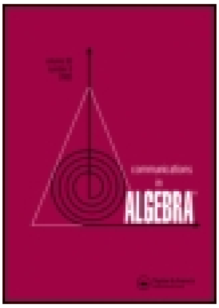

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Endomorphism Rings of Quotient Modules for Spectral Torsion Theories

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Several papers have been written on structural properties of quotient rings. In most cases the hereditary torsion theories considered are in R-*Mod*. See for example [1], [2], [6] and the references of [5]. Now, we are going to study these theories in a more general situation. In order to do so, let M be a left R-module and let $\sigma[M]$ be the full subcategory of all the R-modules that are subgenerated by M, and let M-tors be the lattice of all the hereditary torsion theories in $\sigma[M]$. We will study the structure of the endomorphism rings of quotient modules. In [3], spectral hereditary torsion theories in $\sigma[M]$ are investigated in detail. For these kinds of torsion theories, it has been proved that for all $N \in \sigma[M]$, which are not torsion, the endomorphism ring of the quotient module, $End_R(Q_{\mathcal{T}}(N))$, is a left selfinjective regular ring. Let $\mathcal{T}$ be a spectral torsion theory and $N \in \sigma[M]$ an $(M, \mathcal{T})$-injective torsionfree module. We will establish a bijective correspondence between the lattice of the torsion theories which are larger than the torsion theory cogenerated by N and the lattice of central idempotents of $End_R(N)$. In this context we find out when the endomorphism ring of quotient modules are (products of) prime or full linear rings. This is achieved by considering the behaviour of the spectral torsion theories within the complete lattice of all hereditary torsion theories. In particular, the results proved here can be applied to polyform modules and non-singular rings. Also, our results extend those which appear in [1] and [2] in the case $M = R$.

1 Preliminaries

Let R be an associative ring with unit, *R-Mod* denote the category of left R-modules, and let M be a left R-module. An R-module is said to be *subgenerated by* M if it is isomorphic to a submodule of an M-generated module. By $\sigma[M]$ we denote the full subcategory of *R-Mod* consisting of all modules that are subgenerated by M ([9]). For every $N \in \sigma[M]$, we denote the M-injective hull of N by $\widehat{N}$ or $I_M(N)$. Homomorphisms of left modules will usually be written on the right side of the argument. For notation and terminology on torsion theories in $\sigma[M]$, the reader is referred to [3] or [10].

For a family $\{N_\alpha\}_A$ of modules in $\sigma[M]$, let $\chi(\{N_\alpha\})$ be the maximal element of M-tors for which all the N_α are torsionfree, and let $\xi(\{N_\alpha\})$ denote the minimal element of M-tors for which all the N_α are torsion.

$\chi(\{N_\alpha\})$ is called the torsion theory *cogenerated* by $\{N_\alpha\}_A$ and $\xi(\{N_\alpha\})$ the torsion theory *generated* by $\{N_\alpha\}_A$.

1.1 $\mathcal{T}$-cocritical modules. We recall some definitions (see [5]). Let $\mathcal{T} \in M$-tors and $N \in \sigma[M]$ A submodule of $K \subset N$ is called $\mathcal{T}$-*dense* in N if $N/K \in \mathcal{T}$, and is called $\mathcal{T}$-*closed* in N if N/K is torsionfree. A nonzero module N is called $\mathcal{T}$-*cocritical* if N is $\mathcal{T}$-torsionfree and all the non-zero submodules of N are $\mathcal{T}$-dense. A module N is called *cocritical* if N is $\chi(N)$-cocritical.

A torsion theory $\mathcal{T}$ is called *prime* if it is cogenerated by a cocritical module, and is called *strongly semiprime* if it is cogenerated by all its $\mathcal{T}$-cocritical modules.

For each $\mathcal{T} \in M$-tors, we let $[\mathcal{T}, \sigma[M]] = \{\mathcal{R} \in M\text{-tors} \mid \mathcal{T} \leq \mathcal{R} \leq \sigma[M]\}$.

1.2 Spectral torsion theories. Recall that a Grothendieck category $\mathcal{C}$ is called *spectral* if every short exact sequence in $\mathcal{C}$ splits (see [8]).

Any hereditary torsion class in $\sigma[M]$ allows the construction of a quotient category which may be described in the following way.

