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RADICALS WITH THE α-AMITSUR PROPERTY

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A radical γ of rings is said to have the *Amitsur property* if for all rings A, $\gamma(A[X]) = (\gamma(A[X]) \cap A)[X]$. Let X_α denote a set of indeterminates of cardinality α. We say that γ has the *α-Amitsur property* if for all rings A, $\gamma(A[X_\alpha]) = (\gamma(A[X_\alpha]) \cap A)[X_\alpha]$. We study properties of this type of radicals and show relationships with other known radicals for rings.

A ring A is said to be an *absolute γ-ring* if $A[x_1, \ldots, x_n] \in \gamma$, for any $n \in \mathbb{N}$. We show that A is an absolute $\mathbb{G}$-ring for the Brown–McCoy radical $\mathbb{G}$, if and only if A is in the radical class $\mathbf{S}$ determined by the unitary strongly prime rings. Moreover, A is an absolute nil ring if and only if A is an absolute $\mathbf{J}$-ring, where $\mathbf{J}$ denotes the Jacobson radical.

Keywords: Radical theory (16N20, 16N40, 16N80); polynomial rings; Amitsur property.

1. Introduction

In this paper, rings are associative, not necessarily with identity. The notation $I \trianglelefteq A$ and $L \trianglelefteq_l A$ means that I is an ideal and L is a left ideal in a ring A, respectively.

Recall that a (*Kurosh–Amitsur*) *radical* γ is a class of rings which

(i) is closed under homomorphic images,
(ii) is closed under extensions (I and A/I in γ imply $A \in \gamma$),
(iii) has the inductive property (if $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_\lambda \subseteq \cdots$ is a chain of ideals of the ring $A = \bigcup I_\lambda$ and each I_λ is in γ, then A is in γ).

For a radical γ, the *semisimple class of* γ is defined as

$$S\gamma = \{A \mid A \text{ a ring with } \gamma(A) = 0\}.$$

As usual, $\mathcal{L}(M)$ will stand for the *lower radical* induced by a class M of rings. For basic notions of radical theory we refer to [5] and [20].

Notation 1.1. The following symbols are used:

β	the Baer (prime) radical, [5],
$\mathbf{L}$	the Levitzki radical, [5],
$\mathbf{N}$	the Köthe (nil) radical, [5],
$\mathbf{u}$	the uniformly strongly prime radical, [17]
$\mathbf{S}$	the unitary strongly prime radical, [4]
$\mathbf{J}$	the Jacobson radical,
$\mathbb{G}$	the Brown–McCoy radical.

An important issue in radical theory is to describe the radical of a polynomial ring. This leads to a notion introduced in [10]:

Definition 1.2. A radical γ of rings is said to have the *Amitsur property* if for any ring A,

$$\gamma(A[x]) = (A \cap \gamma(A[x]))[x].$$

This property of radicals was considered in several papers, e.g. [3–5, 8, 10, 12, 17, 19]. We note that all the radicals listed in 1.1 have the Amitsur property.

2. The α-Amitsur Property

Let X be a set of commuting indeterminates. To indicate that card $X = \alpha$ we write X_α. Here α may be infinite.

The following observation from [3] will be needed.

Lemma 2.1. *Let γ be a radical of rings. For any element $f \in \mathbb{Z}[X_\alpha]$,*

$$f\gamma(A[X_\alpha]) \subseteq \gamma(A[X_\alpha]).$$

Proof. Clearly $f\gamma(A[X_\alpha]) \trianglelefteq A[X_\alpha]$. Since $\gamma(A[X_\alpha])$ is an ideal of $A[X_\alpha]$, we have

$$B = \gamma(A[X_\alpha]) + f\gamma(A[X_\alpha]) \trianglelefteq A[X_\alpha].$$

Let φ be the natural homomorphism $\varphi : B \to B/\gamma(A[X_\alpha])$, and define the surjective ring homomorphism

$$\psi : \gamma(A[X_\alpha]) \to B/\gamma(A[X_\alpha]), \quad a \mapsto \varphi(f \cdot a).$$

Thus $B/\gamma(A[X_\alpha]) \in \gamma$ and, by extension closure, $B \in \gamma$ and $B \subseteq \gamma(A[X_\alpha])$. This implies $f\gamma(A[X_\alpha] \subseteq \gamma(A[X_\alpha])$. □

Proposition 2.2. *If γ is a radical then, for any ring A,*

$$(A \cap \gamma(A[X_\alpha]))[X_\alpha] \subseteq \gamma(A[X_\alpha]).$$

Proof. Since $(A \cap \gamma(A[X_\alpha]))[X_\alpha] = (A \cap \gamma(A[X_\alpha]))\mathbb{Z}[X_\alpha]$ and $A \cap \gamma(A[X_\alpha]) \subseteq \gamma(A[X_\alpha])$, Lemma 2.1 yields

$$(A \cap \gamma(A[X_\alpha]))[X_\alpha] \subseteq \gamma(A[X_\alpha]). \qquad \square$$

Definition 2.3. Let α be a cardinal. We say that a radical γ has the *α-Amitsur property* if for all rings A,

$$\gamma(A[X_\alpha]) = (A \cap \gamma(A[X_\alpha]))[X_\alpha],$$

and γ has the *strong α-Amitsur property* if for all rings A,

$$\gamma(A[X_\alpha]) = \gamma(A)[X_\alpha].$$

If $\alpha = 1$, then the 1-Amitsur property is just the Amitsur property and the strong 1-Amitsur property is called the *strong Amitsur property*.

For any radical γ we define the class

$$\gamma_\alpha = \{A \mid A \text{ a ring with } A[X_\alpha] \in \gamma\}.$$

Note that γ_α is a radical for any radical γ and

$$\gamma_1 \supseteq \cdots \supseteq \gamma_\alpha \supseteq \cdots.$$

Theorem 2.4. *For a radical γ and a cardinal α, the following are equivalent.*

(a) *γ has the α-Amitsur property*;

(b) $\gamma(A[X_\alpha]) \cap A = 0$ *implies* $\gamma(A[X_\alpha]) = 0$;

(c) $A[X_\alpha] \in S\gamma$ *for any* $A \in S\gamma_\alpha$.

Proof. (a) $\Rightarrow$ (c). Suppose $A \in S\gamma_\alpha$. Since γ has the α-Amitsur property,

$$\gamma(A[X_\alpha]) = (\gamma(A[X_\alpha]) \cap A)[X_\alpha] \in \gamma,$$

thus $\gamma(A[X_\alpha]) \cap A \in \gamma_\alpha \cap S\gamma_\alpha = 0$. Hence $\gamma(A[X_\alpha]) = 0$ and $A[X_\alpha] \in S\gamma$.

(c) $\Rightarrow$ (b). Let $\gamma(A[X_\alpha]) \cap A = 0$. Clearly $\gamma_\alpha(A) \in \gamma_\alpha$. Therefore $(\gamma_\alpha(A))[X_\alpha] \in \gamma$ and $(\gamma_\alpha(A))[X_\alpha] \subseteq \gamma(A[X_\alpha])$. Hence

$$(\gamma_\alpha(A))[X_\alpha] \cap A \subseteq \gamma(A[X_\alpha]) \cap A = 0.$$

Thus $\gamma_\alpha(A) = 0$ and $A \in S\gamma_\alpha$. By (c) we have $\gamma(A[X_\alpha]) = 0$.

