Strongly and properly semiprime modules and rings

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ABSTRACT

Using torsion theoretic techniques we define ***strongly*** and ***properly semiprime*** modules. Strongly semiprime modules M are characterised by the fact that in the M-injective hull $\widehat{M}$, every fully invariant submodule is a direct summand. They extend Handelman's left strongly semiprime rings.
For properly semiprime modules M, fully invariant submodules of $\widehat{M}$, which are finitely generated by elements of M, are direct summands.
Any (non-associative) ring A is called *SSP* or *PSP ring*, if A is strongly (resp. properly) semiprime as module over its multiplication ring. A is *SSP* if and only if its central closure is a direct sum of simple ideals. The structure of the central closure of *PSP* rings is close to biregular rings.

1 Introduction

Throughout the paper R will be an associative ring with unit and *R-Mod* the category of unital left R-modules. For unexplained notions we refer to [15].

For any left R-module M, the full subcategory of *R-Mod*, whose objects are submodules of M-generated modules, is denoted by $\sigma[M]$.

The M-injective hull of $N \in \sigma[M]$ is written as $\widehat{N}$ or $I_M(N)$.

By $K \subset M$ we usually mean that K is (isomorphic to) a submodule of M, and $K \trianglelefteq M$ indicates that K is an essential submodule of M.

For $X \subset M$ and $b \in M$, we put $(X : b)_R = \{r \in R \mid rb \in X\}$.

$T_K(M)$ is defined in section 2. *Polyform* modules are described in section 3.

Consider the following properties of an R-module M:

(i) $R/An_R(M)$ is cogenerated by every essential submodule of M;

(ii) for every $N \trianglelefteq M$, $M \in \sigma[N]$;

(iii) for every $N \trianglelefteq M$, $M \subset N^{(\Lambda)}$, for some set Λ;

(iv) for every $N \trianglelefteq M$, $R/An_R(M) \subset N^r$, for some $r \in \mathbb{N}$;

(v) M is cogenerated by every essential submodule of M;

(vi) M is polyform;

(vii) for every submodule $K \subset M$, $M/T_K(M) \in \sigma[K]$;

(viii) for every cyclic submodule $K \subset M$, $M/T_K(M) \in \sigma[K]$;

The conditions (i)-(vi) stated for *every* submodule were considered in [13] transferring primeness conditions from rings to modules. In particular, modules satisfying (ii) for *every* submodule $N \subset M$, were called *strongly prime* modules. From the arguments given there it follows that all these conditions are in fact distinct.

Here we have for any module M the implications $(iii) \Rightarrow (v) \Rightarrow (i)$, $(iii) \Rightarrow (ii) \Rightarrow (i)$, $(iv) \Rightarrow (i)$ and $(vii) \Rightarrow (ii)$.

It is easy to prove that for M projective in $\sigma[M]$, (ii) and (iii) are equivalent. With a little more effort we will see that in this case also (vii) is equivalent to M satisfying both (ii) and (vi) (cf. 7.9, 7.10).

For $M = {}_RR$, the conditions (ii), (iii), (iv) and (vii) are equivalent and characterize the *left strongly semiprime rings* introduced by Handelman [6] and further studied by Kutami-Oshiro [9] (cf. 8.2).

For R commutative and $M = R$, any of the properties (i), (v), (vi) and $(viii)$ determine precisely R to be a semiprime ring (hence the complete quotient ring of R is regular). Properties (ii), (iii) (iv) and (vii) imply that the complete quotient ring of R is a finite product of fields.

Basic definitions and technical preparations are provided in Section 2.

General observations on polyform modules are given in Section 3. Hereby their bimodule properties turn out to be of special interest. Using the central idempotents of the self-injective hull, the *idempotent closure* of a polyform module is introduced.

We take (*vii*) to define ***strongly semiprime modules*** in Section 4. They are characterized by the fact that their self-injective hull is semisimple as a bimodule.

Modules satisfying (*viii*) we call ***properly semiprime***. They are treated in Section 5.

Extending the notion of left fully idempotent rings ***pseudo regular modules*** are defined in Section 6.

The structure of modules which are both polyform and SSP (resp. PSP) is investigated in Section 7. For modules M which are projective in $\sigma[M]$ the PSP property in fact implies polyformity. Applying our results to rings we obtain further characterizations of left SSP rings defined in [6] (Section 8).

Section 9 is devoted to the bimodule structure of rings. We consider any ring A as a module over its multiplication ring $M(A)$. As shown in [13], A is a strongly prime $M(A)$-module if and only if the central closure $\widehat{A}$ is a simple ring. Here we observe that A is strongly semiprime as an $M(A)$-module if and only if $\widehat{A}$ is a direct sum of simple ideals. We are also concerned with A being properly semiprime as an-$M(A)$-module. In particular we consider the case when the central closure $\widehat{A}$ is an Azumaya ring.

2 Trace and torsion submodules

We begin with some technical preparations. M always denotes an R-module. For our purpose we need an extended version of the trace $Tr(K, L)$ of a module K in L, namely the trace of the category $\sigma[K]$ in L.

2.1 Trace submodules.

For $K, L \in \sigma[M]$, let $\mathcal{T}^K(L)$ denote the *trace of* $\sigma[K]$ *in* L, i.e.

$$\mathcal{T}^K(L) = \sum\{U \subset L \mid U \in \sigma[K]\}.$$

For cyclic submodules $Ra \subset L$, put $\mathcal{T}^a(L) = \mathcal{T}^{Ra}(L)$.

Notice the following properties:

(1) $\mathcal{T}^K(L)$ *is a fully invariant submodule of* L, $\mathcal{T}^K(L)End_R(L) \subset \mathcal{T}^K(L)$.

(2) $\mathcal{T}^K(\widehat{L}) = Tr(K, \widehat{L}) = K\,Hom_R(K, \widehat{L})$ *is a* K*-injective module.*

(3) $\mathcal{T}^K(L) = L \cap \mathcal{T}^K(\widehat{L})$.

(4) If $K = K_1 + K_2$, *then* $\mathcal{T}^K(\widehat{L}) = \mathcal{T}^{K_1}(\widehat{L}) + \mathcal{T}^{K_2}(\widehat{L})$.

(5) If $K \subset L$, $\mathcal{T}^K(\widehat{L}) = K\,End_R(\widehat{L})$ *and* $\mathcal{T}^K(L) = L \cap K\,End_R(\widehat{L})$.

For any injective object $Q \in \sigma[M]$, the modules $X \in \sigma[M]$ with $Hom_R(X, Q) = 0$, form a ***torsion class*** in $\sigma[M]$ (cf. [12]). A module is called ***torsionfree*** with respect to this torsion theory, if its torsion submodule is zero.

With the notation above, the M-injective hull of $\mathcal{T}^K(M)$ may be regarded as submodule of $\widehat{M}$. Denoting $T = End_R(\widehat{M})$, we have $I_M(\mathcal{T}^K(M)) = I_M(\mathcal{T}^K(\widehat{M})) = I_M(KT)$.

We fix the following notation. For the definition of *reject* $Re(-,-)$ see [15], 14.4.

2.2 Torsion submodules.

For $K, L \in \sigma[M]$, let $\mathcal{T}_{K,M}(L)$ denote the *torsion submodule of* L, with respect to the torsion theory (in $\sigma[M]$) determined by $I_M(\mathcal{T}^K(M))$, i.e.

$$\begin{aligned}\mathcal{T}_{K,M}(L) &= \sum\{U \subset L \mid Hom_R(U, I_M(\mathcal{T}^K(M)) = 0\} \\ &= \bigcap\{Ke\,f \mid f \in Hom_R(L, I_M(\mathcal{T}^K(M))\} = Re\,(L, I_M(\mathcal{T}^K(M)).\end{aligned}$$

The index 'M' refers to the category $\sigma[M]$ we are working in. In case this category is fixed, we simply write $\mathcal{T}_K(L)$ instead of $\mathcal{T}_{K,M}(L)$.

