-bimodule is H-generated as right H-comodule.

Given now an epimorphism $\theta : H^{(\Delta)} \to M$ in $\mathcal{M}^H$ for $M \in \mathcal{M}^H_A$ we get by 1.2 (4) an epimorphism

$$(H \otimes^c_R A)^{(\Delta)} \simeq H^{(\Delta)} \otimes^c_R A \xrightarrow{\theta \otimes id} M \otimes^c_R A \xrightarrow{\mu_M} M$$

in $\mathcal{M}^H_A$. Now using the bijectivity of the antipode, which gives the isomorphism $H \otimes^c_R A \simeq A \otimes_R H$ by 1.4 we see that $A \otimes_R H$ is a generator in $\mathcal{M}^H_A$. □

Before formulating the main result we need some technical machinery.

Recall that for an R-coalgebra C, the cotensor product of a right C-comodule $\varrho_M : M \to M \otimes_R C$ and a left C-comodule $\varrho_N : N \to C \otimes_R N$ (denoted by $M \Box_C N$) is defined by the exact sequence of R-modules

$$0 \to M \Box_C N \to M \otimes_R N \xrightarrow{\alpha} M \otimes_R C \otimes_R N,$$

where $\alpha = \varrho_M \otimes id_N - id_M \otimes \varrho_N$.

Clearly for a right C-comodule M the cotensor product gives rise to a functor

$$M \Box_C - : {}^C\mathcal{M} \longrightarrow R - \text{Mod}.$$

In general this is neither left nor right exact. If M is flat over R the cotensor functor is left exact. If it is right exact and ${}_RM$ is flat we call M *coflat*. If R is a QF-ring and $M \in \mathcal{M}^C$ is flat over R, the properties of M being coflat as right C-comodule and being injective in $\mathcal{M}^C$ coincide ([21]).

Recall that for an H-comodule M there exist two different comodule structures on the tensor product of M with H.

The module $M\otimes_R H$ is endowed with comodule structure $\varrho_{M\otimes_R H} = id_M\otimes\Delta$ (trivial right comodule structure) and the module $H\otimes^c_R M$ with structure map $\varrho_{H\otimes^c_R M} = \mu_{34}\circ\tau_{23}\circ(\Delta\otimes\varrho_M)$ (crossed comodule sructure). In general these two structures are different, but under the hypothesis that the antipode of H is bijective with inverse $\bar{S}$ we can state some canonical isomorphisms between these comodule structures. In this case there exists an H-colinear isomorphism

$$\begin{array}{cccccc} H\otimes^c_R M & \longrightarrow & M\otimes_R H, & h\otimes m & \mapsto & \sum_i m_i\otimes h\tilde{m}_i; \\ M\otimes_R H & \longrightarrow & H\otimes^c_R M, & m\otimes h & \mapsto & \sum_i h\bar{S}(\tilde{m}_i)\otimes m_i. \end{array}$$

There is a close relation between the cotensor functor $A\Box_H-$ and the functor of coinvariants. In order to state the next result we give another natural isomorphism.

Recall that the antipode of H gives rise to a functor $\mathcal{S}: {}^H\mathcal{M}\to\mathcal{M}^H$ which assigns to a left H-comodule $\varrho_U: U\to H\otimes_R U$ the right H-comodule

$$\varrho_{U^S} = (id\otimes S)\circ\tau\circ\varrho_U : U^S \longrightarrow U^S\otimes_R H,$$

leaving the morphisms unchanged.

If the antipode S of H is bijective, $\mathcal{S}$ is a categorical equivalence with inverse functor given by $\bar{\mathcal{S}}: \mathcal{M}^H\to {}^H\mathcal{M}$,

$$\varrho_V : V\to V\otimes_R H \quad\longmapsto\quad \varrho_{\bar{S}V} = (\bar{S}\otimes id)\circ\tau\circ\varrho_{\bar{S}V} : {}^{\bar{S}}V\to H\otimes_R {}^{\bar{S}}V.$$