(b) $\Rightarrow$ (a). Put $I = \gamma(A[X_\alpha]) \cap A$. By Proposition 2.2, $I[X_\alpha] \subseteq \gamma(A[X_\alpha])$. We know that for any radical γ,

$$\gamma(A[X_\alpha])/I[X_\alpha] = \gamma(A[X_\alpha]/I[X_\alpha]) \cong \gamma(B[X_\alpha]),$$

where $B = A/I$. Thus

$$\begin{aligned}
\gamma(B[X_\alpha]) \cap B &\cong (\gamma(A[X_\alpha])/I[X_\alpha]) \cap ((A + I[X_\alpha])/I[X_\alpha]) \\
&= (\gamma(A[X_\alpha]) \cap (A + I[X_\alpha]))/I[X_\alpha] \\
&= ((\gamma(A[X_\alpha]) \cap A) + I[X_\alpha])/I[X_\alpha] \\
&= (I + I[X_\alpha])/I[X_\alpha] = 0.
\end{aligned}$$

Thus $\gamma(B[X_\alpha]) \cap B = 0$. Hence $I[X_\alpha] \supseteq \gamma(A[X_\alpha])$ and

$$\gamma(A[X_\alpha]) = I[X_\alpha] = (\gamma(A[X_\alpha]) \cap A)[X_\alpha]. \qquad \square$$

Corollary 2.5. *Let γ be a radical. Then the following are equivalent.*

(a) *γ has the α-Amitsur property;*

(b) *If $\gamma(A[X_\alpha]) \neq 0$, then $\gamma(A[X_\alpha]) \cap A \neq 0$, for any ring A.*

(c) *If $\gamma(A[X_\alpha]) \neq 0$ then $\gamma_\alpha(A) \neq 0$, for any ring A.*

Proposition 2.6. *Let γ have the α-Amitsur property. Then γ has the β-Amitsur property for all β with $\alpha \leq \beta$.*

Proof. We claim that $\gamma(A[X_\beta]) \neq 0$ implies $A \cap \gamma(A[X_\beta]) \neq 0$. Since X_β consists of commuting indeterminates,

$$0 \neq \gamma(A[X_\beta]) = \gamma((A[X_\beta \backslash Y_\alpha])[Y_\alpha]) = (A[X_\beta \backslash Y_\alpha] \cap \gamma(A[X_\beta]))[Y_\alpha]$$

for any subset Y_α of X_β with card $Y_\alpha = \alpha$. Clearly, for any $0 \neq a \in \gamma(A[X_\beta])$, there exist elements $x_{\iota_1}, \dots, x_{\iota_{n(a)}}$ of X_β. such that

$$a = \sum a_{i_1,\dots,i_{n(a)}} x_{i_1}^{\alpha_1} \dots x_{i_{n(a)}}^{\alpha_{n(a)}},$$

where $a_{i_1,\dots,i_{n(a)}} \in A$ and for each x_{i_j} there exists a component $\alpha_j \neq 0$ such that $a_{i_1,\dots,i_{n(a)}} x_{i_1}^{\alpha_1} \dots x_{i_{n(a)}}^{\alpha_{i_{n(a)}}} \neq 0$ or $a \in A$.

We call the number of nonzero summands of a the *length of* a and denote it by $l(a)$. Suppose that $A \cap \gamma(A[X_\beta]) = 0$. Clearly $1 \leq l(a)$ for each nonzero $a \in \gamma(A[X_\beta])$. Now choose a nonzero element $a \in \gamma(A[X_\beta])$ with $l(a)$ minimal.

If all $x_{i_1}, \dots, x_{i_{n(a)}} \in X'_\alpha$ of a, for a subset $X'_\alpha \subseteq X_\beta$ such that card $X'_\alpha = \alpha$, then all $a_{i_1,\dots,i_{n(a)}} \in A[X_\beta \backslash X'_\alpha] \cap \gamma(A[X_\beta])$. Since $a_{i_1,\dots,i_{n(a)}} \in A$,

$$a_{i_1,\dots,i_{n(a)}} \in A[X_\beta \backslash X'_\alpha] \cap \gamma(A[X_\beta]) \cap A \subseteq \gamma(A[X_\beta]) \cap A = 0.$$

Therefore $a = 0$, a contradiction. Hence $\{x_{i_1}, \dots, x_{i_{n(a)}}\} \nsubseteq X'_\alpha$ for each X'_α.

Now consider $x_{i_1}, \dots, x_{i_s} \notin X'_\alpha$, where $1 \leq s < n(a)$ and $x_{i_{s+1}}, \dots, x_{i_{n(a)}} \in X'_\alpha$, without loss of generality. Then

$$\begin{aligned} 0 \neq a &= \sum f_s(x_{i_1}, \dots, x_{i_s}) x_{i_{s+1}}^{\alpha_{s+1}} \dots x_{i_{n(a)}}^{\alpha_n} \\ &\in \gamma(A[X_\beta] = ((A[X_\beta \backslash X'_\alpha]) \cap \gamma(A[X_\beta]))[X'_\alpha] \end{aligned}$$

and we also have

$$f_s = f_s(x_{i_1}, \dots, x_{i_s}) \in (A[X_\beta \backslash X'_\alpha]) \cap \gamma(A[X_\beta]).$$

If $l(f_s) = l(a)$ we can take f_s instead of a. Then the number of indeterminates in f_s is less than that in a. Continuing the above procedure we can find some $f_k \in \gamma(A[X_\beta])$ and $f_k \in \gamma(A[X'_\alpha])$ for some α, contradicting the above construction. Therefore $l(f_s) < l(a)$ and $f_s \in \gamma(A[X_\beta])$. Since $a \neq 0$, there exists some $f_s \neq 0$. This is a contradiction to $l(a)$ being minimal. Thus $A \cap \gamma(A[X_\beta]) \neq 0$. By Corollary 2.5, γ has the β-Amitsur property. $\square$

Corollary 2.7. *Let γ be a radical.*

(i) *If γ has the Amitsur property, then it has the α-Amitsur property, for any cardinal α.*
(ii) *If γ does not have the α-Amitsur property, then it does not have the α'-Amitsur property for $\alpha' \leq \alpha$.*

In [12], Puczyłowski proved that $\tilde{\gamma}(A[X_\alpha]) = (A \cap \tilde{\gamma}(A[X_\alpha]))[X_\alpha]$ for any cardinal α and commuting indeterminates, where $\tilde{\gamma}$ is one of $\beta, \mathbf{L}, \mathbf{J}$ or $\mathbb{G}$. Our Corollary 2.7(i) is a proper generalization of these results.

Definition 2.8. Let γ be a radical. A ring A is said to be an *absolute γ-ring* if $A[x_1, \ldots, x_n] \in \gamma$, for any $0 \neq n \in \mathbb{N}$. It is easy to see that the class $\bar{\gamma}$ of all absolute γ-rings is a radical class and $\bar{\gamma} = \bigcap_{n \in \mathbb{N}} \gamma_n$.

Definition 2.9. Let $\mathbf{P}$ denote a property an element of a ring may possess. It will be assumed that the element zero always has this property.