In particular, for cyclic modules $K = Ra$, we put $\mathcal{T}_a(L) = \mathcal{T}_{Ra}(L)$.

$\mathcal{T}_K(-)$ is a left exact radical in $\sigma[M]$ and so $\mathcal{T}_K(L)$ is fully invariant in L (see [12]).

Let us list some properties in the form we will mostly use them:

2.3 Properties.

Let M *be an* R*-module,* $K \subset M$ *a submodule and* $T = End_R(\widehat{M})$.

(1) $\mathcal{T}^K(\widehat{M}) = KT$ *is a* K*-injective module.*

(2) $\mathcal{T}^K(M) = \mathcal{T}^K(\widehat{M}) \cap M$.

(3) $\mathcal{T}^K(M) \cap \mathcal{T}_K(M) = 0$.

(4) For submodules $L, N \subset M$ *with* $K = L + N$,
$\mathcal{T}^K(\widehat{M}) = \mathcal{T}^L(\widehat{M}) + \mathcal{T}^N(\widehat{M})$ *and* $\mathcal{T}_K(M) = \mathcal{T}_L(M) \cap \mathcal{T}_N(M)$.

(5) If K *is generated by* $a_1, \ldots, a_n \in K$, $\mathcal{T}_K(M) = \bigcap_{i=1}^n \mathcal{T}_{a_i}(M)$.

Proof. (1),(2) are special cases of the preceding observations.

(3) For $X \subset \mathcal{T}^K(M) \cap \mathcal{T}_K(M)$, $X \subset KT$ and $Hom_R(X, I_M(KT)) = 0$. This implies $X = 0$.

(4) For $K = L + N$, $KT = LT + NT$. Hence without restriction assume $K = KT$, $L = LT$ and $N = NT$. Clearly $\mathcal{T}_K(M) \subset \mathcal{T}_L(M) \cap \mathcal{T}_N(M)$. Consider an additive complement $N_o \subset N$ of L in K. Then $L \oplus N_o$ is an essential submodule of K and

$$\widehat{K} = \widehat{L} \oplus \widehat{N}_o \subset \widehat{L} \oplus \widehat{N}.$$

From this we conclude $\mathcal{T}_K(M) \supset \mathcal{T}_L(M) \cap \mathcal{T}_N(M)$.

(5) This is derived from (4) by induction. □

3 Polyform modules

A module $N \in \sigma[M]$ is called *singular in* $\sigma[M]$ or *M-singular* if $N \simeq L/K$ for some $L \in \sigma[M]$ and $K \trianglelefteq L$ (see [12]). For $M = R$, instead of *R-singular* we just say *singular*.

A submodule $U \subset M$ is called *rational in* M if $Hom(M/U, \widehat{M}) = 0$. Every rational submodule is in particular essential in M.

M is called *polyform* if every essential submodule is rational. The following characterizations of these modules are easy to verify (e.g. [16], [12]).

3.1 Polyform modules. Characterizations.
For a module M with M-injective hull $\widehat{M}$ the following are equivalent:

(a) M is polyform;

(b) for any submodules $K \trianglelefteq L \subset M$, $Hom_R(L/K, \widehat{M}) = 0$;

(c) for every $N \subset M$, the canonical map $Hom_R(\widehat{N}, \widehat{M}) \to Hom_R(N, \widehat{M})$ is an isomorphism;

(d) $\widehat{M}$ is polyform;

(e) $End_R(\widehat{M})$ is regular.

For such modules M, $End_R(M)$ is a subring of $End_R(\widehat{M})$.
Every monic $f \in End_R(M)$ with $Im\, f \trianglelefteq M$ is invertible in $End_R(\widehat{M})$.

If the M-singular submodule of M is zero, then M is polyform. A ring R is left polyform if and only if it is left non-singular.

3.2 Properties of polyform modules.
Let M be a polyform R-module, $\widehat{M}$ its M-injective hull $\widehat{M}$ and $T = End_R(\widehat{M})$. Then for any submodule $K \subset M$:

(1) $\mathcal{T}_K(\widehat{M})$ is M-injective.

(2) $\mathcal{T}_{\mathcal{T}_K(\widehat{M})}(\widehat{M}) = I_M(KT)$.

(3) $\widehat{M} = \mathcal{T}_K(\widehat{M}) \oplus I_M(KT)$.

(4) $\mathcal{T}_K(M) + \mathcal{T}^K(M) \trianglelefteq M$.

(5) for any $m \in \widehat{M}$, $An_T(m)$ is generated by an idempotent and $\widehat{M}$ is a non-singular right T-module.

Proof. Put $L = \mathcal{T}_K(\widehat{M})$.

(1) Since $\widehat{M}$ is polyform, $Hom_R(I_M(L), I_M(KT)) = 0$ and hence $I_M(L) \subset L$, i.e. $I_M(L) = L$ is M-injective.

(2) Consider $g \in Hom_R(K, L)$. Since $Ke\,g$ is not essential in K, there exists $0 \neq X \subset K$ with $X \cap Ke\,g = 0$ and $X \simeq (X)g \subset L$, a contradiction. Hence $K \subset \mathcal{T}_L(\widehat{M})$ and $KT \subset \mathcal{T}_L(\widehat{M})$ since $\mathcal{T}_L(\widehat{M})$ is fully invariant.

Now we show that $KT \trianglelefteq \mathcal{T}_L(\widehat{M})$. Assume there is a non-zero submodule $U \subset \mathcal{T}_L(\widehat{M})$ with $U \cap KT = 0$. If $Hom_R(U, I_M(KT)) = 0$, then $U \subset L$ and $U \subset \mathcal{T}^L(\widehat{M}) \cap \mathcal{T}_L(\widehat{M}) = 0$, a contradiction.

Hence there is a non-zero $g : U \to I_M(KT)$. Since $Ke\,g$ is not essential in U, there exists a non-zero submodule $V \subset U$ with $V \cap Ke\,g = 0$. Because of $KT \trianglelefteq I_M(KT)$, we may assume $V \simeq (V)g \subset KT$. Now for some $t \in T$, we have $V = (V)gt \subset U \cap KT = 0$, a contradiction.

(3) By (1), L and $\mathcal{T}_L(\widehat{M})$ are M-injective. Since $L \cap \mathcal{T}_L(\widehat{M}) = 0$ by definiton, $\widehat{M} = L \oplus \mathcal{T}_L(\widehat{M}) \oplus W$ for some $W \subset \widehat{M}$.

Assume there is a non-zero $h \in Hom_R(W, L)$ and $Q \cap Ke\,h = 0$ for some non-zero $Q \subset W$. Then $Q \simeq (Q)h \subset L$ and there is some $t \in T$ with $Q = (Q)ht \subset W \cap L = 0$. This implies $Hom_R(W, L) = 0$ and $W \subset \mathcal{T}_L(\widehat{M})$.

Hence $W = 0$ and by (2), $\widehat{M} = L \oplus \mathcal{T}_L(\widehat{M}) = L \oplus I_M(KT)$.

(4) is an immediate consequence of (3).

(5) For $m \in \widehat{M}$ consider $t \in T$ with $(m)t = 0$. Then $(Rm)t = 0$ and - $\widehat{M}$ being polyform - $I_M(Rm)t = 0$. We have $\widehat{M} = I_M(Rm) \oplus U$ for some R-submodule $U \subset \widehat{M}$. For the related projection (idempotent) $g : \widehat{M} \to I_M(Rm)$, $\widehat{M}gt = 0$. This means $gt = 0$ and $t = (1-g)t \in (1-g)T$. Therefore $An_T(m) = (1-g)T$ implying that $\widehat{M}$ is a non-singular T-module. □

As already shown above, the condition on an R-module to be polyform has a strong influence on the structure of its fully invariant submodules. We collect information about this in our next lemma:

3.3 Bimodule properties of polyform modules.
Let M be a polyform R-module, $\widehat{M}$ its M-injective hull and $T = End_R(\widehat{M})$. Denote by C the center of T (i.e., the endomorphism ring of $\widehat{M}$ as an (R,T)-bimodule). Then:

(1) Every essential (R,T)-submodule of $\widehat{M}$ is essential as an R-submodule.