3.2 Canonical isomorphism.

Let H be a Hopf algebra over R with bijective antipode S and U and V be right H-comodules. Then there exists a canonical R-linear isomorphism (functorial in U and V)

$$U\Box_H {}^{\bar{S}}V \longrightarrow (U\otimes^c_R V)^{coH}.$$

Proof. Recall that the module $U\otimes^c_R V$ becomes a right H-comodule with right crossed comodule structure $\varrho_{U\otimes^c_R V}$. Now consider the following diagram of R-modules, where the horizontal lines are the defining sequences for the cotensor product and coinvariants, respectively:

$$\begin{array}{ccccccc} 0 & \longrightarrow & U\Box_H {}^{\bar{S}}V & \longrightarrow & U\otimes_R {}^{\bar{S}}V & \stackrel{\varrho_U\otimes id_V - id_U\otimes\varrho_{\bar{S}V}}{\longrightarrow} & U\otimes_R H\otimes_R {}^{\bar{S}}V \\ & & \vdots\,\gamma & & \downarrow id_{U\otimes V} & & \downarrow id_U\otimes\psi \\ 0 & \longrightarrow & (U\otimes^c_R V)^{coH} & \longrightarrow & U\otimes^c_R V & \stackrel{\varrho_{U\otimes^c V}-\iota}{\longrightarrow} & (U\otimes^c_R V)\otimes_R H, \end{array}$$

where $\psi: H\otimes_R {}^{\bar{S}}V\to V\otimes_R H,\ h\otimes v\mapsto \sum_i v_i\otimes h\tilde{v}_i$ is an R-isomorphism and $\iota: U\otimes_R V\to U\otimes_R V\otimes_R H,\ u\otimes v\mapsto u\otimes v\otimes 1_H$ is the canonical embedding.

Then the right rectangle commutes with downward R-isomorphisms $id_{U\otimes V}$ and $id_U \otimes \psi$, since the elements are mapped via

$$\begin{array}{ccc} u\otimes v & \longmapsto & \sum_i u_i \otimes \tilde{u}_i \otimes v - \sum_j u \otimes \bar{S}(\tilde{v}_j) \otimes v_j \\ \downarrow & & \downarrow \\ u\otimes v & \longmapsto & \sum_i u_i \otimes v_i \otimes \tilde{u}_i \tilde{v}_i - u \otimes v \otimes 1_H \end{array}$$

By the kernel property of $(U \otimes_R V)^{coH}$ there exists an R-linear isomorphism $\gamma : U\square_H{}^{\bar{S}}V \longrightarrow (U \otimes_R V)^{coH}$ which makes the diagram commute. □

The next result will be needed in the proof of 3.7 but it is interesting in its own right. It relates coflatness of A as an object in $\mathcal{M}^H$ to projectivity of A as an object in $\mathcal{M}_A^H$. It refines Corollary 3.2 in [14].

3.3 A **coflat in** $\mathcal{M}^H$**.**

Let H be a Hopf algebra over R with ${}_RH$ projective. Assume that the antipode of H is bijective. Let A be a right H-comodule algebra which is flat over R. Then the following are equivalent:

(a) A is a projective in $\mathcal{M}_A^H$;

(b) the functor $\mathrm{Com}^H(R,-) : \mathcal{M}_A^H \longrightarrow R\text{-Mod}$ *is exact;*

(c) the functor $(-\square_H R) : \mathcal{M}_A^H \longrightarrow R\text{-Mod}$ *is exact;*

(d) the functor $(R\square_H -) \circ \bar{\mathcal{S}} : \mathcal{M}_A^H \longrightarrow R\text{-Mod}$ *is exact;*

(e) ${}^{\bar{S}}A$ is coflat as left H-comodule;

(f) A is coflat as right H-comodule;

(g) A^{op} is coflat as right H^{op}-comodule;

(h) A is projective in ${}_A\mathcal{M}^H$.

Proof. Recall that projectivity of A in $\mathcal{M}_A^H$ is equivalent to the exactness of the functor $\mathrm{Bim}_A^H(A,-) : \mathcal{M}_A^H \to \mathrm{Mod} - A^{coH}$ which is the same as exactness of the functor $(-)^{coH} : \mathcal{M}_A^H \to \mathrm{Mod} - A^{coH}$ by 2.3.