A ring A is called a $\mathbf{P}$-*ring* (see [18]), if each element $a \in A$ is a $\mathbf{P}$-element, that is, a has property $\mathbf{P}$. The fact that $a \in A$ is a $\mathbf{P}$-element in A we denote by $a \in \mathbf{P}_A$.

A radical γ is said to be $\mathbf{P}$-*radical* if for any ring A,

$$\gamma(A) = \sum \{I \trianglelefteq A \mid I \text{ is a } \mathbf{P}\text{-ring}\}.$$

Lemma 2.10. *Let γ be a $\mathbf{P}$-radical. Then $\bar{\gamma} = \gamma_{\mathbb{N}}$.*

If γ has the $\mathbb{N}$-Amitsur property, then for every ring A,

$$\gamma(A[X_{\mathbb{N}}]) = \bar{\gamma}(A[X_{\mathbb{N}}]) = \bar{\gamma}(A)[X_{\mathbb{N}}],$$

and $\bar{\gamma}$ is the unique minimal radical such that $\gamma(A[X_{\mathbb{N}}]) = \bar{\gamma}(A[X_{\mathbb{N}}])$.

Proof. Clearly $\bar{\gamma} \supseteq \gamma_{\mathbb{N}}$. Let $A \in \bar{\gamma}$ and consider the ring $A[X_{\mathbb{N}}]$. Then $A[X_n] \in \gamma$ for any $n \in \mathbb{N}$. For any element $a \in A[X_{\mathbb{N}}]$, there exists $n \in \mathbb{N}$ such that $a \in A[X_n] \in \gamma$. Since γ is a $\mathbf{P}$-radical, every element $a \in A[X_{\mathbb{N}}]$ is a $\mathbf{P}$-element, and $a \in \mathbf{P}_{A[X_{\mathbb{N}}]}$. Thus $A[X_{\mathbb{N}}] \in \gamma$ and so $A \in \gamma_{\mathbb{N}}$.

Now assume γ to have the $\mathbb{N}$-Amitsur property. Then $\gamma(A[X_{\mathbb{N}}]) = (A \cap \gamma(A[X_{\mathbb{N}}]))[X_{\mathbb{N}}] \in \gamma$. Hence $A \cap \gamma(A[X_{\mathbb{N}}]) \in \bar{\gamma}$. Since the elements in $X_{\mathbb{N}}$ are commuting, $(A \cap \gamma(A[X_{\mathbb{N}}]))[X_{\mathbb{N}}] \in \bar{\gamma}$ and therefore

$$(A \cap \gamma(A[X_{\mathbb{N}}]))[X_{\mathbb{N}}] \subseteq \bar{\gamma}(A[X_{\mathbb{N}}]) \subseteq \gamma(A[X_{\mathbb{N}}]).$$

Thus $\gamma(A[X_{\mathbb{N}}]) = \bar{\gamma}(A[X_{\mathbb{N}}]) = (A \cap \gamma(A[X_{\mathbb{N}}))[X_{\mathbb{N}}]$.

Clearly $(A \cap \gamma(A[X_{\mathbb{N}}]))[X_{\mathbb{N}}] \subseteq \bar{\gamma}(A)[X_{\mathbb{N}}] \subseteq \bar{\gamma}(A[X_{\mathbb{N}}])$. Suppose $\gamma(A[X_{\mathbb{N}}]) = \sigma(A[X_{\mathbb{N}}])$, for some radical σ. From above, $\bar{\gamma}(A) \in \sigma$ for every ring A. Thus $\bar{\gamma} \subseteq \sigma$. The proof is complete. □

Corollary 2.11. *Let γ be a $\mathbf{P}$-radical. If $\gamma(A[X_{\mathbb{N}}]) = \sigma(A[X_{\mathbb{N}}]) = \sigma(A)[X_{\mathbb{N}}]$ for a radical σ, then $\sigma = \bar{\gamma}$.*

Proof. Clearly $\sigma \subseteq \gamma_{\mathbb{N}} = \bar{\gamma}$. By assumption, γ has the $\mathbb{N}$-Amitsur property. By Lemma 2.10, $\bar{\gamma}$ is unique minimal, hence $\sigma = \bar{\gamma}$. □

Proposition 2.12. *Let γ be a* **P***-radical with the $\mathbb{N}$-Amitsur property. Then any radical σ such that $\bar{\gamma} \subseteq \sigma \subseteq \gamma$ has the $\mathbb{N}$-Amitsur property and, for any ring A,*

$$\gamma(A[X_{\mathbb{N}}]) = \sigma(A[X_{\mathbb{N}}]) = \bar{\sigma}(A)[X_{\mathbb{N}}] = \bar{\gamma}(A)[X_{\mathbb{N}}].$$

Proof. By assumption, $\bar{\gamma}(A[X_{\mathbb{N}}]) \subseteq \sigma((A[X_{\mathbb{N}}]) \subseteq \gamma(A[X_{\mathbb{N}}])$. In view of Lemma 2.10 and Corollary 2.11,

$$\gamma(A[X_{\mathbb{N}}]) = \bar{\gamma}(A[X_{\mathbb{N}}]) = \bar{\sigma}(A)[X_{\mathbb{N}}].$$

Therefore $\gamma(A[X_{\mathbb{N}}]) = \bar{\gamma}(A[X_{\mathbb{N}}]) = \bar{\sigma}(A[X_{\mathbb{N}}])$.

If $\gamma(A[X_{\mathbb{N}}]) \neq 0$, then $0 \neq \bar{\gamma}(A) \subseteq \sigma(A[X]) \cap A$. By Corollary 2.5, σ has the $\mathbb{N}$-Amitsur property. □

Definition 2.13. We say that a class M of rings is *α-polynomially extensible* if for all rings $A \in M$, $A[X_\alpha] \in M$.

Clearly, if a class M of rings is 1-polynomially extensible, then M is n-polynomially extensible for any $n \in \mathbb{N}$.

Proposition 2.14. *The class of all polynomially extensible radicals form a complete sublattice of the lattice of all radical classes.*

Proof. The intersection property is clear.

Let $A \in \mathcal{L}(\bigcup_{i\in I} \gamma_i) = \sigma$, where each γ_i is a polynomially extensible radical. Suppose $A[x] \notin \sigma$, i.e., $A[x]/\sigma(A[x]) \neq 0$. Then $\bar{A} = (A + \sigma(A[x]))/\sigma(A[x]) \neq 0$ since otherwise, by Lemma 2.1, $A[x] \subseteq \sigma(A[x])$, a contradiction.