(2) $\widehat{M}$ is self-injective and polyform as an (R,T)-bimodule.
C is a regular self-injective ring.

(3) For every submodule (subset) $K \subset \widehat{M}$, there exists an idempotent $\varepsilon(K) \in C$, such that $An_C(K) = (1 - \varepsilon(K))C$.

(4) If $K \trianglelefteq L \subset \widehat{M}$, then $\varepsilon(K) = \varepsilon(L)$.

(5) Every finitely generated C-submodule of $\widehat{M}$ is C-injective.

(6) If $\widehat{M}$ is a finitely generated (R,T)-module, $\widehat{M}$ is a generator in C-Mod.

Proof. (1) Let $N \subset \widehat{M}$ be an essential (R,T)-submodule. Then $N \cap T_N(\widehat{M}) = 0$ implies $T_N(\widehat{M}) = 0$ and $\widehat{M} = I_M(NT) = I_M(N)$ by 3.2.

So $N \trianglelefteq \widehat{M}$ as an R-submodule.

(2) Again let $N \subset \widehat{M}$ be an essential (R,T)-submodule and $h : N \to \widehat{M}$ an (R,T)-morphism. Since $\widehat{M}$ is a self-injective R-module, there is an $f \in T$ which extends h from N to $\widehat{M}$.

For any $t \in T$ and $n \in N$, $(nt)f - (n)ft = (nt)h - (n)ht = 0$. Hence $N \subset Ke(tf - ft)$. By (1), N is an essential R-submodule and since $\widehat{M}$ is polyform, $tf - ft = 0$, implying that f is an (R,T)-morphism and $\widehat{M}$ is a self-injective (R,T)-module.

The endomorphism ring of the self-injective (R,T)-module $\widehat{M}$ is the center of the regular ring T and hence is also regular. So $\widehat{M}$ is a polyform (R,T)-module by 3.1. This in turn implies that C is self-injective.

(3) By 3.2, there is a bimodule decomposition $\widehat{M} = T_K(\widehat{M}) \oplus I_M(KT)$. Then the projection $\varepsilon(K) : \widehat{M} \to I_M(KT)$ is an idempotent in C and

$An_C(K) = An_C(I_M(KT)) = (1 - \varepsilon(K))C$.

(4) This property is obvious since $\widehat{M}$ is polyform.

(5) As shown in 3.2, every cyclic C-submodule of $\widehat{M}$ is isomorphic to a direct summand of C and hence is C-injective. Since $\widehat{M}$ is a non-singular C-module, any finite sum of C-injective submodules is again C-injective.

(6) Let $\widehat{M}$ be generated as (R,T)-module by $m_1, \dots, m_k$. Then the map $C \to \widehat{M}^k$, $c \to (m_1, \dots, m_k)c$, is a monomorphism. Since C is injective, it is a direct summand of $\widehat{M}^k$ and so $\widehat{M}$ is a generator in C-Mod. □

For later use we state some linear dependence properties of elements in M with respect to $End_R(M)$.

3.4 Independence over the endomorphism ring.

Let M be a self-injective R-module, $T = End_R(M)$ and $m_1, \dots, m_n \in M$.

(1) Assume $m_1T \cap \sum_{i=2}^n m_iT = 0$. Then $An_R(m_2, \dots, m_n)m_1 \trianglelefteq Rm_1$.

(2) If M is polyform and $An_R(m_2, \dots, m_n)m_1 \trianglelefteq Rm_1$, then $m_1T \cap \sum_{i=2}^n m_iT = 0$.

Proof. Put $U = An_R(m_2, \dots, m_n)$.

(1) Assume there exists a non-zero submodule $V \subset Rm_1$ satisfying $V \cap Um_1 = 0$. Consider the canonical projection $\alpha : V \oplus Um_1 \to V$. M being self-injective α extends to an endomorphism t of M. From $Rm_1t \supset (V + Um_1)t = V \neq 0$ we conclude $m_1t \neq 0$.

We also have $U(m_1t) = (Um_1)t = (Um_1)\alpha = 0$. By Corollary 2.2 in [8], this implies $m_1t \in \sum_{i=2}^n m_iT$, a contradiction.

(2) Assume $0 \neq m_1t \in \sum_{i=2}^n m_iT$ for some $t \in T$. Then $Um_1t \subset U\sum_{i=2}^n m_iT = 0$. Since M is polyform and $Um_1 \trianglelefteq Rm_1$ we have $Rm_1t = 0$, a contradiction. □

There is an extension of a module M contained in the M-injective hull $\widehat{M}$ which turns out to be of some interest.

Definition. Let M be an R-module, $T = End_R(\widehat{M})$ and B the Boolean ring of all central idempotents of T. Then we call $\widetilde{M} = MB$ the *idempotent closure of M*.

This notion is closely related to the *π-injective hull* of M defined in Goel-Jain [5], which can be written as MU, with U the subring generated by all idempotents in T. Hence if all idempotents in T are central, $\widetilde{M}$ is just the π-injective hull of M.

3.5 Idempotent closure of polyform modules.

We use the above notation. Let M be an R-module with idempotent closure $\widetilde{M}$. Then for every $a \in \widetilde{M}$, there exist $m_1, \dots, m_k \in M$ and pairwise orthogonal $c_1, \dots, c_k \in B$ such that $a = \sum_{i=1}^k m_ic_i$.

If M is polyform module, there exist pairwise orthogonal $e_1, \dots, e_k \in B$ such that

(1) $a = \sum_{i=1}^k m_ie_i$;

(2) $e_i = \varepsilon(m_i)e_i$ for $i = 1, \dots, k$;

(3) $\varepsilon(a) = \sum_{i=1}^k e_i$.

Proof. Write $a = \sum_{j=1}^r u_jb_j$, with $u_1, \dots, u_r \in M$ and $b_1, \dots, b_r \in B$.

The Boolean subring of B generated by $b_1, \dots, b_r$ is finite and hence isomorphic to $(\mathbb{Z}_2)^k$ for some $k \in \mathbb{N}$. So it contains a subset of pairwise orthogonal idempotents $c_1, \dots, c_k$ such that, for all $j = 1, \dots, r$, $b_j = \sum_{i \in S(j)} c_i$ with $S(j) \subset \{1, \dots k\}$. Therefore

$$a = \sum_{i=1}^k m_ic_i \quad \text{with } m_i = \sum \{u_l \mid i \in S(l)\}.$$

Assume M is polyform. Put $e_i = \varepsilon(m_i)c_i\varepsilon(a)$. Since $a\varepsilon(a) = a$ and $m_i\varepsilon(m_i) = m_i$,

$$\sum_{i=1}^k m_ie_i = \sum_{i=1}^k m_i\varepsilon(m_i)c_i\varepsilon(a) = \sum_{i=1}^k m_ic_i\varepsilon(a) = (\sum_{i=1}^k m_ic_i)\varepsilon(a) = a.$$

(2) follows from the definition of the idempotents e_i above.

(3) Clearly $\varepsilon(a)e_i = e_i$ for all $i = 1,\dots,k$ and hence $\varepsilon(a)\sum_{i=1}^k e_i = \sum_{i=1}^k e_i$.