$(a) \Leftrightarrow (b)$ follows from the Bim-Com relations $\mathrm{Bim}_A^H(A,-) \simeq \mathrm{Com}^H(R,-)$ in 1.6.

$(a) \Leftrightarrow (c)$ By 3.2, the functor $(-\square_H R)$ is canonically isomorphic to the functor $(-)^{coH}$.

$(c) \Leftrightarrow (d)$ is clear since S is bijective.

$(a) \Rightarrow (e)$ By the canonical isomorphism for A and a right H-comodule M from 3.2 we know that $M\square_H{}^{\bar{S}}A \simeq (M \otimes_R A)^{coH}$. But this isomorphism is functorial in M and by assumption the composition of functors $(- \otimes_R A)^{coH}$ is exact (A flat over R). So is $-\Box_H {}^{\bar{S}}A$ which means that ${}^{\bar{S}}A$ is coflat as left H-comodule.

$(e) \Rightarrow (a)$ Note that for each $M \in \mathcal{M}_A^H$ the A-multiplication map $\mu_M : M \otimes_R^c A \to M$ is H-colinear and splits in $\mathcal{M}^H$ by $\nu_M : M \to M \otimes_R^c A$, $m \mapsto m \otimes 1_A$. We have to show that under the assumption on ${}^{\bar{S}}A$ being coflat the left exact functor $(-)^{coH}$ preserves epimorphisms in $\mathcal{M}_A^H$.

Let $f : M \to N$ be an epimorphism in $\mathcal{M}_A^H$ and consider the commutative diagram with the vertical arrows epimorphisms.

$$\begin{array}{ccc} M \otimes^c A & \xrightarrow{f \otimes id} & N \otimes^c A \\ \downarrow \mu_M & & \downarrow \mu_N \\ M & \xrightarrow{f} & N. \end{array}$$

Since $(-)^{coH}$ is left exact and μ_M and μ_N split in $\mathcal{M}^H$ we have the commuting diagram with vertical arrows still epimorphic.

$$\begin{array}{ccc} (M \otimes^c A)^{coH} & \xrightarrow{(f \otimes id)^{coH}} & (N \otimes^c A)^{coH}. \\ \downarrow \mu_M^{coH} & & \downarrow \mu_N^{coH} \\ M^{coH} & \xrightarrow{f^{coH}} & N^{coH} \end{array}$$

We know that f^{coH} is surjective if $(f \otimes id)^{coH}$ is. But by the functorial isomorphism $(- \otimes_R^c A)^{coH} \simeq -\Box_H {}^{\bar{S}}A$ from 3.2, we have $(f \otimes id)^{coH} = (f)\Box_H {}^{\bar{S}}A$.

By assumption $-\Box_H {}^{\bar{S}}A$ is exact so it preserves epimorphisms and the proof is complete.

$(e) \Leftrightarrow (f)$ is clear since S is bijective.

$(f) \Leftrightarrow (g)$ is trivial since H^{op} has the same coalgebra structure as H and A^{op} has the same right comodule structure as A.

$(g) \Leftrightarrow (h)$ follows from $(a) \Leftrightarrow (e)$ applied to the H^{op}-right comodule algebra A^{op}. $\Box$

Note that the proof given here for $(a) \Leftrightarrow (e)$ follows the proof of *Satz 5.8* in Oberst [12]. Studying the proof carefully we obtain in particular the following corollary for (not necessarily flat) comodule algebras:

3.4 A relatively coflat.

Let H be a Hopf algebra projective over R with bijective antipode. Let A be a right H-comodule algebra. Then the following are equivalent:

(a) A is relatively projective in $\mathcal{M}_A^H$, i.e. the functor $\mathrm{Bim}_A^H(A, -)$ *is exact on R-split, exact sequences in $\mathcal{M}_A^H$;*

(b) A is relatively coflat, i.e. the functor $A\Box_H -$ is (left and right) exact with respect to R-split, exact sequences in $\mathcal{M}^H$.