Since $A \in \sigma$, $0 \neq \bar{A} \in \sigma$. We have $\gamma_i(A) \neq 0$ for some i. By assumption $\bar{0} \neq \gamma_i(\bar{A})[x] \in \gamma_i \subseteq \sigma$. Thus $\sigma(A[x]/\sigma(A[x])) \neq \bar{0}$, a contradiction. □

Corollary 2.15. *For any radical class γ, the radical class of absolute γ-rings is equal to the largest polynomially extensible subradical σ in γ.*

Proof. By Definition 2.8, the class of all absolute γ-rings is equal to $\bar{\gamma} = \bigcap_{n\in\mathbb{N}} \gamma_n$. Let $A \in \bar{\gamma}$ and $A[x] \notin \bar{\gamma}$. Then there exists some $n \in \mathbb{N}$ such that $A[x] \notin \gamma_n$. But for any $s \in \mathbb{N}$, $A[x_1, \ldots, x_s] \in \gamma$. Choosing $n + 1 < s$ we get $A[x_1] \in \gamma_n$, a contradiction. Thus $\bar{\gamma}$ is polynomially extensible. By Proposition 2.14, there exists a unique largest polynomially extensible subradical $\sigma \subseteq \gamma$. Hence $\bar{\gamma} \subseteq \sigma$. Let $A \in \sigma$. Since σ is polynomially extensible, $A[x_1] \in \sigma$ and in fact $A[x_1, \ldots, x_n] \in \sigma$ for all $n \in \mathbb{N}$. Thus A is an absolute γ-ring and therefore $A \in \bar{\gamma}$. □

Definition 2.16. Let $\mathbb{Z}\langle x, y_1, \ldots, y_n, \ldots\rangle$ and $\mathbb{Z}\langle x, y_1, \ldots, y_n\rangle$ denote the polynomial rings over the integers $\mathbb{Z}$ in non-commuting indeterminates $x, y_1, \ldots, y_n, \ldots$ and

$x, y_1, \dots, y_n$, respectively. The subrings of polynomials with zero constant term will be denoted by H and H_n, for $n = 0, 1, 2, \dots$, respectively. Furthermore, let G be a nonempty subset of H and $G_n = G \cap H_n$. For any ring A consider the set

$$P_G(A) = \{f(x, b_1, \dots, b_n) \mid n = 0, 1, 2, \dots, f \in G_n,\ b_i \in A\}.$$

For an element $a \in A$, let P_G denote the property that there exists a polynomial $f(x, b_1, \dots, b_n) \in P_G(A)$ such that $f(a, b_1, \dots, b_n) = 0$.

Theorem 2.17. *For a ring A the following are equivalent*:

(a) *A is an absolute $\mathbb{G}$-ring*;

(b) *A is an absolute $\mathbf{S}$-ring*;

(c) $A \in \bar{\mathbb{G}} = \bigcap_{n\in\mathbb{N}} \mathbb{G}_n$;

(d) $A \in \mathbf{S}$;

(e) *$A \in \sigma$ where σ is the unique largest polynomially extensible subradical of* $\mathbb{G}$;

(f) *for any $f \in A[X_k]$, $k \in \mathbb{N}$, there exist $n, m \in \mathbb{N}$ and elements $a_1, \dots, a_m$, $b_1, \dots, b_m \in A[X_k]$ such that $f^n = \sum_{i=1}^m a_i[f, b_i]$, where $[f, b_i]$ is the commutator of f and b_i*;

(g) *for any $f \in A[X_\mathbb{N}]$, there exist $n, m \in \mathbb{N}$ and elements $a_1, \dots, a_m$, $b_1, \dots, b_m \in A[X_\mathbb{N}]$ such that $f^n = \sum_{i=1}^m a_i[f, b_i]$*;

(h) $A \in \mathbb{G}_\mathbb{N} = \{A \mid A$ *a ring with* $A[X_\mathbb{N}] \in \mathbb{G}\}$.

The assertions (f), (g) *and* (h) *also hold for non-commuting indeterminates.*

Proof. Define

$$G = \{x + \textstyle\sum_{i=1}^l y_i z_i + \sum_{i=1}^l y_i x z_i \mid x, y_i, z_i \text{ are free generators of } \mathbb{Z}\langle x, y_i, z_i\rangle,\ i, l \in \mathbb{N}\}$$

and consider P_G as defined in Definition 2.16. We know that $\mathbb{G}$ is a P_G-radical and since $\mathbb{G}$ has the Amistur property, by Corollary 2.7(i), $\mathbb{G}$ has the $\mathbb{N}$-Amitsur property. Therefore, in view Lemma of 2.10, Corollary 2.11, 2.15 and [4, Theorem 5.1], we have $\bar{\mathbb{G}} = \mathbf{S} = \sigma = \mathbb{G}_\mathbb{N}$, and A is an absolute $\mathbb{G}$-ring if and only if it is an absolute $\mathbf{S}$-ring. Thus the assertions (a) to (e) are equivalent.

(c) $\Rightarrow$ (h). Suppose $A \in \bar{\mathbb{G}}$. Then by Lemma 2.10,

$$\mathbb{G}(A[X_\mathbb{N}]) = \bar{\mathbb{G}}(A)[X_\mathbb{N}] = A[X_\mathbb{N}],$$

and we have $A \in \mathbb{G}_\mathbb{N}$.

(h) $\Rightarrow$ (g). Let $A \in \mathbb{G}_\mathbb{N}$. Then $A[X_\mathbb{N}] \in \mathbb{G}$ and since $A[X_\mathbb{N} \cup x] \cong A[X_\mathbb{N}] \in \mathbb{G}$, we have $A[X_\mathbb{N}][x] \in \mathbb{G}$, therefore $A[X_\mathbb{N}] \in \mathbb{G}_1$. By [18, Corollary 7.6], for any $f \in A[X_\mathbb{N}]$, there exist $n, m \in \mathbb{N}$ and elements $a_1, \dots, a_m$, $b_1, \dots, b_m \in A[X_\mathbb{N}]$, such that $f^n = \sum_{i=1}^m a_i[f, b_i]$.

(g) $\Rightarrow$ (f). Let A be a ring satisfying the condition in (f). $A[X_k]$ is a homomorphic image of $A[X_\mathbb{N}]$, hence for $f \in A[X_k]$, there exists $g \in A[X_\mathbb{N}]$ such that $f = \bar{g}$. Therefore $f^n = \bar{g}^n = \sum \bar{a}_i[\bar{g}, \bar{b}_i]$ where $\bar{a}_i, \bar{b}_i \in A[X_k]$.

(f) $\Rightarrow$ (c). Let A be a ring with the condition in (f). Then, for every $f \in A[X_k]$, there exist $n, m \in \mathbb{N}$ and elements $a_i, b_i \in A[X_k]$, $i \leq m$, satisfying the condition in (f). Therefore, again by [18, Corollary 7.6], $A[X_k] \in \mathbb{G}_1$ for each $k \in \mathbb{N}$. Thus $A[X_k] \in \mathbb{G}$ for every $k \in \mathbf{N}$. This shows that A is an absolute $\mathbb{G}$-ring, hence $A \in \bar{\mathbb{G}}$.

The fact that (f), (g) and (h) also hold for non-commuting indeterminates was observed in [4, Remark 3.2] (referring to [12, 1.6] and [8, Corollary 13]).

This completes the proof of Theorem 2.17. □

Denote by $\mathbf{S}_1$ the subclass of $\mathbf{S}$ consisting of all unitary strongly prime rings R with nonzero *pseudo radical* (see [4]),

$$0 \neq ps(R) = \bigcap \{I \trianglelefteq R \mid 0 \neq I \text{ prime ideal in } R\}.$$

Putting $\mathbf{S}_2 = \mathbf{S} \backslash \mathbf{S}_1$, Ferrero and Wisbauer proved in [4] that for $R \in \mathbf{S}_2$ and any $0 \neq a \in R$, there exists some $n \in \mathbb{N}$ and an ideal M of $R[X_n]$, such that $R[X_n]/M$ is a simple ring with unit and $a \notin M \cap R$. If R is a simple ring without unit, then $A[x] \in \mathbb{G}$ (see [14]). In [4] this result was extended: If A is a simple ring without unit, then $A[X_n] \in \mathbb{G}$ for any $n \in \mathbb{N}$. This is again extended by the next corollary.