The category $\mathcal{E}_{\mathcal{T}}[M]$. Let $\mathcal{E}_{\mathcal{T}}[M]$ denote the full subcategory of $\sigma[M]$ whose objects are all $(M, \mathcal{T})$-injective and $\mathcal{T}$-torsionfree modules (see [10]). $\mathcal{E}_{\mathcal{T}}[M]$ is a reflective subcategory of $\sigma[M]$ and hence a Grothendieck category (see [8, Chap. X]).

Definition. A hereditary torsion theory $\mathcal{T}$ in $\sigma[M]$ is called *spectral* if $\mathcal{E}_{\mathcal{T}}[M]$ is a spectral category.

In [3] a characterization of spectral torsion theories was given.

1.3 Lattice Theory. Let $(L, \wedge, \vee; 0, 1)$ be a complete lattice. A non-zero element $a \in L$ is an *atom* if $b < a$ implies $b = 0$. We say that L is an *atomic lattice* if for all $0 \neq b \in L$, there is an atom $a \in L$ such that $a \leq b$. L is *locally atomic* if each non-zero element is a join of atoms. A Boolean lattice is a complemented distributive lattice. We note that if L is a complete Boolean lattice, then L is atomic if and only if L is locally atomic.

2 Lattice properties of spectral torsion theories

Given a spectral torsion theory $\mathcal{T}$ in $\sigma[M]$, let $\mathcal{T}''$ denote the spectral torsion theory that is cogenerated by all the $\mathcal{T}$-cocritical modules,

$$\mathcal{T}'' = \chi(\{N \in \sigma[M] \mid N \text{ is } \mathcal{T}\text{-cocritical}\}).$$

Note that $\mathcal{T} \leq \mathcal{T}''$. We will prove that for every proper torsion theory $\mathcal{R}$ such that $\mathcal{T} \leq \mathcal{R}$, there exist $\mathcal{R}$-cocritical modules.

Remark. Let $\mathcal{R} \in M$-tors and $N \in \sigma[M]$ an $\mathcal{R}$-cocritical module. Then $\mathcal{R} \vee \xi(N)$ is an atom in $[\mathcal{R}, \sigma[M]]$.

If $N \in \sigma[M]$, then $\{K \subset N \mid K$ is cocritical and $\mathcal{T}$-torsionfree $\} = \{K \subset N \mid K$ is $\mathcal{T}$-cocritical$\}$. Notice that since $\mathcal{T}$ is spectral, the uniform submodules of a $\mathcal{T}$-torsionfree module are exactly the cocritical submodules.

2.1 Lemma. *Let $\mathcal{T} \in M$-tors be spectral.*

(1) If $\mathcal{T}'' < \sigma[M]$, then $\mathcal{T}''$ is strongly semiprime.

(2) Let $\mathcal{T}'' \leq \mathcal{R} < \sigma[M]$. If N is $\mathcal{R}$-torsionfree, then N contains a non-zero $\mathcal{R}$-cocritical submodule.

(3) If $\mathcal{T}'' \leq \mathcal{R} < \sigma[M]$, then $\mathcal{R}$ is strongly semiprime.

(4) If $\mathcal{T}'' < \mathcal{R}$, then $\mathcal{R} = \mathcal{T}'' \vee \xi(\{N \in \sigma[M] \mid N$ is $\mathcal{T}''$-cocritical and $N \in \mathcal{R}\})$.

(5) $\mathcal{T}$ is strongly semiprime if and only if $\mathcal{T} = \mathcal{T}''$.

Proof. (1) This follows by the previous remark.

(2) Let N be $\mathcal{R}$-torsionfree. Since N is $\mathcal{T}''$-torsionfree, there exists a $\mathcal{T}''$-torsionfree cocritical module C and a morphism $0 \neq f : \widehat{N} \to \widehat{C}$. We have that $Im\ f$ is $\mathcal{T}''$-torsionfree; thus $Ker\ f$ is a direct summand of $\widehat{N}$. Therefore $Im\ f$

is M-injective. Since C is cocritical, $\widehat{C}$ is uniform. Hence $Im\ f = \widehat{C}$. Thus $\widehat{C}$ is isomorphic to a submodule N' of $\widehat{N}$, and N' is an $\mathcal{R}$-cocritical submodule of $\widehat{N}$. We have that $0 \neq N \cap N'$. Since the submodules of $\mathcal{R}$-cocritical modules are always $\mathcal{R}$-cocritical, we obtain (2).