Since $ae_i = m_ie_i$, $a = \sum_{i=1}^k m_ie_i = a(\sum_{i=1}^k e_i)$. Therefore $id - \sum_{i=1}^k e_i \in An_T(a) = (id - \varepsilon(a))T$ (see 3.3). So $\varepsilon(a)\sum_{i=1}^k e_i = \varepsilon(a)$ and $\varepsilon(a) = \sum_{i=1}^k e_i$. □

4 Strongly semiprime modules

The two following technical lemmas will be crucial for our investigations.

4.1 Lemma.

Let M be an R-module with submodules $K, L \subset M$. Then the following are equivalent:

(a) $M/L \in \sigma[K]$;

(b) for any $b \in M$, there exists a finite subset $X \subset K$, with $An_R(X)b \subset L$.

Proof. $(a) \Rightarrow (b)$ Assume $M/L \in \sigma[K]$ and $b \in M$. Then $Rb + L/L \subset M/L$ is a cyclic module in $\sigma[K]$ and hence a factor module of a cyclic submodule of $K^{(\mathbb{N})}$. So there exist $x_1,\dots,x_k \in K$ and a morphism

$$R(x_1,\dots,x_k) \to M/L,\ (x_1,\dots,x_k) \mapsto (b+L)/L.$$

This implies $An_R(x_1,\dots,x_k)b \subset L$.

$(b) \Rightarrow (a)$ Consider $b \in M$ and chose $x_1,\dots,x_k \in K$ with $An_R(x_1,\dots,x_k)b \subset L$. Then we can define a map as given above and so $(Rb+L)/L \in \sigma[K]$. Hence $M/L \in \sigma[K]$. □

4.2 Lemma.

Let M be a left R-module, $K \subset M$ a submodule and $T = End_R(\widehat{M})$. Then the following conditions are equivalent:

(a) $M/\mathcal{T}_K(M) \in \sigma[K]$;

(b) for any $b \in M$, there exists a finite subset $X \subset K$, with $An_R(X)b \subset \mathcal{T}_K(M)$.

(c) every K-injective, $\mathcal{T}_K$-torsionfree module in $\sigma[M]$ is M-injective and K-generated;

(d) $I_M(\mathcal{T}^K(M)) \in \sigma[K]$ and $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$;

(e) $\widehat{M} = KT \oplus I_M(\mathcal{T}_K(\widehat{M}))$.

Notice that the decomposition of $\widehat{M}$ given in (e) is in R-Mod. Though KT obviously is a fully invariant submodule, in general this need not be true for $I_M(\mathcal{T}_K(\widehat{M}))$.

Proof. $(a) \Leftrightarrow (b)$ follows from 4.1.

$(a) \Rightarrow (c)$ Assume $Q \in \sigma[M]$ is K-injective and $\mathcal{T}_K$-torsionfree. For any submodule $L \subset M$, consider a morphism $f : L \to Q$. Since $\mathcal{T}_K(Q) = 0$, f factorizes through $f' : L/\mathcal{T}_K(L) \to Q$. We have the commutative diagram (with canonical mappings)

$$\begin{array}{ccccc} 0 & \to & L & \to & M \\ & & \downarrow & & \downarrow \\ 0 & \to & L/\mathcal{T}_K(L) & \to & M/\mathcal{T}_K(M) \\ & & \downarrow_{f'} & & \\ & & Q & & \end{array}$$

Since Q is K-injective and $M/\mathcal{T}_K(M) \in \sigma[K]$, there exists some $M/\mathcal{T}_K(M) \to Q$ yielding a commutative diagram. Hence Q is M-injective and

$$Q = Tr(M,Q) = Tr(M/\mathcal{T}_K(M),Q) = Tr(K,Q).$$

$(c) \Rightarrow (a)$ Since $I_M(M/\mathcal{T}_K(M))$ is M-injective and $\mathcal{T}_K$-torsionfree, it is K-generated by (c) and so $M/\mathcal{T}_K(M) \in \sigma[K]$.

$(a) \Rightarrow (d)$ As shown above, $I_M(M/\mathcal{T}_K(M)) \in \sigma[K]$. For a complement U of $\mathcal{T}_K(M)$ in M, $U \oplus \mathcal{T}_K(M) \trianglelefteq M$ and U is isomorphic to a submodule of $M/\mathcal{T}_K(M)$. Hence $U \subset \mathcal{T}^K(M)$ and $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$.

$(d) \Rightarrow (a)$ Since $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$ and $\mathcal{T}^K(M) \cap \mathcal{T}_K(M) = 0$,

$$\mathcal{T}^K(M) \simeq [\mathcal{T}^K(M) + \mathcal{T}_K(M)]/\mathcal{T}_K(M) \trianglelefteq M/\mathcal{T}_K(M).$$

Hence $M/\mathcal{T}_K(M)$ is isomorphic to a submodule of $I_M(\mathcal{T}^K(M)) \in \sigma[K]$.

$(d) \Rightarrow (e)$ The assumptions imply $\widehat{M} = I_M(\mathcal{T}^K(M)) \oplus I_M(\mathcal{T}_K(M))$. As an injective object, $I_M(\mathcal{T}^K(M)) \in \sigma[K]$ is K-generated and hence $I_M(\mathcal{T}^K(M)) = KT$.

Since $\mathcal{T}^K(M) \trianglelefteq \mathcal{T}^K(\widehat{M})$, $I_M(\mathcal{T}^K(M)) = I_M(\mathcal{T}^K(\widehat{M}))$.

$(e) \Rightarrow (d)$ As a direct summand of $\widehat{M}$, KT is M-injective. Hence

$$\mathcal{T}^K(M) \trianglelefteq I_M(\mathcal{T}^K(M)) = I_M(\mathcal{T}^K(\widehat{M})) = KT.$$

Since $\mathcal{T}_K(M) \trianglelefteq \mathcal{T}_K(\widehat{M}) \trianglelefteq I_M(\mathcal{T}_K(\widehat{M}))$, we conclude $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$. □

The above relations are used for our next definition:

4.3 Strongly semiprime modules.
Let M be a left R-module and $T = End_R(\widehat{M})$. We call M strongly semiprime (SSP) *if it satisfies the following equivalent conditions for every submodule $K \subset M$:*

(a) $M/\mathcal{T}_K(M) \in \sigma[K]$;

(b) *for any $b \in M$, there exists a finite subset $X \subset K$, with $An_R(X)b \subset \mathcal{T}_K(M)$;*

(c) *every K-injective $\mathcal{T}_K$-torsionfree module in $\sigma[M]$ is M-injective and K-generated;*

(d) $I_M(\mathcal{T}^K(M)) \in \sigma[K]$ *and* $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$;

(e) $\widehat{M} = KT \oplus I_M(\mathcal{T}_K(\widehat{M}))$.

For the relationship with strongly prime modules see 5.4. We state some important properties for these modules.

4.4 Basic properties of SSP modules.
Let M be an R-module and $S = End_R(M)$.

(1) *Assume M is SSP. Then for every $N \trianglelefteq M$, $M \in \sigma[N]$.*

(2) *Assume M is self-injective. Then M is SSP if and only if M is semisimple as (R,S)-bimodule.*

Proof. (1) For $N \trianglelefteq M$, $\mathcal{T}^N(M) \cap \mathcal{T}_N(M) = 0$ implies $\mathcal{T}_N(M) = 0$ and $NT = \widehat{M}$, in particular $M \in \sigma[N]$.

(2) Let M be self-injective and $U \subset M$ an essential (R,S)-submodule. Then $U = \mathcal{T}^U(M)$, and $\mathcal{T}^U(M) \cap \mathcal{T}_U(M) = 0$ implies $\mathcal{T}_U(M) = 0$. We see from 4.3 that $U = M$. So M has no proper essential (R,S)-submodule. Hence it is a semisimple (R,S)-module.