Corollary 2.18.

(i) *If A is an $\mathbf{S}$-semisimple ring, then for any $0 \neq a \in A$ there exists an $n \in \mathbb{N}$ and an ideal M of $A[X_n]$ for which $A[X_n]/M$ is a simple ring with unit and $a \notin M \cap A$.*

(ii) *If $A[X_n] \in \mathbb{G}$, for all $n \in \mathbb{N}$ for commuting indeterminates X_n, then $A[X] \in \mathbb{G}$ for any commuting or non-commuting indeterminates X.*

(iii) *The unitary strongly prime radical $\mathbf{S}$ is the unique largest polynomially extensible subradical in $\mathbb{G}$.*

Proof. (i) Assume that A is an $\mathbf{S}$-semisimple ring. Consider the chain of ideals of A,

$$A \supseteq \mathbb{G}_1(A) \supseteq \cdots \supseteq \mathbb{G}_n(A) \supseteq \cdots.$$

Since A is $\mathbf{S}$-semisimple, $\bigcap_{\mathbb{N}} \mathbb{G}_n(A) = 0$. Therefore, for any $0 \neq a \in A$, there exists $n \in \mathbb{N}$ such that $a \in \mathbb{G}_{n-1}(A)$ and $a \notin \mathbb{G}_n(A)$. Since $\mathbb{G}$ has the Amitsur property, in view of Corollary 2.7(i) and Theorem 2.4, $A_n[X_n] = (A/\mathbb{G}_n(A))[X_n] \in S\mathbb{G}$. Therefore $A_n[X_n]$ is a subdirect product of simple rings with unit. Choose $b \neq 0$ as an image of a. Then $A_n[X_n]$ has a factor ring $A_n[X_n]/K$ which is a simple ring with unit and contains a nonzero image c of b. Since $A_n[X_n]/K$ is a homomorphic image of $A[X_n]/\mathbb{G}_n(A)[X_n]$ such that $K \supseteq (\mathbb{G}_n(A))[X_n]$ and also $A_n[X_n]/K \cong A[X_n]/K \ni c \neq 0$ we conclude that $a \notin A \cap K$.

(ii) Let $A[X_n] \in \mathbb{G}$, for all $n \in \mathbb{N}$. Then by the above theorem $A \in S$. Therefore $A[X] \in \mathbb{G}$ for all X.

(iii) This follows from Corollary 2.15. □

Now we consider the Jacobson radical $\mathbf{J}$ and the nil radical $\mathbf{N}$.

Theorem 2.19.

(i) *For any ring* A, $\mathbf{J}(A[X_{\mathbb{N}}]) = \mathbf{N}(A[X_{\mathbb{N}}]) = \bar{\mathbf{N}}(A)[X_{\mathbb{N}}] = \bar{\mathbf{J}}(A)[X_{\mathbb{N}}]$ *for commuting indeterminates* $X_{\mathbb{N}}$.

(ii) $\mathbf{J}(A[X_{\mathbb{N}}]) = \mathbf{N}(A[X_{\mathbb{N}}])$ *for non-commuting* $X_{\mathbb{N}}$ *and in general*

$$\mathbf{J}(A[X_{\mathbb{N}}]) \neq \bar{\mathbf{J}}(A)[X_{\mathbb{N}}].$$

(iii) $\bar{\mathbf{N}}$ *is the unique largest polynomially extensible subradical in* $\mathbf{N}$ *and* $\mathbf{J}$.

(iv) $\beta = \bar{\beta} \subset \bar{\mathbf{L}} = \mathbf{L} \subset \bar{\mathbf{N}} = \bar{\mathbf{J}} \subset \bar{\mathbf{S}} = \mathbf{S} = \bar{\mathbb{G}} \subset \mathbb{G}$.

Proof. (i) We know that $\mathbf{J}_1 \subseteq \mathbf{N}$, therefore we have $\bar{\mathbf{J}} \subseteq \mathbf{N}$. Put $G = \{x + y + xy\}$ and consider P_G as defined in Definition 2.16. Then $\mathbf{J}$ is P_G-radical and has the Amitsur property and also the $\mathbb{N}$-Amitsur property. By Proposition 2.12, we have

$$\mathbf{J}(A[X_{\mathbb{N}}]) = \mathbf{N}(A[X_{\mathbb{N}}]) = \bar{\mathbf{N}}(A)[X_{\mathbb{N}}] = \bar{\mathbf{J}}(A)[X_{\mathbb{N}}].$$

(ii) Puczyłowski proved in [12] that $\mathbf{J}(A[X_{\mathbb{N}}]) = \mathbf{L}(A)[X_{\mathbb{N}}]$ for non-commuting indeterminates. Therefore $\mathbf{J}(A[X_{\mathbb{N}}]) = \mathbf{N}(A[X_{\mathbb{N}}]) = \mathbf{L}(A)[X_{\mathbb{N}}]$. But in [7], Golod constructed an absolutely nil and non-locally nilpotent algebra B. Therefore $\mathbf{J}(B[X_{\mathbb{N}}]) \neq \bar{\mathbf{J}}(B)[X_{\mathbb{N}}]$.

(iii) follows from (i) and Corollary 2.15.

(iv) $\mathbf{L} \subsetneq \bar{\mathbf{N}}$ follows from Golod's example and $\bar{\mathbf{J}} \subsetneq \bar{\mathbf{S}}$ is a consequence of Corollary 2.18. □

We note that Theorem 2.19(iv) is a proper generalization of [9, Theorem 3.3].

Corollary 2.20. *For a ring* A, *the following are equivalent*:

(a) A *is an absolute nil ring*;

(b) A *is an absolute* $\mathbf{J}$*-ring*;

(c) *for any* $f \in A[X_n]$, $n \in \mathbb{N}$, *there exists* $g \in A[X_n]$ *with* $f + g + fg = 0$;

(d) *for all* $n \in \mathbb{N}$, $A[X_n]$ *is a nil ring*;

(e) $A[X_{\mathbb{N}}]$ *is a nil ring*;

(f) $A[X_{\mathbb{N}}]$ *is a Jacobson radical ring.*

Proof. (a) $\Leftrightarrow$ (b). It follows from Theorem 2.19(i) that $\bar{\mathbf{N}} = \bar{\mathbf{J}}$.

(b) $\Rightarrow$ (f). Let $A \in \bar{\mathbf{J}}$. Then by Theorem 2.19(i), $A[X_{\mathbb{N}}] \in \mathbf{J}$.

(f) $\Rightarrow$ (e). If $A[X_{\mathbb{N}}] \in \mathbf{J}$, then $A[X_{\mathbb{N}} \cup x] \cong A[X_{\mathbb{N}}] \in \mathbf{J}$. Thus $A[X_{\mathbb{N}}] \in \mathbf{N}$.