(3) Let $\mathcal{R}' = \chi(\{N \in \sigma[M] \mid N \text{ is } \mathcal{R}\text{-cocritical}\})$. Suppose $\mathcal{R} < \mathcal{R}'$. There exists a non-zero $\mathcal{R}$-torsionfree module K, such that $\mathcal{R}'(K) = K$. By (2), there exists $0 \neq K' \subset K$ such that K' is $\mathcal{R}$-cocritical. Hence $K' \in \mathcal{R}'$ and K' is $\mathcal{R}'$-torsionfree, which is a contradiction.

(4) follows in the same way as (3).

(5) is clear. □

Observe that the condition (4) of 2.1 implies that if $\mathcal{T}$ is spectral and $\mathcal{T}'' \neq \sigma[M]$, then $[\mathcal{T}'', \sigma[M]]$ is locally atomic.

For any ring R, let $B(R)$ denote the Boolean algebra of central idempotents in R. If R is a left selfinjective regular ring, then $B(R)$ is a complete Boolean algebra. The following observation enables us to construct a close relationship between the central idempotents of the endomorphism ring of $N \in \mathcal{E}_{\mathcal{T}}[M]$ and the (spectral) hereditary torsion theories containing $\chi(N)$.

2.2 Lemma. *Let $\mathcal{T} \in M$-tors be spectral and $N \in \mathcal{E}_{\mathcal{T}}[M]$. Then for any fully invariant $K \subset N$, there exists $e \in B(End_R(N))$ such that $\widehat{K} = Ne$.*

Proof. Since N is polyform (by [3, 2.4(1)]), the unique M-injective hull $\widehat{K}$ of K is fully invariant and $N = \widehat{K} \oplus X$, where X is also fully invariant (see [10, 11.11]). Now the projection $N \to \widehat{K}$ yields a central idempotent in T with $\widehat{K} = Ne$. □

2.3 Theorem. *Let $\mathcal{T} \in M$-tors be spectral, $N \in \mathcal{E}_\mathcal{T}[M]$ and $S := End_R(N)$. The correspondence*

$$\mu : B(S) \to [\chi(N), \sigma[M]], \quad e \mapsto \chi(N(1-e)),$$

is an isomorphism of complete lattices.

Proof. By [3, 2.4(1)], S is a left self-injective regular ring; thus $B(S)$ is a complete lattice. Let $e \in B(S)$. Since $N \in \mathcal{E}_\mathcal{T}[M]$, $N(1-e) \in \mathcal{E}_\mathcal{T}[M]$; thus $\chi(N(1-e)) \in [\chi(N), \sigma[M]]$.

It is clear that the correspondence is order preserving. It remains to prove that μ is onto. Notice that all torsion theories in $[\chi(N), \sigma[M]]$ are spectral.

Let $\mathcal{P} \in [\chi(N), \sigma[M]]$. By [10, 9.5(b)], there exists an M-injective, $\chi(N)$-torsionfree module E such that $\mathcal{P} = \chi(E)$. Then $K := Tr(E, N)$, the trace of E in N, is a fully invariant submodule of N. By 2.2, there exists $e \in B(S)$ such that $\widehat{K} = Ne$. We claim that $\chi(\widehat{K}) = \mathcal{P}$.

Since E is $\chi(N)$-torsionfree, E is cogenerated by N and hence by $\widehat{K}$. Therefore, $\mathcal{P} = \chi(E) \geq \chi(\widehat{K})$. It remains to prove that $\widehat{K}$ is cogenerated by E. Since E is M-injective, it suffices to show that every finitely E-generated submodule of K is cogenerated by E.

Consider any homomorphism $\varphi : E^k \to N$. Then $Ker\,\varphi$ is a $\chi(N)$-closed submodule of E^k and hence is a direct summand by [3, 2.2]. So $Im\,\varphi$ is cogenerated by E.