Now assume M is a semisimple (R,S)-module and $K \subset M$ an R-submodule. Then $M = KS \oplus L$ for some fully invariant $L \subset M$. This implies $Hom_R(L, KS) = 0$ and so $L \subset \mathcal{T}_K(M)$. Hence $M/\mathcal{T}_K(M) \in \sigma[M/L] = \sigma[K]$ showing that M is SSP. □

4.5 More characterizations of SSP.
Let M be an R-module and $T = End_R(\widehat{M})$. Then the following are equivalent:

(a) ${}_RM$ *is SSP;*

(b) *for any essential submodule $N \subset {}_RM$, $M \in \sigma[N]$ and for every submodule $K \subset {}_RM$, $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq_R M$;*

(c) ${}_R\widehat{M}$ *is SSP;*

(d) $\widehat{M}$ *is a semisimple (R,T)-bimodule.*

Proof. (a) ⇒ (b) This follows from 4.3 and 4.4.

(b) ⇒ (a) Let $K \subset M$ be any submodule. Since $\mathcal{T}_K(M) + \mathcal{T}^K(M) \trianglelefteq M$, $M \in \sigma[\mathcal{T}_K(M) + \mathcal{T}^K(M)] = \sigma[M]$ by assumption. So $\mathcal{T}_K(M) + K$ is a subgenerator in $\sigma[M]$ and hence it generates the M-injective module $I_M(\mathcal{T}^K(M))$. However, since $Hom_R(\mathcal{T}_K(M), I_M(\mathcal{T}^K(M))) = 0$ we conclude that $I_M(\mathcal{T}^K(M))$ is K-generated and M is an SSP module.

(a) ⇒ (c) For any submodule $N \subset \widehat{M}$, put $K = NT \cap M$. Then $KT \trianglelefteq NT$ and $\mathcal{T}_K(\widehat{M}) = \mathcal{T}_N(\widehat{M})$. Consider any N-injective and $\mathcal{T}_N$-torsionfree module $Q \in \sigma[M]$. Then Q is K-injective and $\mathcal{T}_K$-torsionfree, and hence M-injective and K-generated since M is SSP (cf. 4.3). As easily seen, Q is also N-generated and so $\widehat{M}$ is SSP by 4.3.

(c) ⇒ (a) Essential submodules of SSP modules are SSP (cf. 4.4).

(c) ⇔ (d) This is shown in 4.4. □

4.6 SSP and semisimple modules.
Let M be an R-module.

(1) *Assume M has essential socle and for every $N \trianglelefteq M$, $M \in \sigma[N]$. Then M is semisimple.*

(2) *M is semisimple if and only if every module in $\sigma[M]$ is SSP.*

Proof. (1) By assumption, $M \in \sigma[Soc(M)]$ and every module in $\sigma[Soc(M)]$ is semisimple.

(2) We see from (1) that every finitely cogenerated module in $\sigma[M]$ is semisimple and hence every simple module in $\sigma[M]$ is M-injective, i.e. M is co-semisimple.

Let N be the sum of all non-isomorphic simple modules in $\sigma[M]$ and consider $L = M \oplus N$. Then $\mathcal{T}_N(L) \subset Rad\,(L) = 0$ (cf. [15], 23.1). Since L is SSP, this implies $L/\mathcal{T}_N(L) \in \sigma[N]$. Hence L and M are semisimple modules. □

5 Properly semiprime modules

Weakening the conditions for strongly semiprime modules we define:

5.1 Properly semiprime modules.
Let M be a left R-module and $T = End_R(\widehat{M})$. We call M properly semiprime (PSP) *if it satisfies the following equivalent conditions:*

(a) *For every element $a \in M$, $M/\mathcal{T}_a(M) \in \sigma[Ra]$;*

(b) for any $a, b \in M$, there exist $r_1, \dots, r_n \in R$ such that

$$An_R(r_1 a, r_2 a, \dots, r_n a)b \subset \mathcal{T}_a(M);$$

(c) for every finitely generated submodule $K \subset M$, $M/\mathcal{T}_K(M) \in \sigma[K]$;

(d) for any cyclic $K \subset M$, every K-injective $\mathcal{T}_K$-torsionfree module in $\sigma[M]$ is M-injective and K-generated;

(e) for any cyclic $K \subset M$, $I_M(\mathcal{T}^K(M)) \in \sigma[K]$ and $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$;

(f) for any cyclic $K \subset M$, $\widehat{M} = KT \oplus I_M(\mathcal{T}_K(\widehat{M}))$.

Conditions (d)-(f) also hold for finitely generated submodules.

Proof. The equivalence of (a), (c), (d), (e) and (f) follows from 4.2.

$(b) \Leftrightarrow (c)$ One direction is trivial.

Let $a_1, \dots, a_k$ be a generating subset of K. By 2.3, $\mathcal{T}_K(M) = \bigcap_{i=1}^k \mathcal{T}_{a_i}(M)$. Hence $M/\mathcal{T}_K(M) \subset \bigoplus_{i=1}^k M/\mathcal{T}_{a_i}(M)$. But $M/\mathcal{T}_{a_i}(M) \in \sigma[Ra_i] \subset \sigma[K]$.

Thus $M/\mathcal{T}_K(M) \in \sigma[K]$. □

5.2 Properties of PSP-modules.

Let M be an R-module and $T = End_R(\widehat{M})$.

(1) Assume M is a PSP-module and $U \subset \widehat{M}$ an (R,T)-submodule of finite uniform dimension. Then U is a semisimple (R,T)-bimodule.

(2) Assume $\widehat{M}$ has finite uniform dimension as (R,T)-bimodule. Then M is an SSP-module if and only if M is a PSP-module.

Proof. (1) First assume U is a uniform (R,T)-bimodule. Let $V \subset U$ be an (R,T)-submodule, and $N \subset M \cap V$ a finitely generated R-submodule. Then NT is an essential (R,T)-submodule of U. $\mathcal{T}^N(U) \cap \mathcal{T}_N(U) = 0$ implies $\mathcal{T}_N(U) = 0$.

From this we deduce $NT \trianglelefteq V \trianglelefteq U$ as R-modules. Since NT is M-injective, we conclude $NT = V = U$ and U is a simple (R,T)-bimodule.

Now assume U has finite uniform dimension as (R,T)-bimodule. Hence there exist uniform submodules $V_i \subset U$, $i = 1, \dots, n$, such that $\bigoplus_{i=1}^n V_i \trianglelefteq U$ as (R,T)-submodule.

Let N_i be finitely generated submodule of the left R-module $V_i \cap M$ and $N = \bigoplus_{i=1}^n N_i$. As shown above, all $V_i = N_i T$ are simple (R,T)-bimodules. Hence $NT = \sum_{i=1}^n N_i T = \bigoplus_{i=1}^n V_i \trianglelefteq U$ as R-submodule. Therefore $U = NT = \bigoplus_{i=1}^n V_i$ is a finitely generated semisimple (R,T)-bimodule.

(2) If M is SSP then obviously M is PSP.

If M is PSP and $\widehat{M}$ has finite uniform dimension as (R,T)-bimodule, $\widehat{M}$ is a semisimple (R,T)-bimodule by (1) and hence M is SSP by 4.5. □

5.3 Corollary.

Let M be a finitely generated left R-module and $T = End_R(\widehat{M})$. Then the following are equivalent:

(a) M is an SSP-module;

(b) $\widehat{M}$ is (finitely generated and) semisimple as an (R,T)-bimodule;

(c) M is PSP and $\widehat{M}$ has finite uniform dimension as an (R,T)-bimodule.

Proof. Since M is a finitely generated R-module, $\widehat{M} = MT$ is a finitely generated (R,T)-bimodule. Hence the assertions follow from 4.5 and 5.2. □

Recall that an R-module M is called *strongly prime* if for every submodule $N \subset M$, $M \in \sigma[N]$ (see [13]). With similar arguments as used above we can show:

5.4 Strongly prime modules.

For an R-module M with $T = End_R(\widehat{M})$ the following are equivalent:

(a) M is strongly prime;

(b) M is PSP (or SSP) and $\widehat{M}$ is a uniform (R,T)-bimodule.