(e) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (b) is clear. □

Amitsur proved that a nil algebra A over an infinite field F is absolutely nil if and only if A is an LBL-algebra (every finitely generated submodule of A is of bounded index, see [1]).

Remark 2.21. Let γ be a radical. If $A[X_\alpha] \in \gamma$ for some α, then $A[x] \in \gamma$.

Recall from Definition 2.3 that a radical γ has the *strong α-Amitsur property* if $\gamma(A[X_\alpha]) = \gamma(A)[X_\alpha]$.

Proposition 2.22. *For a radical γ and a cardinal α, the following are equivalent.*
(a) *$S\gamma$ and γ are α-polynomially extensible*;
(b) $\gamma(A[X_\alpha]) = \gamma(A)[X_\alpha]$.

Proof. (a) $\Rightarrow$ (b). From (a) we have $\gamma(A)[X_\alpha] \in \gamma$. Since

$$\gamma(A)[X_\alpha] \trianglelefteq A[X_\alpha], \quad \gamma(A)[X_\alpha] \subseteq \gamma(A[X_\alpha]),$$

we know that $\gamma(A[X_\alpha]/\gamma(A)[X_\alpha]) = \gamma(A[X_\alpha])/\gamma(A)[X_\alpha]$. Again by (a),

$$0 = \gamma(A/\gamma(A))[X_\alpha] \cong \gamma(A[X_\alpha]/\gamma(A)[X_\alpha])$$

and thus $\gamma(A[X_\alpha]) = \gamma(A)[X_\alpha]$.

(b) $\Rightarrow$ (a). This is clear, since for $A \in \gamma$, $\gamma(A[X_\alpha]) = \gamma(A)[X_\alpha] = A[X_\alpha]$ and for $A \in S\gamma$, $\gamma(A[X_\alpha]) = \gamma(A)[X_\alpha] = 0[X_\alpha] = 0$. □

Theorem 2.23. *Let γ be a* **P***-radical. Then γ has the strong Amitsur property if and only if γ has the strong α-Amitsur property for some (or any) cardinal α.*

Proof. $\Rightarrow$. By Proposition 2.22, if $A \in \gamma$, then $A[X_n] \in \gamma$, and $A \in S\gamma$ implies $A[X_n] \in S\gamma$, for any $n \in \mathbb{N}$. Hence, again by Proposition 2.22, γ has the strong n-Amitsur property. Suppose α is infinite and $A \in S\gamma$. For any $a \in A[X_\alpha]$, there exist $x_{i_1}, \dots, x_{i_n} \in X_\alpha$ and $a_{i_1,\dots,i_n} \in A$ such that $a = \sum a_{i_1,\dots,i_n} x_{i_1}^{\alpha_1} \dots x_{i_n}^{\alpha_n}$. Then $A[x_{i_1}, \dots, x_{i_n}] \in S\gamma$ as above. But $A[x_{i_1}, \dots, x_{i_n}]$ is a homomorphic image of $A[X_\alpha]$. Since a is an arbitrary element, $A[X_\alpha]$ is a subdirect product of γ-semisimple rings. Since $S\gamma$ is subdirectly closed, $A[X_\alpha] \in S\gamma$.

Let $A \in \gamma$. For every $a \in A[X_\alpha]$, there exists $n \in \mathbb{N}$ such that $a \in A[X_n]$. Since γ is a **P**-radical, for every element a of $A[X_\alpha]$, $a \in \mathbf{P}_{A[X_\alpha]}$. Thus $A[X_\alpha] \in \gamma$. By Proposition 2.22, γ has the strong α-Amitsur property.

$\Leftarrow$. Suppose γ has the strong α-Amitsur property and $1 < \alpha$. Suppose $A \in \gamma$. Then $A[X_\alpha] \in \gamma$. By Remark 2.21, $A[X_1] \in \gamma$.

Suppose $A \in S\gamma$ and $\gamma(A[x_1]) \neq 0$. Then from above

$$\gamma(A[x_1])[x_2, \dots, x_{n-1}] \in \gamma \text{ and } 0 \neq \gamma(A[x_1])[x_2, \dots, x_{n-1}] \trianglelefteq A[X_n]$$

for any $n \in \mathbb{N}$. If α is infinite, since γ is **P**-radical, $\gamma(\gamma(A[x_1])[X_\alpha]) \neq 0$. Therefore $\gamma(A[X_\alpha]) \neq 0$. Since γ has the strong α-Amitsur property, $A[X_\alpha] \in S\gamma$, a contradiction. □

The following corollary extends [4, Theorem 3.3] for commuting indeterminates. In general it need not be true for non-commuting indeterminates.

Corollary 2.24. *Let γ be a* $\mathbf{P}$*-radical with the α-Amitsur property. Then*:

(i) $\bar{\gamma}(A[X]) = \bar{\gamma}(A)[X]$, *for all A, where X are commuting indeterminates.*
(ii) $\bar{\gamma}$ *is the unique largest strong Amitsur radical in γ.*

Proposition 2.25. *The semisimple class of idempotent radicals* (*that is all radical rings are idempotent*) *is polynomially extensible.*

Proof. Let γ be an idempotent radical and $\gamma(A[x]) \neq 0$ and $A \in S\gamma$. It is clear that $xA[x]$ is a subdirect product of nilpotent rings. Therefore $\gamma(A[x]) \nsubseteq xA[x]$, because $\gamma(A[x])$ is an idempotent ring. Note that

$$((\gamma(A[x]) + xA[x])/xA[x]) \trianglelefteq (A[x])/xA[x] \cong A \in S\gamma.$$

Since $\gamma(A[x]) \nsubseteq xA[x]$, we have

$$0 \neq (\gamma(A[x]) + xA[x])/xA[x] \cong \gamma(A[x])/(\gamma(A[x]) \cap xA[x]) \in \gamma.$$

Therefore A contains a nonzero γ radical ideal, a contradiction. □

Corollary 2.26.

(i) *Every polynomially extensible idempotent radical has the strong Amitsur property.*
(ii) *Every nonzero hereditary idempotent radical has the Amitsur property but not the strong Amitsur property.*

Proof. (i) This follows from Propositons 2.25 and 2.22.

(ii) If $\gamma(A[x]) \neq 0$ then $\gamma(A[x]) \cap xA[x] \neq 0$ which is not idempotent. This is impossible. □

Theorem 2.27. *Let γ be an idempotent radical. Then γ has the Amitsur property if and only if it satisfies the condition*

$$(T) \quad f(x) \in \gamma(A[x]) \ \textit{ implies } f(0) \in \gamma(A[x]).$$

Proof. $\Rightarrow$. This follows from [11].

$\Leftarrow$. Suppose $\gamma(A[x]) \neq 0$. By a proof similar to that of Proposition 2.25, we get $\gamma(A[x]) \subsetneqq xA[x]$. Therefore there exists $0 \neq a_0 + a_1x + \cdots + a_nx^n \in \gamma(A[x])$ such that $a_0 \neq 0$. Hence by condition (T), $f(0) \in \gamma(A[x])$ and by Corollary 2.5, γ has the Amitsur property. □

3. Lattice of Radicals with α-Amitsur Property

Recall that $\mathcal{L}(M)$ denotes the lower radical generated by a class M of rings and let Λ be any index set.