Now we have $\chi(E) \leq \chi(\widehat{K})$ and therefore $\mathcal{P} = \chi(\widehat{K})$. □

2.4 Corollary. *Let $\mathcal{T} \in M$-tors be spectral and $M \notin \mathcal{T}$.*

(1) For $T = End_R(Q_\mathcal{T}(M))$, $B(T) \simeq [\mathcal{T}, \sigma[M]]$ (lattice isomorphism).

In particular, $[\mathcal{T}, \sigma[M]]$ is a complete Boolean lattice.

(2) If N is any subgenerator of $\sigma[M]$, then $B(T) \simeq B(End_R(Q_{\mathcal{T}}(N)))$.

Proof. (1) This follows from 2.3 since $Q_{\mathcal{T}}(M)$ is a cogenerator of $\mathcal{T}$ (by [3, 2.5]).

The other assertions are obvious. □

Let $\mathcal{S}_M^2$ denote the Goldie torsion theory in $\sigma[M]$, which is always spectral (see [10], [3] for details).

2.5 Corollary. *If $\mathcal{T} \in [\mathcal{S}_M^2, \sigma[M]]$, then $[\mathcal{T}, \sigma[M]]$ is a complete Boolean lattice.*

3 The structure of $End_R(Q_{\mathcal{T}}(N))$

In this section we present the main results of this paper. For convenience, we recall the following results from [7, Chapter 9].

3.1 Proposition. *Let R be a left selfinjective regular ring. Then:*

(1) R is indecomposable (as a ring) if and only if R is a prime ring.

(2) R is isomorphic to a direct product of prime rings if and only if $B(R)$ is atomic.

(3) R is isomorphic to a full linear ring if and only if R is prime and $Soc(_RR) \neq 0$.

(4) R is isomorphic to a direct product of left full linear rings if and only if $Soc(_RR)$ is essential in R.

3.2 Theorem. *Let $\mathcal{T} \in M$-tors be spectral and $N \in \mathcal{E}_{\mathcal{T}}[M]$. Then the following conditions are equivalent:*

(a) $\chi(N)$ is a coatom in M-tors;

(b) $End_R(N)$ is a prime ring;

(c) the only fully invariant $\mathcal{T}$-closed submodules of N are 0 and N.

Proof. $(a) \Rightarrow (b)$ We have $\chi(N)$ is spectral. By 2.3, $B(End_R(N)) \simeq [\chi(N), \sigma[M]]$. Since $B(End_R(N))$ has only two elements, $End_R(N)$ is an indecomposable ring. By 3.1(1), $End_R(N)$ is a prime ring.

$(b) \Rightarrow (c)$ Let $0 \neq L \subset N$ be a fully invariant $\mathcal{T}$-closed submodule. Since $\mathcal{T}$ is spectral, L is a direct summand of N. By 2.2, there exists $0 \neq e \in B(End_R(N))$ such that $Ne = L$. If $e \neq 1$, then $End_R(N)$ is not a prime ring. Therefore $N = L$.

$(c) \Rightarrow (a)$ Let $\mathcal{R} \in [\chi(N), \sigma[M]]$ with $\mathcal{R} \neq \chi(N)$ in M-tors. We will prove that N is $\mathcal{R}$-torsion. Since $\chi(N) < \mathcal{R}$, N is not $\mathcal{R}$-torsionfree; thus $\mathcal{R}(N) \neq 0$. Now $N/\mathcal{R}(N)$ is $\mathcal{R}$-torsionfree and hence $\mathcal{R}(N)$ is $\mathcal{T}$-closed. However, $\mathcal{R}(N)$ is also a fully invariant submodule of N and so, by (c), $\mathcal{R}(N) = N$.

Let $\mathcal{R}$ be such that $\chi(N) < \mathcal{R} < \sigma[M]$. Since $[(\chi(N), \sigma[M]]$ is a complete Boolean lattice, $\mathcal{R}$ has a complement, $\mathcal{R}^c$ say, with $\chi(N) < \mathcal{R}^c < \sigma[M]$. Therefore $N \in \mathcal{R} \cap \mathcal{R}^c$, which is a contradiction. Hence $\chi(N)$ is a coatom. $\square$

3.3 Theorem. *Let $\mathcal{T} \in M$-tors be spectral and $N \in \sigma[M]$ such that $\mathcal{T}(N) = 0$. Then the following conditions are equivalent:*

(a) $\chi(Q_{\mathcal{T}}(N))$ is a coatom and is prime in M-tors;

(b) $T := End_R(Q_{\mathcal{T}}(N))$ is a full linear ring;

(c) N contains a uniform submodule and the only fully invariant $\mathcal{T}$-closed submodules in $Q_{\mathcal{T}}(N)$ are 0 and $Q_{\mathcal{T}}(N)$.