(c) $\widehat{M}$ is a simple (R,T)-bimodule.

In particular, for a uniform R-module M, the conditions strongly prime, SSP, and PSP are equivalent.

Proof. $(a) \Rightarrow (b)$ For every submodule $K \subset M$, $KT = \widehat{M}$ and hence the assertion is clear.

$(b) \Rightarrow (a)$ Assume M is PSP and $K \subset M$ is a finitely generated submodule. By the uniformity condition, $\mathcal{T}^K(\widehat{M}) \cap \mathcal{T}_K(\widehat{M}) = 0$ implies $\mathcal{T}_K(\widehat{M}) = 0$ and $\widehat{M} = KT$. So M is strongly prime.

$(a) \Leftrightarrow (c)$ This follows with the same arguments applied in the proof of 4.4. It is also shown in [13], Proposition 2.1. □

Both the definitions of SSP and PSP modules M refer to the category $\sigma[M]$. Nevertheless we can show that submodules of such modules are of the same type.

5.5 Submodules of SSP and PSP modules.
Let M be an R-module and $L \subset M$ a submodule. Then:
If M is SSP (resp. PSP), then L is also SSP (resp. PSP).

Proof. Refer to the notation in 2.2. Let $L \subset M$ be a submodule and $K \in \sigma[M]$. Then $\mathcal{T}^K(L) \subset \mathcal{T}^K(M)$ and $I_L(\mathcal{T}^K(L)) \subset I_M(\mathcal{T}^K(L)) \subset I_M(\mathcal{T}^K(M))$. Therefore

$$\mathcal{T}_{K,M}(L) = Re(L, I_M(\mathcal{T}^K(M)) \subset Re(L, I_L(\mathcal{T}^K(L)) = \mathcal{T}_{K,L}(L).$$

Now assume M is PSP and $K \subset L$ is a submodule. By the above relation, $L/\mathcal{T}_{K,L}(L)$ is a homomorphic image of $L/\mathcal{T}_{K,M}(L)$. Since $\mathcal{T}_{K,M}(-)$ is a left exact functor,

$$\mathcal{T}_{K,M}(L) = \mathcal{T}_{K,M}(M) \cap L, \text{ and } L/\mathcal{T}_{K,M}(L) \subset M/\mathcal{T}_{K,M}(M) \in \sigma[K].$$

Therefore $L/\mathcal{T}_{K,L}(L) \in \sigma[K]$ and L is PSP. □

6 Pseudo regular modules

The class of modules we define now is related to regularity properties of modules and includes left fully idempotent rings.

6.1 Pseudo regular modules.
Let M be a left R-module, $S = End_R(M)$, $T = End_R(\widehat{M})$ and $D = End_T(\widehat{M})$. Then the following conditions are equivalent:

(a) for every $m \in M$, there exists $h \in D$ such that $hM \subset RmS$ and $hm = m$;

(b) for any (R,S)-submodule $N \subset M$ and $m_1, \dots, m_k \in N$, there exists $h \in D$ such that $hM \subset N$ and $hm_i = m_i$ for all $i = 1, \dots, k$.

A module satisfying these conditions is called pseudo regular.

Proof. $(a) \Rightarrow (b)$ We can follow an induction argument in the proof of [4], Proposition 11.27. Let $N \subset M$ be an (R,S)-submodule of M and $m_1, \dots, m_k \in N$. For $k = 1$ there is nothing to show.

Assume the assertion holds for any $k-1$ elements in N. Choose $f \in D$ such that $fm_k = m_k$ and $fM \subset Rm_kS \subset N$.

Clearly $m_i - fm_i \in N$ for all $i = 1, \dots, k-1$, and by hypothesis, there exists $g \in D$ with $gM \subset N$ and $g(m_i - fm_i) = m_i - fm_i$ for $i = 1, \dots, k-1$. Put $h = 1-(1-g)(1-f)$. Then $hm = gm - gfm + fm \in N$ for all $m \in M$, i.e. $hM \subset N$, and for $i = 1, \dots, k$,

$$hm_i = m_i - (1-g)(1-f)m_i = m_i - (1-g)(m_i - fm_i) = m_i.$$

$(b) \Rightarrow (a)$ is obvious. □

Let us check what pseudo regularity means for rings:

6.2 Left fully idempotent rings.
A ring R is left fully idempotent if and only if the left R-module R is pseudo regular.

Proof. Recall $End_R(R) = R$. Assume R is left fully idempotent. Let N be an ideal of R and $m \in N$. Since $(Rm)^2 = Rm$, there exist $h \in RmR$ such that $hm = m$. Clearly $hR \subset N$ and $h \in R \subset Biend_R(\widehat{R})$. Hence ${}_RR$ is pseudo regular.

Now assume ${}_RR$ to be pseudo regular and $m \in R$. Then there exists $h \in Biend_R(\widehat{R})$ such that $hm = m$ and $hR \subset RmR$. Hence $r = h1 \in RmR$ and $rm = (h1)m = hm = m$. So $m = rm \in (Rm)^2$, $(Rm)^2 = Rm$ and the ring R is left fully idempotent. □

It follows from the above observation that a commutative ring R is von Neumann regular if and only if ${}_RR$ is pseudo regular.

Of particular interest for our investigations is the following relationship:

6.3 Pseudo regular and PSP modules.
Let M be a left R-module, $S = End_R(M)$ and $T = End_R(\widehat{M})$. The following conditions are equivalent:

(a) for every $m \in M$, there is a central idempotent $e \in T$ with $RmS = Me$;

(b) for every $m \in M$, $M = RmS \oplus \mathcal{T}_m(M)$ and $\widehat{M} = RmT \oplus \mathcal{T}_m(\widehat{M})$;

(c) for every finitely generated submodule $N \subset M$, there is a central idempotent $e \in T$ with $NS = Me$;

(d) for every finitely generated submodule $N \subset M$, $M = NS \oplus \mathcal{T}_N(M)$ and $\widehat{M} = NT \oplus \mathcal{T}_N(\widehat{M})$;

(e) M is a pseudo regular PSP-module over R.

Proof. Denote $D = End_T(\widehat{M})$.

$(a) \Rightarrow (b)$ For $m \in M$ choose a central idempotent $e \in T$ with $RmS = Me$. Since $\widehat{M} = MT$ we have

$$\begin{aligned} \widehat{M} &= MTe \oplus MT(1-e) = MeT \oplus MT(1-e) \\ &= RmST \oplus MT(1-e) = RmT \oplus MT(1-e). \end{aligned}$$

Hence RmT is M-injective and $MTe = RmT = \mathcal{T}_m(\widehat{M}) = I_M(\mathcal{T}_m(\widehat{M}))$ (see 2.3). Since $\widehat{M}$ is M-injective,

$$\begin{aligned} Hom_R(MT(1-e), I_M(\mathcal{T}_m(\widehat{M}))) &= \{t \in T \mid MT(1-e)t \subset I_M(\mathcal{T}_m(\widehat{M}))\} \\ &= \{t \in T \mid MT(1-e)t \subset MTe\} = 0. \end{aligned}$$

So $MT(1-e) \subset \mathcal{T}_m(\widehat{M})$. But $MTe = RmT$, $\widehat{M} = MTe \oplus MT(1-e)$ and

$$MTe \cap \mathcal{T}_m(\widehat{M}) = \mathcal{T}^m(\widehat{M}) \cap \mathcal{T}_m(\widehat{M}) = 0.$$

Hence $\mathcal{T}_m(\widehat{M}) = MT(1-e)$ and $\widehat{M} = RmT \oplus \mathcal{T}_m(\widehat{M})$. Clearly $RmS = Me = M \cap MTe$ and

$$\mathcal{T}_m(M) = M \cap \mathcal{T}_m(\widehat{M}) = M \cap MT(1-e) = M(1-e).$$

So $M = RmT \oplus \mathcal{T}_m(M)$.