Proposition 3.1. *Let $\{\gamma_i\}_{i\in\Lambda}$ be a family of radicals with the α-Amitsur property. Then the radicals*

$$\gamma = \mathcal{L}(\bigcup_\Lambda \gamma_i) \quad \text{and} \quad \gamma_o = \bigwedge_\Lambda \gamma_i = \bigcap_\Lambda \gamma_i$$

have the α-Amitsur property.

Thus, for any cardinal α, the class of radicals with α-Amitsur property is a complete sublattice in the lattice of all radicals.

Proof. Let A be a ring such that $\gamma(A[X_\alpha]) \neq 0$. Then there exists an $i \in \Lambda$, with $\gamma_i(A[X_\alpha]) \neq 0$. Since γ_i has the α-Amitsur property,

$$0 \neq (A \cap \gamma_i(A[X_\alpha])[X_\alpha] = \gamma_i(A[X_\alpha]) \subseteq \gamma(A[X_\alpha]).$$

Thus $0 \neq A \cap \gamma(A[X_\alpha])$ and, by Corollary 2.5, γ has the α-Amitsur property.

Now suppose $\gamma_o(A[X_\alpha]) \neq 0$ for a ring A, and set

$$I_\alpha = \left\{ a_{i_1,\ldots,i_n} \in A \,\middle|\, \sum a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \gamma_o(A[X_\alpha]) \right\}.$$

Denote by I^α the ideal of A generated by I_α. Then clearly

$$\gamma_o(A[X_\alpha]) \subseteq I^\alpha[X_\alpha] \quad \text{and} \quad \gamma_i(I^\alpha[X_\alpha]) \neq 0 \text{ for every } i \in \Lambda.$$

We set $J_i = I^\alpha \cap \gamma_i(I^\alpha[X_\alpha])$ and by the α-Amitsur property,

$$\gamma_i(I^\alpha[X_\alpha]) = J_i[X_\alpha].$$

Since $I^\alpha \trianglelefteq A$ we have $J_i \trianglelefteq A$.

Now we claim that $I^\alpha = J_i$ for all $i \in \Lambda$. Clearly $J_i \subseteq I^\alpha$ and suppose $I^\alpha \neq J_i$ for some i. Since $J_i \trianglelefteq A$, there exists $a_{i_1,\ldots,i_n} \in I^\alpha \backslash J_i$. Thus there is some $0 \neq f(X_\alpha) \in \gamma_i(I^\alpha[X_\alpha])$ such that $f(X_\alpha) = a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} + g(X_\alpha)$ where $g(X_\alpha)$ has no member of $x_1^{i_1} \cdots x_n^{i_n}$.

Since $\gamma_o(A[X_\alpha]) \subseteq \gamma_i(I^\alpha[X_\alpha]) = J_i[X_\alpha]$, all coefficients of $f(X_\alpha)$ are in J_i, a contradiction. Thus $I^\alpha = J_i$, for all i. Hence by Lemma 2.1, $I^\alpha[X_\alpha] \subseteq \gamma_i(I^\alpha[X_\alpha])$. From this we have $\gamma_o(I^\alpha[X_\alpha]) = I^\alpha[X_\alpha]$. Since $\gamma_o(A[X_\alpha]) \neq 0$, also $I^\alpha \neq 0$ and so $0 \neq I^\alpha \cap I^\alpha[X_\alpha] \subseteq A \cap \gamma(A[X_\alpha])$. By Corollary 2.5, γ_o has the α-Amitsur property. □

Corollary 3.2. *Let α be a cardinal.*

(i) *Any radical contains a unique largest subradical with the α-Amitsur property.*
(ii) *For any radical γ there exists a unique minimal radical σ with α-Amitsur property such that $\gamma \subseteq \sigma$.*

Remark 3.3. Denote by $\mathbb{L}_\alpha$ the lattice of radicals with the α-Amitsur property. Then, by Corollary 2.7, we have the ascending chain

$$\mathbb{L}_1 \subseteq \mathbb{L}_2 \subseteq \cdots \subseteq \mathbb{L}_\alpha \subseteq \cdots.$$

A radical γ is said to be *hereditary* (*left*, *right*, *strongly hereditary*), if $I \trianglelefteq \gamma(A)$ ($L \trianglelefteq_l \gamma(A)$, $R \trianglelefteq_r \gamma(A)$, subring $S \subseteq \gamma(A)$) implies $I \in \gamma$ ($L \in \gamma$, $R \in \gamma$, $S \in \gamma$), respectively (see [5]).

We note the following (see [2, 5, 13, 16] and [15]).

Proposition 3.4. *The class of all* (*left*, *right*, *strongly*) *hereditary radicals is a complete sublattice in the lattice of all radical classes.*

Corollary 3.5. *The class of all* (*left*, *right*, *strongly*) *hereditary radicals with the* α*-Amitsur property is a complete sublattice in the lattice of all radical classes.*

Proof. This follows from the Propositions 3.1 and 3.4. □

Proposition 3.6. *The class of all strong Amitsur radicals is a complete sublattice.*

Proof. A radical γ is strongly Amitsur if and only if $S\gamma$ and γ are polynomially extensible. Thus the assertion follows from Theorem 2.4 and the Propositions 3.1 and 2.14. □

Proposition 3.7. *The class of all* (*left*, *right*, *strongly*) *hereditary radicals with the strong Amitsur property is a complete sublattice in the lattice of all radical classes.*

Proof. This is a consequence of the Propositions 3.4 and 3.6. □

From the above results it is natural to ask which of the radicals are coatomic. We will give an answer to this question. Let X be an infinite set of indeterminates and denote by $|M|$ the cardinality of any set M.

Lemma 3.8. *Let A be a semiprime ring. Then there exists a nonzero ideal I_J of A such that $|I_J| \leq |J|$ for $0 \neq J \trianglelefteq A[X]$.*

Proof. Let $0 \neq J \trianglelefteq A[X]$. Then there exists $0 \neq f(X) \in J$ such that the degree of $f(X)$ is minimial. From those elements, we can choose an element $f_0(X) = \sum a_{i_1,\dots,i_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with minimial length. Since $f_0(X) \neq 0$, there exists $b = a_{i_1,\dots,i_n} \neq 0$. We consider the ideal $I_J = A^1 b A^1 \trianglelefteq A$ and the map

$$\phi : I_J \to J, \quad c = \sum\nolimits_{i=1}^{m} a_i b b_i \mapsto \bar{c} = \sum\nolimits_{i=1}^{m} a_i f_0(X) b_i.$$

Indeed if $c, d \in I_J$ such that $c \neq d$ then $\bar{c} \neq \bar{d}$ because the length of $f_0(X)$ is minimial. Clearly that if $c = d$ then $\bar{c} = \bar{d}$. Thus $| I_J | \leq | J |$. □

Theorem 3.9. *Let X be an infinite set of indeterminates. Then the lower radical $\gamma = \mathcal{L}(\mathbb{Z}\langle X\rangle)$ determined by the free ring $\mathbb{Z}\langle X\rangle$ is strongly hereditary and has the strong Amitsur property.*