Proof. $(a) \Rightarrow (b)$ By 3.2, T is a prime ring. By 3.1(3), it suffices to prove that T has non-zero socle. Since $\chi(Q_{\mathcal{T}}(N))$ is prime, there exists a $\chi(Q_{\mathcal{T}}(N))$-cocritical module $C \in \sigma[M]$ such that $\chi(Q_{\mathcal{T}}(N)) = \chi(C)$. Let $0 \neq f : C \to Q_{\mathcal{T}}(N)$. Since C is $\chi(Q_{\mathcal{T}}(N))$-cocritical and $Q_{\mathcal{T}}(N)$ is $\chi(Q_{\mathcal{T}}(N))$-torsionfree, f is a monomorphism and we may consider $\widehat{C}$ as a direct summand of $Q_{\mathcal{T}}(N)$. There exists an idempotent

$0 \neq e \in T$ such that $Q_{\mathcal{T}}(N)e = \widehat{C}$. Since $\widehat{C}$ is an indecomposable M-injective module and T is regular, $0 \neq Te$ is a minimal left ideal of T, and we have (b).

$(b) \Rightarrow (c)$ T is a prime ring and hence there are no proper fully invariant $\mathcal{T}$-closed submodules in $Q_{\mathcal{T}}(N)$ (by 3.2). Let S be a minimal left ideal of T. Since T is regular there exists an idempotent $0 \neq e \in T$ such that $S = Te$. It is clear that $Q_{\mathcal{T}}(N)e$ is an indecomposable M-injective submodule of $Q_{\mathcal{T}}(N)$. Thus $Q_{\mathcal{T}}(N)e$ is uniform. Therefore $N \cap Q_{\mathcal{T}}(N)e$ is uniform.

$(c) \Rightarrow (a)$ By 3.2, $\chi(Q_{\mathcal{T}}(N))$ is a coatom. Let $C \subset N$ be uniform. Then $\widehat{C}$ is a uniform submodule of $Q_{\mathcal{T}}(N)$. Since $\mathcal{T}$ is spectral, $\widehat{C}$ is $\mathcal{T}$-cocritical.

Since $\chi(Q_{\mathcal{T}}(N)) \leq \chi(\widehat{C}) < \sigma[M]$ and $\chi(Q_{\mathcal{T}}(N))$ is a coatom, we have that $\chi(Q_{\mathcal{T}}(N)) = \chi(\widehat{C})$ and $\chi(Q_{\mathcal{T}}(N))$ is prime. □

3.4 Theorem. *Let $\mathcal{T} \in M$-tors be spectral and $M \notin \mathcal{T}$. Then the following conditions are equivalent:*

(a) $End_R(Q_{\mathcal{T}}(M))$ is a direct product of prime rings;

(b) $[\mathcal{T}, \sigma[M]]$ is an atomic lattice;

(c) $End_R(Q_{\mathcal{T}}(N))$ is a direct product of prime rings, for all $N \in \sigma[M]$, $N \notin \mathcal{T}$;

(d) $[\chi(Q_{\mathcal{T}}(N)), \sigma[M]]$ is an atomic lattice, for all $N \in \sigma[M]$, $N \notin \mathcal{T}$.

Proof. $(a) \Rightarrow (b)$ follows by 3.1(2) and 2.3.

$(b) \Rightarrow (d)$ We always have $\chi(Q_{\mathcal{T}}(M)) \leq \chi(Q_{\mathcal{T}}(N))$. By general lattice theory, for the atomic lattice $[\mathcal{T}, \sigma[M]]$ and any $\mathcal{R} \in [\mathcal{T}, \sigma[M]]$, the interval $[\mathcal{R}, \sigma[M]]$ is also an atomic lattice.

$(d) \Rightarrow (c)$ follows by 3.1(2) and 2.3.

$(c) \Rightarrow (a)$ is clear. □

For the next result, we need the following technical lemma.