$(b) \Rightarrow (a)$ Let e be the projection of $\widehat{M}$ onto RmT along $\mathcal{T}_m(\widehat{M})$. Since both submodules are fully invariant, e is a central idempotent of T. From $RmS \subset RmT$ and $\mathcal{T}_m(M) \subset \mathcal{T}_m(\widehat{M})$ we conclude $Me = RmS$.

$(c) \Leftrightarrow (d)$ can also be shown with the above proof.

$(c) \Rightarrow (a)$ is obvious.

$(a) \Rightarrow (e)$ Since $M/\mathcal{T}_m(M) = RmS \in \sigma[Rm]$, M is PSP.

Let π be the projection of $\widehat{M}$ onto RmT along $\mathcal{T}_m(\widehat{M})$. Obviously, $\pi \in D$. Since $RmS \subset RmT$ and $\mathcal{T}_m(M) \subset \mathcal{T}_m(\widehat{M})$, $\pi M = RmS$. Hence M is pseudo regular.

$(e) \Rightarrow (c)$ Let $N \subset M$ be a finitely generated submodule. Since M is PSP, $\widehat{M} = NT \oplus I_M(\mathcal{T}_N(\widehat{M}))$.

Suppose $I_M(\mathcal{T}_N(\widehat{M})) \neq \mathcal{T}_N(\widehat{M})$ and consider $x \in I_M(\mathcal{T}_N(\widehat{M})) \setminus \mathcal{T}_N(\widehat{M})$. Then $Hom_R(Rx, NT) \neq 0$ and - by M-injectivity of $\widehat{M}$ - there exists $t \in T$ with $0 \neq xt \in NT$. So $xt = \sum_{i=1}^n m_i s_i$ for some $m_i \in N$, $s_i \in T$. M being pseudo regular, there exists $h \in D$ such that $hM \subset NS$ and $hm_i = m_i$ for $i = 1, \ldots, n$. Clearly

$$h(xt) = \sum_{i=1}^n h(m_i s_i) = \sum_{i=1}^n (hm_i) s_i = \sum_{i=1}^n m_i s_i = xt.$$

Since $\widehat{M} = MT$, $h\widehat{M} = (hM)T \subset NT$ and $hx \in NT$. Let π be the projection of $\widehat{M}$ onto $I_M(\mathcal{T}_N(\widehat{M}))$ along NT. Then $0 = (hx)\pi = h(x\pi) = hx$ and $xt = h(xt) = (hx)t = 0$, contradicting $xt \neq 0$. Hence $I_M(\mathcal{T}_N(\widehat{M}) = \mathcal{T}_N(\widehat{M})$ and $\widehat{M} = NT \oplus \mathcal{T}_N(\widehat{M})$.

Now consider $\alpha = 1 - \pi : \widehat{M} \to NT$ and $y \in M$. Then $y\alpha = \sum_{i=1}^n n_i t_i$ for some $m_i \in N$, $t_i \in T$. Since M is pseudo regular, there exists $h \in D$ such that $hM \subset NS$ and $hn_i = n_i$ for $i = 1, \ldots, n$. Clearly $h(y\alpha) = y\alpha$ and $hy \in NS \subset NT$. So $y\alpha = h(y\alpha) = (hy)\alpha = hy \in NS$ and $y - y\alpha \in \mathcal{T}_N(\widehat{M}) \cap M = \mathcal{T}_N(M)$. Therefore $M = NS \oplus \mathcal{T}_N(M)$. □

7 Polyform PSP and SSP modules

In this section we are concerned with the interplay between polyform and SSP properties. One of the crucial observations is that projective PSP modules are polyform. We begin with general modules enjoying both properties.

7.1 Polyform SSP modules.
Let M be a polyform R-module and $T = End_R(\widehat{M})$. Then the following are equivalent:

(a) M is an SSP-module;

(b) for every $N \trianglelefteq M$, $M \in \sigma[N]$;

(c) $I_M(KT) \in \sigma[K]$;

(d) for any submodule $K \subset M$, $\widehat{M} = KT \oplus \mathcal{T}_K(\widehat{M})$.

Proof. The statements follow from 3.2, 4.3 and 4.5. □

In [12], Theorem 3.7 it is shown, that any polyform module M is SSP if M has finite uniform dimension, $Hom_R(M,N) \neq 0$ for non-zero $N \subset M$, and $End_R(M)$ is semiprime. We can extend this result to a more general class of modules considered in Zelmanowitz [17]. There an R-module M is called *weakly semisimple* if

(i) M is polyform,

(ii) every finitely generated submodule has finite uniform dimension, and

(iii) for every non-zero submodule $N \subset M$, there exists $f \in Hom_R(M,N)$ with $f|_N \neq 0$ (weakly compressible).

7.2 Weakly semisimple modules.
Every weakly semisimple module is SSP.

Proof. We have to show that for every $N \trianglelefteq M$, $M \in \sigma[N]$.

For any $m \in M$, Rm has finite uniform dimension and $N \cap Rm \trianglelefteq Rm$. Hence by Proposition 1.1 in [17], there exists a monomorphism $Rm \to N \cap Rm$ and so $Rm \in \sigma[N]$. This implies $M \in \sigma[N]$. □

We know that a module M is SSP if and only if $\widehat{M}$ is SSP. In general, for a PSP module M, $\widehat{M}$ need not be PSP. For polyform modules the PSP property extends at least to the idempotent closure:

7.3 Polyform PSP modules.
Let M be a polyform R-module, $T = End_R(\widehat{M})$, B the Boolean ring of central idempotents of T and $\widetilde{M}$ the idempotent closure of M. Then the following are equivalent:

(a) M is a PSP module;

(b) for every $m \in M$, $RmT = \widehat{M}\varepsilon(m)$;

(c) for every finitely generated submodule $K \subset M$, $KT = \widehat{M}\varepsilon(K)$;

(d) $\widehat{M}$ is a PSP module.

Under the given conditions, $\mathcal{T}_m(\widehat{M}) = \widehat{M}(id - \varepsilon(m))$.

Proof. $(a) \Rightarrow (b)$ By 5.1, (f) and 3.2, (1), $\widehat{M} = RmT \oplus \mathcal{T}_m(\widehat{M})$. Now (b) follows from the definition of the idempotent $\varepsilon(m)$.

$(b) \Rightarrow (a)$ Since $\widehat{M}\varepsilon(m)$ is M-injective,

$$I_M(\mathcal{T}^m(M)) = I_M(\mathcal{T}^m(\widehat{M})) = I_M(RmT) = RmT \in \sigma[Rm].$$

M being polyform, we have $\mathcal{T}_m(M) + \mathcal{T}^m(M) \trianglelefteq M$ and the assertion follows by 5.1.

$(b) \Leftrightarrow (c)$ This is obvious by 5.1 and 3.3.