Proof. Of course the functor $-\otimes_R A$ is always relative exact (even for non-flat A) and the functor $-\Box_H{}^{\bar{S}}A$ is always left exact with respect to R-split sequences. Starting with split sequences, we can imitate the proof of $(a) \Leftrightarrow (e)$ from the previous result. □

Recall that a right H-comodule N is called *relative injective* (in $\mathcal{M}^H$), if the functor $\mathrm{Com}(-, N)$ is exact with respect to R-split, exact sequences in $\mathcal{M}^H$ (see [6, 1.4]). Now we are able to state the main result. It is a supplement to 2.6 for the special case of a bijective antipode and an H-generated right H-comodule algebra A.

3.5 A as a projective generator in $\mathcal{M}_A^H$.

Let H be a Hopf algebra over R with ${}_RH$ projective. Assume that the antipode of H is bijective. Let A be a right H-comodule algebra which is H-generated as a right H-comodule. Then the following are equivalent:

(a) A is a projective generator in $\mathcal{M}_A^H$;

(b) the functor $\mathrm{Bim}_A^H(A, -) : \mathcal{M}_A^H \to \text{Mod-}A^{coH}$ is an equivalence;

(c) the functor $\mathrm{Bim}_A^H(A, -) : {}_A\mathcal{M}^H \to A^{coH}\text{-Mod}$ is an equivalence;

(d) the functor $\mathrm{Bim}_A^H(A, -) : \mathcal{M}_A^H \to \text{Mod-}A^{coH}$ is exact and the map

$$\Psi_{A\otimes_R H} \quad : \quad \mathrm{Bim}_A^H(A, A\otimes_R H) \otimes_{A^{coH}} A \to A\otimes_R H,$$

is surjective.

(e) A is relative injective as right H-comodule and the canonical map

$$\beta^A : A\otimes_{A^{coH}} A \to A\otimes_R H,\ a\otimes b \mapsto (b\otimes x)\varrho_A(a)$$

is surjective.

Proof. $(a) \Leftrightarrow (b)$ This is a well known fact for abelian categories (see for example [20, 46.2]).

$(a) \Rightarrow (d)$ Since A is projective in $\mathcal{M}_A^H$, the functor $\mathrm{Bim}_A^H(A.-)$ is exact. The surjectivity of $\Psi_{A\otimes_R H}$ stems from the fact, that A is a generator by assumption.

$(d) \Rightarrow (a)$ Under the hypotheses A is H-generated as right H-comodule. Now by 3.1 we obtain that the module $A\otimes_R H$ is a generator in $\mathcal{M}_A^H$. The surjectivity of $\Psi_{A\otimes H}$ shows that A generates $A\otimes_R H$ in $\mathcal{M}_A^H$ and therefore A is a generator. Since A is always finitely generated in $\mathcal{M}_A^H$ we have by exactness of $\mathrm{Bim}_A^H(A,-)$ that A is a progenerator in $\mathcal{M}_A^H$.

$(a) \Rightarrow (e)$ As mentioned above $A\otimes_R H$ is a generator in $\mathcal{M}_A^H$. But this module is always relative injective as a right H-comodule since H has this property. A is always finitely generated in $\mathcal{M}_A^H$ and by assumption A is projective in $\mathcal{M}_A^H$ so it is a direct summand of $(A\otimes_R H)^k$ in $\mathcal{M}_A^H$ for some k. Forgetting the A-module-structure, A is an H-colinear direct summand of $(A\otimes_R H)^k$. Of course $(A\otimes_R H)^k$ is relative injective, since $A\otimes_R H$ is. Hence A as a direct summand is relative injective in $\mathcal{M}^H$. The surjectivity of β is clear by 2.5 since A is a generator in $\mathcal{M}_A^H$.

$(e) \Rightarrow (a)$ The surjectivity of β shows by 2.5 that A generates $A\otimes_R H$ in $\mathcal{M}_A^H$. By assumption and 3.1 $A\otimes_R H$ is a generator in $\mathcal{M}_A^H$ and so is A. Moreover relative injectivity of A in $\mathcal{M}^H$ implies projectivity of A in $\mathcal{M}_A^H$.