3.5 Lemma. *Let $\mathcal{T} \in M$-tors be spectral and let $N \in \mathcal{E}_{\mathcal{T}}[M]$. Let $\{T_\lambda\}_\Lambda$ be a family of rings such that $\prod_\Lambda T_\lambda \overset{\varphi}{\simeq} End_R(N)$. Let $i_\lambda : T_\lambda \to \prod_\Lambda T_\lambda$ denote the inclusions, $f_\lambda = (e_\lambda)\varphi$, where $e_\lambda = (1)i_\lambda$, and $N_\lambda = (N)f_\lambda$. Then:*

(1) $T_\lambda \simeq End_R(N_\lambda)$ for all $\lambda \in \Lambda$.

(2) $\{N_\lambda\}_\Lambda$ is an independent family of $\mathcal{T}$-closed fully invariant submodules of N and $Hom_R(N_\lambda, N_\beta) = 0$ for all $\lambda \neq \beta$.

(3) $N = I_M(\bigoplus_\Lambda N_\lambda)$.

Proof. Note first that $\{f\}_{\lambda\in\Lambda}$ is a set of central idempotents of $End_R(N)$.

(1) Let $\varphi_\lambda : T_\lambda \to End_R(N_\lambda)$ be defined by $(x)(t)\varphi_\lambda = (x)((t)i_\lambda\varphi$. It is easy to see that φ_λ is a ring isomorphism.

(2) is clear.

(3) As every N_λ is a fully invariant submodule of N, so is $\bigoplus_\Lambda N_\lambda$. By 2.2 there exists $e \in B(End_R(N))$ such that $N = I_M(\bigoplus_\Lambda N_\lambda) \bigoplus Ne$. Since φ is an isomorphism, $e = ((t_\lambda))\varphi$, where $(t_\lambda) \in B(\prod_\Lambda T_\lambda)$. For a fixed $\beta \in \Lambda$, we define $(t'_\lambda) \in B(\prod_\Lambda T_\lambda)$ as follows:

$$t'_\lambda = \begin{cases} t_\lambda & \text{if } \lambda \neq \beta \\ 0 & \text{if } \lambda = \beta \end{cases} .$$

Let $f = ((t'_\lambda))\varphi$. We note that $f = e-((t_\beta)i_\beta)\varphi$. Then, $fe = (e-((t_\beta)i_\beta)\varphi)e = e$ since $(((t_\beta)i_\beta)\varphi)e \subset I_M(\bigoplus_\Lambda N_\lambda) \cap Ne = 0$. On the other hand, $fe = ((t'_\lambda))\varphi \cdot ((t_\lambda))\varphi = f$ and so, we have $t_\beta = 0$; therefore $e = 0$, which proves (3). □

3.6 Theorem. *Let $\mathcal{T} \in M$-tors be spectral and $M \notin \mathcal{T}$. Then the following conditions are equivalent:*

(a) $End_R(Q_{\mathcal{T}}(M))$ is a direct product of full linear rings;

(b) $\mathcal{T}$ *is a strongly semiprime torsion theory;*

(c) $\chi(Q_{\mathcal{T}}(N))$ *is a strongly semiprime torsion theory, for all* $N \in \sigma[M]$, $N \notin \mathcal{T}$;

(d) $End_R(Q_{\mathcal{T}}(N))$ *is a direct product of full linear rings, for all* $N \in \sigma[M]$, $N \notin \mathcal{T}$;

(e) for all $\mathcal{T}$*-torsionfree* $N \in \sigma[M]$, $\sum\{K \subset N \mid K\, is\, uniform\} \trianglelefteq N$;

(f) for all $\mathcal{T}$*-torsionfree* $N \in \sigma[M]$, *every* $0 \neq K \subset N$ *contains a non-zero uniform submodule.*

Proof. $(a) \Rightarrow (b)$. By 2.1(5), it is enough to prove that $\mathcal{T} = \mathcal{T}''$. Assume $\mathcal{T} < \mathcal{T}''$. Let $T = End_R(Q_{\mathcal{T}}(M)) = \prod_\Lambda T_\lambda$ with T_λ a full linear ring for each $\lambda \in \Lambda$.