Proof. First we show that γ has the strong Amitsur property. We claim that if a ring $A \in S\gamma$, then $A[x] \in S\gamma$. Suppose $A \in S\gamma$ and $J = \gamma(A[x]) \neq 0$. Since $J \in \gamma$, J has a nonzero accessible subring J_1 (that is, $J_1 \trianglelefteq J_2 \trianglelefteq \cdots \trianglelefteq J_n = J \trianglelefteq A[x]$) such that $J_1 \cong \mathbb{Z}\langle X\rangle/I$ for some ideal I of $\mathbb{Z}\langle X\rangle$. Hence

$$|J_1| = |\mathbb{Z}\langle X\rangle/I| \leq |\mathbb{Z}\langle X\rangle|.$$

Denoting by $\langle J_1\rangle$ the ideal of J_3 generated by J_1, we have

$$\langle J_1\rangle \trianglelefteq J_3 \trianglelefteq \cdots \trianglelefteq J_n = J \trianglelefteq A[x],$$

and by Andrunakievich's Lemma, $\langle J_1\rangle^3 \subseteq J_1$. Continuing this procedure we get that $J_0 = \langle \cdots \langle\langle J_1\rangle^3\rangle^3 \cdots\rangle^3$ is an ideal of $A[x]$ and $J_0 \subseteq J_1$ for some $n \in \mathbb{N}$. Therefore $|J_0| \leq |J_1| \leq |\mathbb{Z}\langle X\rangle|$.

It is easy to see that the Baer radical $\beta \subseteq \gamma$. Since $A \in S\gamma$, A is a semiprime ring. Hence $J_0 \neq 0$. By Lemma 3.8, there exists a nonzero ideal I_0 of A such that $|I_0| \leq |J_0|$. Hence I_0 is a homomorphic image of $\mathbb{Z}\langle X\rangle$. Since $A \in S\gamma$ and $I_0 \trianglelefteq A$, $I_0 \in S\gamma \cap \gamma = 0$. This is a contradiction.

Next we show that $A \in \gamma$ implies $A[x] \in \gamma$. We may assume that A is a semiprime ring. Since $A \in \gamma$ there exists an accessible subring J_1 such that J_1 is a homomorphic image of $\mathbb{Z}\langle X\rangle$. In the same way as above we can find a nonzero ideal I_λ of A such that $I_\lambda \cong \mathbb{Z}\langle X\rangle/I_\lambda^0$ for some $I_\lambda^0 \trianglelefteq \mathbb{Z}\langle X\rangle$. Therefore $I_\lambda[x] \in \gamma$.

Put $I = \sum\{I_\lambda \trianglelefteq A \mid \text{such that } I_\lambda[x] \in \gamma\}$. Then $I[x] = (\sum I_\lambda)[x] = \sum(I_\lambda[x]) \in \gamma$. Suppose that $I \neq A$. Then $A[x]/I[x] \cong (A/I)[x] = \bar{A}[x]$ and $0 \neq \bar{A} \in \gamma$. Thus, as above, there exists $0 \neq \bar{B} \trianglelefteq \bar{A}$ such that $\bar{B}[x] \in \gamma$ where $\bar{B} = B/I$ for some $B \trianglelefteq A$. Therefore $B[x] \in \gamma$, a contradiction. Hence we have $\gamma(A[x]) = A[x]$ and, by Proposition 2.22, γ has the strong Amitsur property. The hereditariness follows from [16, Proposition 8]. □

Theorem 3.10. *There is no coatom in the lattice of all radicals with the α-Amitsur property* (*which are hereditary, left hereditary, or strongly hereditary*).

Moreover, there is no coatom in the lattice of all strong Amitsur radicals (*which are hereditary, left hereditary, strongly hereditary*).

Proof. Let γ be a coatom in the lattice of radicals with the α-Amistur property radicals. Then there exists a free ring $\mathbb{Z}\langle X\rangle \notin \gamma$ with $|X|$ infinite. We consider the lower radical σ generated by γ and $\mathbb{Z}\langle X\rangle$. Then, by Corollary 1 in [6], $\sigma \neq$ all rings. Hence, by Theorems 3.1 and 3.9, it has the α-Amitsur property.

The other cases are covered by Corollary 3.5. □

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References

[1] S. Amitsur, Algebras over infinite fields, *Proc. Amer. Mxath. Soc.* **7** (1956) 35–48.
[2] N. J. Divinsky, *Rings and Radicals* (Allen and Unwin, London, 1965).
[3] N. J. Divinsky and A. Sulinski, Kurosh radicals of rings with operators, *Can. J. Math.* **17** (1965) 278–288.
[4] M. Ferrero and R. Wisbauer, Unitary strongly prime rings and related radicals, *J. Pure. Applied Alg.* **181** (2003) 209–226.
[5] B. J. Gardner and R. Wiegandt, *Radical Theory of Rings* (Marcel Dekker, 2004).
[6] B. J. Gardner and L. Zhian, Small and large radical classes, *Commun. Algebra* **20** (1992) 2533–2551.
[7] E. S. Golod, On nil-algebras and finitely approximable p-groups, *Izv. Akad. Nauk SSSR, Ser. Mat.* **28** (1964) 273–276 (Russian), translation: *Am. Math. Soc., Translat., II. Ser.* **48** (1965) 103–106.
[8] E. Jespers and E. R. Puczyłowski, The Jacobson and Brown–McCoy radicals of rings graded by free groups, *Commun. Algebra* **19**(2) (1991) 551–558.
[9] A. Kaučikas and R. Wisbauer, On strongly prime rings and ideals, *Commun. Algebra* **28**(11) (2000) 5461–5473.
[10] J. Krempa, On properties of polynomial rings, *Bull. Acad. Polon. Sci.* **20** (1972) 545–548.
[11] N. V. Loi and R. Wiegandt, On the Amitsur property of radicals, *Algebra Discrete Mathematics (Kiev)* **3** (2006) 92–100.
[12] E. R. Puczyłowski, Behaviour of radical properties of rings under some algebraic constructions, *Proc. Radical Theory, Eger (Hungary), Coll. Math. Janos Bolyai*, (1982) 449–480.
[13] E. R. Puczyłowski, Hereditariness of strong and stable radicals, *Glasgow. Math. J.* **23** (1982) 85–90.
[14] E. R. Puczyłowski and A. Smoktunowicz, On maximal ideals and the Brown–McCoy radical of polynomial rings, *Commun. Algebra* **26**(8) (1998) 2473–2484.
[15] R. L. Snider, Lattice of radicals, *Pacif. J. Math.* **46** (1972) 207–220.
[16] S. Tumurbat and R. Wiegandt, On the lattice of strongly hereditary radicals, *Contrib. Gen. Alg.* **9** (1995) 309–312.
[17] S. Tumurbat and R. Wiegandt, Radicals of polynomial rings, *Soochow. J. Math.* **29** (2003) 425–434.
[18] S. Tumurbat and R. Wiegandt, On polynomial and multiplicative radicals, *Quaest. Math.* **26** (2003) 453–469.
[19] S. Tumurbat and R. Wiegandt, On radicals with Amitsur property, *Commun. Algebra* **32**(3) (2004) 1219–1227.
[20] R. Wiegandt, Radical and semisimple classes of rings, *Queen's Papers Pure Appl. Math.* **37** (1974).