$(e) \Leftrightarrow (c)$ Since S is bijective, it is easy to show that the canonical map $\beta^{A^{op}}$ is surjective too. So the assertion follows from $(e) \Leftrightarrow (b)$ applied to A^{op} as a right H^{op}-comodule algebra. □

Note that $(e) \Rightarrow (a)$ corresponds to Schneiders theorem [14, 3.5] and our techniques provide a fairly simple proof of this fact. The proof in [14] shows the isomorphism of the adjunction $M \simeq M^{coH}\otimes_B A$ for $M\in\mathcal{M}_A^H$ and $B = A^{coH}$. However, under the assumptions in (e) this isomorphism follows from the fact, that A is a generator in $\mathcal{M}_A^H$.

It is an interesting problem whether (relative) coflatness of A is sufficient to make A H-generated as a right H-comodule, which would make it possible to weaken the hypotheses of our previous result. Nevertheless the result applies in many cases:

3.6 Applications.

Let A be a right H-comodule algebra. Then A is H-generated as right H-comodule provided one of the following holds:

(1) A is weakly H-injective in $\mathcal{M}^H$;

(2) A is relative injective in $\mathcal{M}^H$;

(3) H is finitely generated over the ring R;

(4) R is a QF-ring and H is co-Frobenius.

Proof. (1) is proved in [20], 16.11.

(2) is equivalent to the fact that A is an H-colinear direct summand in $A \otimes_R H$, and the last module is H-generated as a trivial right H-comodule.

Under the assumptions in (3) and (4), H is a generator in $\mathcal{M}^H$ by [9, 3.9]. In both cases the antipode S is always bijective. □

Note that the main result could be compared with other Hopf algebraic proofs of the theorem on affine quotients. In particular we obtain a pure module theoretic proof of Schneider's Theorem I in [14] for the case of QF-rings.

3.7 Schneiders theorem over QF-rings.

Let H be a Hopf algebra over a QF-ring R with ${}_RH$ projective. Assume that the antipode of H is bijective. Let A be a right H-comodule algebra which is flat over R. Then the following are equivalent:

(a) A is a projective generator in $\mathcal{M}_A^H$;

(b) the functor $\mathrm{Bim}_A^H(A,-) : \mathcal{M}_A^H \to \text{Mod-}A^{coH}$ *is an equivalence;*

(c) the functor $\mathrm{Bim}_A^H(A,-) : {}_A\mathcal{M}^H \to A^{coH}\text{-Mod}$ *is an equivalence;*

(d) A is injective in $\mathcal{M}^H$ and we have a surjective map

$$\beta^A \quad : \quad A \otimes_{A^{coH}} A \to A \otimes_R H, \;\; a \otimes b \mapsto (b \otimes 1)\varrho_A(a).$$

Moreover, if one of these conditions is satisfied, A is H-generated as a right H-comodule.

Proof. $(a) \Leftrightarrow (b)$ is well known for abelian categories.

$(a) \Rightarrow (d)$ Since A is projective the functor $\mathrm{Bim}_A^H(A,-)$ is exact on $\mathcal{M}_A^H$. By 3.3 this is equivalent to A being coflat in $\mathcal{M}^H$. Now A is flat and R is a QF-ring, so A is an injective object in $\mathcal{M}^H$ (see [21]). The surjectivity of β^A stems again from the fact that A generates $A \otimes_R H$.

$(d) \Rightarrow (a)$ By [21] injectivity of A is equivalent to A being coflat over H. By the previous result 3.3 this is the same as exactness of the functor $\mathrm{Bim}_A^H(A,-)$, which means A is a projective object in $\mathcal{M}_A^H$. But injectivity of A in $\mathcal{M}^H$ implies that A is H-generated as a right H-comodule (by [20, 16.11]). Hence our previous result 3.5 applies.

$(b) \Rightarrow (c)$ Since (b) is equivalent to (d), A is H-generated under the assumption (b) and we can use our previous result 3.5.

$(c) \Rightarrow (b)$ By applying the equivalence of (b) and (d) to A^{op} and H^{op} we get that A is H-generated as right H-comodule, so 3.5 applies. □

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