By 3.5, there exists a family $\{M_\lambda\}_\Lambda$ of fully invariant submodules of $Q_{\mathcal{T}}(M)$, with $Q_{\mathcal{T}}(M) = I_M(\bigoplus_\Lambda M_\lambda)$ and $T_\lambda = End_R(M_\lambda)$. For each $\lambda \in \Lambda$, there exists a central idempotent $e_\lambda \in T$ such that $Q_{\mathcal{T}}(M)e_\lambda = \widehat{M_\lambda}$. By 3.1(4), $Te_\lambda \cap Soc(T) \neq 0$. Hence there exist idempotents $f_\lambda \in T$ such that the Tf_λ are minimal left ideals of T and $Tf_\lambda \subset Te_\lambda$. We have that for each $\lambda \in \Lambda$, $Q_{\mathcal{T}}(M)f_\lambda \subset Q_{\mathcal{T}}(M_\lambda)$ and $Q_{\mathcal{T}}(M)f_\lambda$ is uniform. By [3, 2.5],

$$\mathcal{T} = \chi(Q_{\mathcal{T}}(M)) = \chi(I_M(\bigoplus\nolimits_\Lambda M_\lambda)) \leq \bigwedge\nolimits_\Lambda \chi(M_\lambda).$$

Notice that for all λ, M_λ is $\bigwedge_\Lambda \chi(M_\lambda)$-torsionfree and so too is $Q_{\mathcal{T}}(M) = I_M(\bigoplus_\Lambda M_\lambda)$. Hence $\mathcal{T} = \bigwedge_\Lambda \chi(M_\lambda)$. Since each T_λ is a prime ring, $\chi(M_\lambda)$ is a coatom by 3.2. Since $\chi(M_\lambda) \leq \chi(Q_{\mathcal{T}}(M)f_\lambda)$, we get $\chi(Q_{\mathcal{T}}(M)f_\lambda) = \chi(M_\lambda)$. Since $Q_{\mathcal{T}}(M)f_\lambda$ is uniform and $\mathcal{T}$-torsionfree, $Q_{\mathcal{T}}(M)f_\lambda$ is cocritical. This proves (b).

$(b) \Rightarrow (c)$ follows by 2.1(3), since $\mathcal{T} \leq \chi(Q_{\mathcal{T}}(N))$.

$(c) \Rightarrow (d)$ $[\chi(Q_{\mathcal{T}}(N)), \sigma[M]]$ is an atomic lattice. By 3.4, $T = End_R(Q_{\mathcal{T}}(N)) = \prod_\Lambda T_\lambda$, where each T_λ is a prime ring.

We prove that every T_λ is a full linear ring. For each λ, there exists an idempotent $e_\lambda \in T$ such that $Te_\lambda = T_\lambda$. By 3.5, $End_R((Q_{\mathcal{T}}(N))e_\lambda) = T_\lambda$.

Thus $Q_{\mathcal{T}}(N)e_\lambda$ is $\mathcal{T}$-torsionfree, M-injective, and $End_R((Q_{\mathcal{T}}(N))e_\lambda)$ is prime. By 3.2, $\chi((Q_{\mathcal{T}}(N))e_\lambda)$ is a coatom. Since $\chi(Q_{\mathcal{T}}(N))$ is a strongly semiprime torsion theory, $\chi((Q_{\mathcal{T}}(N))e_\lambda)$ is a prime torsion theory. By 3.3, $End_R(Q_{\mathcal{T}}(N)e_\lambda) \simeq End_R(M_\lambda) \simeq T_\lambda$ is a full linear ring.

$(d) \Rightarrow (a)$ is clear.

$(d) \Rightarrow (e)$ Let $T = End_R(Q_{\mathcal{T}}(N)) = \prod_\Lambda T_\lambda$, where each T_λ is a full linear ring. Then $Soc(T) \trianglelefteq T$. Assume that there exists $K_0 \subset N$ such that $\sum\{K \subset N \mid K$ is uniform$\} \cap K_0 = 0$. Since $\widehat{K}_0 \subset Q_{\mathcal{T}}(N)$, there exists $f \in T$ such that $Q_{\mathcal{T}}(N)f = \widehat{K}_0$. Since $Tf \cap Soc(T) \neq 0$, there exists $0 \neq e \in T$ such that Te is simple and $Te \subset Tf$. Since $Q_{\mathcal{T}}(N)e$ is uniform and contained in $\widehat{K}_0$, we get $K_0 \cap Q_{\mathcal{T}}(N)e \neq 0$, which is a contradiction.

$(e) \Rightarrow (f)$ Suppose $A = \sum\{K \subset N \mid K$ is uniform $\} \trianglelefteq N$. Thus,

$$B = \sum\{\widehat{K} \subset N \mid K \text{ is uniform } \} \trianglelefteq Q_{\mathcal{T}}(N).$$

Let $0 \neq K_0 \subset N$. We have that $\widehat{K}_0 \cap B \neq 0$. Let $0 \neq y \in \widehat{K}_0 \cap B$. There exist uniform modules $K_1, \ldots, K_n$ with $y \in \sum_{i=1}^{n} \widehat{K}_i$. Let $\varphi : \bigoplus_{i=i}^{n} \widehat{K}_i \rightarrow \sum_{i=1}^{n} \widehat{K}_i$ denote the natural epimorphism. Since $\mathcal{T}$ is spectral, φ splits. Let f be the monomorphism obtained from the splitting. If π_i is the natural projection, then there exists some $i \leq n$, such that $(Ry)f\pi_i \neq 0$. For this i, we have a non-zero morphism $g : Ry \rightarrow \widehat{K}_i$. Since $\widehat{K}_i$ is M-injective, there exists an extension of g, $\overline{g} : \widehat{K}_0 \rightarrow \widehat{K}_i$. Again, since $\widehat{K}_0$ is M-injective and $\widehat{K}_i$ is $\mathcal{T}$-torsionfree, $Im\, \overline{g}$ is M-injective. This implies that $\overline{g}$ is onto. Therefore $\widehat{K}_i \simeq K' \subset \widehat{K}_0$. Hence $K_0 \cap K'$ is the uniform module that we are looking for.

$(f) \Rightarrow (d)$ By 3.1(4), it suffices to show that if $T = End(Q_{\mathcal{T}}(N))$, then $Soc(T) \trianglelefteq T$. Assume there exists $0 \neq K \subset T$ such that $Soc(T) \cap K = 0$. Let $0 \neq f \in K$. By (f), there exists a uniform module $U \subset Q_{\mathcal{T}}(N)f$. Thus there exists an idempotent

$0 \neq e \in T$ with $\hat{U} = Q_{\mathcal{T}}(N)e \subset Q_{\mathcal{T}}(N)f$. Thus $Te \subset K$. Since $\hat{U}$ is uniform, Te is simple, which is a contradiction. □

Remarks. (1) Consider any algebra A as module over the multiplication algebra $M(A)$. If A is semiprime, the singular torsion theory in $\sigma[_{M(A)}A]$ is spectral (see [10, 32.1]). In this setting 3.6 characterizes the case when the central closure of A is a direct product of fields (see Exercise 32.12(11) in [10]). This is closely related to Theorems 2 and 4 of [4].

(2) By [10], $\mathcal{S}_M^2$ is always spectral. If M is polyform it is $\mathcal{S}_M^2$-torsionfree and $Q_{\mathcal{S}_M^2}(M) = \widehat{M}$ ([10, 9.13]). Therefore our results in this section apply in particular to polyform modules M and $End_R(\widehat{M})$. Specializing to $M = R$, we obtain nice properties of the left non-singular rings R and $Q_{max}(R)$. Notice that 3.2, 3.3, 3.4, and 3.6 give generalizations of analogous results in [1] and [2] in the case $M = R$.

An R-module M is called *strongly prime* if for every non-zero $K \subset M$, $M \in \sigma[K]$. By [10, 12.3], M is strongly prime if and only if $\widehat{M}$ has no non-trivial fully invariant submodules. If M is strongly prime and projective in $\sigma[M]$, then M is non-M-singular. By [10, 13.4], for such modules, $End_R(\widehat{M})$ is the maximal left ring of quotients of $End_R(M)$. By the previous results we obtain the following:

3.7 Proposition. *Let M be strongly prime and projective in $\sigma[M]$ and $T = End_R(\widehat{M})$. Then:*

(1) T is a prime ring.

(2) T is a full linear ring if and only if M contains a uniform submodule.

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