$(b) \Rightarrow (d)$ Any $a \in \widehat{M}$ can be written as $a = \sum_{i=1}^k m_i e_i$, with $m_1, \ldots, m_k \in M$ and pairwise orthogonal $e_1, \ldots, e_k \in B$, satisfying

$$\varepsilon(a) = \sum_{i=1}^k e_i \text{ and } e_i = \varepsilon(m_i)e_i \text{ for } i = 1, \ldots, k.$$

By (b), $Rm_iT = \widehat{M}\varepsilon(m_i)$ for $i = 1, \ldots, k$, and

$$RaT = R(\sum_{i=1}^k m_i e_i)T = \sum_{i=1}^k (Rm_iT)e_i = \sum_{i=1}^k \widehat{M}\varepsilon(m_i)e_i = \widehat{M}(\sum_{i=1}^k e_i) = \widehat{M}\varepsilon(a).$$

$(d) \Rightarrow (a)$ By 5.5, submodules of PSP modules are again PSP. □

By 4.5, any self-injective SSP module is semisimple as a bimodule. For self-injective polyform PSP modules we get a weaker structure theorem:

7.4 Self-injective *PSP* modules.

Let M be a self-injective polyform R-module and $T = End_R(\widehat{M})$. Denote $\Lambda = R \otimes_{\mathbb{Z}} T^o$ and $C = End_\Lambda(M)$. Then the following conditions are equivalent:

(a) ${}_RM$ is a PSP R-module;

(b) every cyclic Λ-submodule of M is a direct summand;

(c) every finitely generated Λ-submodule of M is a direct summand;

(d) as a Λ-module, M is a selfgenerator;

(e) for any $m \in M$ and $f \in End_C(M)$, there exists $h \in \Lambda$ with $f(m) = hm$.

Proof. Notice that the Λ-submodules of M are just the fully invariant submodules and that C can be identified with the center of T. Hence C is a commutative regular ring.

$(a) \Leftrightarrow (b) \Leftrightarrow (c)$ is clear by 7.3, $(c) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (e)$ This follows from the proof of the Density Theorem (e.g. [15], 15.7).

$(e) \Rightarrow (b)$ Choose any $m \in M$ and $\varepsilon(m) \in C$ as defined in 3.3. Since $mC \simeq \varepsilon(m)C$ is a direct summand, for any $n \in M$, there exists $f \in End_C(M)$ with $f(m) = n\varepsilon(m)$.

By (e), $f(m) = hm$ for some $h \in \Lambda$ and hence $M\varepsilon(m) \subset \Lambda m$.

On the other hand, $m = m\varepsilon(m)$ and $\Lambda m \subset (\Lambda M)\varepsilon(m) \subset M\varepsilon(m)$. So $\Lambda m = M\varepsilon(m)$ is a direct summand and the assertion is proved. □

Modules M, whose finitely generated submodules are direct summands, are closely related to M being regular in $\sigma[M]$. In fact, if M is finitely presented in $\sigma[M]$, these two notions coincide (e.g. [15], 37.3, 37.4). As a special case we derive from the above theorem:

7.5 Self-injective finitely presented *PSP* modules.

Let M be a self-injective polyform R-module and $T = End_R(\widehat{M})$. Denote $\Lambda = R \otimes_{\mathbb{Z}} T^o$ and $C = End_\Lambda(M)$. Assume M is finitely generated as a Λ-module. Then the following conditions are equivalent:

(a) M is a PSP R-module and finitely presented in $\sigma[{}_\Lambda M]$;

(b) ${}_\Lambda M$ is regular and projective in $\sigma[{}_\Lambda M]$;

(c) ${}_\Lambda M$ is a (projective) generator in $\sigma[{}_\Lambda M]$;

(d) for any $m_1, \ldots, m_n \in M$ and $f \in End_C(M)$, there exists $h \in \Lambda$ with $f(m_i) = hm_i$ for $i = 1, \ldots, n$ (density property).

Proof. Since ${}_\Lambda M$ is finitely generated, M_C is a generator in C-*Mod* by 3.3. So M_C is a faithfully flat C-module.

$(a) \Leftrightarrow (b)$ By 7.4, every finitely generated Λ-submodule of M is a direct summand. Now the assertion follows from [15], 37.4.

$(b) \Rightarrow (c) \Rightarrow (d)$ are obvious (Density Theorem, [15], 15.7).

$(d) \Rightarrow (a)$ Since M_C is a generator in C-*Mod*, M is (finitely generated and) projective as an $End_C(M)$-module (e.g. [15], 18.8).

By the Density Property, the categories $\sigma[{}_\Lambda M]$ and $\sigma[{}_{End_C(M)}M]$ coincide (see [15], 15.8). So M is projective (hence finitely presented) in $\sigma[{}_\Lambda M]$.

By 7.4, M is a PSP R-module. □

We have seen in 7.4 and 7.5 that self-injective polyform PSP modules have nice structural properties. Hence we may ask for which modules M, the M-injective hull $\widehat{M}$ is a PSP module. For this we need some properties of self-injective polyform modules.

7.6 Self-injective polyform modules.
Let M be a self-injective polyform R-module, $T = End_R(\widehat{M})$ and $\Lambda = R \otimes_{\mathbb{Z}} T^o$. Then:

(1) For any submodule $N \subset M$ and $m \in M$, $(\mathcal{T}_N(M) : m)_R = An_R(m\varepsilon(N))$.

(2) For $m_1, \ldots, m_n, m \in M$ and $U = An_R(m_2, \ldots, m_n)$ the following are equivalent:

(a) There exists $h \in \Lambda$ with $hm_1 = m\varepsilon(Um_1)$ and $hm_i = 0$ for $i = 2, 3, \ldots, n$;

(b) there exist $r_1, \ldots, r_k \in R$ such that for any $s_1, \ldots, s_n \in R$ the relations

$$\sum_{l=1}^{n} s_l r_j m_l = 0 \ \textit{for}\ j = 1, \ldots, k,$$

imply $s_1 m \in \mathcal{T}_{Um_1}(M)$.

Proof. (1) By definition of the idempotent $\varepsilon(N)$ (see 3.3), $M\varepsilon(N) = I_M(NT)$ and $M(1 - \varepsilon(N)) = \mathcal{T}_N(M)$. Hence for $r \in R$, $rm \in \mathcal{T}_N(M)$ if and only if $(rm)\varepsilon(N) = 0$. Now our assertion follows from $(rm)\varepsilon(N) = r(m\varepsilon(N))$.

(2) $(a) \Rightarrow (b)$ Put $e = \varepsilon(Um_1)$. By (1), $(\mathcal{T}_{Um_1}(M) : m)_R = An_R(me)$.
Choose an element $h = \sum_{j=1}^{k} r_j \otimes t_j$ in Λ with

$$hm_1 = me \ \text{and}\ hm_i = 0 \ \text{for all}\ i = 2, \ldots, n.$$

Assume for $s_1, \ldots, s_n \in R$, $\sum_{l=1}^{n} s_l r_j m_l = 0$ for $j = 1, \ldots, k$. Then

$$s_1 me = s_1 hm_1 = \sum_{l=1}^{n} s_l hm_l = \sum_{l=1}^{n} s_l \sum_{j=1}^{k} r_j m_l t_j = \sum_{j=1}^{k} (\sum_{l=1}^{n} s_l r_j m_l) t_j = 0.$$

Hence $s_1 \in An_R(me) = (\mathcal{T}_{Um_1}(M) : m)_R$.

$(b) \Rightarrow (a)$ Put $N = \sum_{i=1}^{n} R(r_1 m_i, \ldots, r_k m_i) \subset M^k$ and consider the assignement

$$\psi : N \to M, \quad \sum_{i=1}^{n} s_i (r_1 m_i, \ldots, r_k m_i) \mapsto s_1 me.$$

To show that ψ is well-defined assume $\sum_{i=1}^{n} s_i (r_1 m_i, \ldots, r_k m_i) = 0$.
Then $\sum_{i=1}^{n} s_i r_j m_i = 0$, for all $j = 1, \ldots, k$.
By assumption, $s_1 \in (\mathcal{T}_{Um_1}(M) : m)_R = An_R(me)$. Hence $s_1 me = 0$ proving that ψ is a well-defined morphism.

M being M-injective, ψ can be extended to a morphism $M^k \to M$, also denoted by ψ. Since $Hom_R(M^k, M) = T^k$ there exist $t_1, \ldots, t_k \in T$ such that

$$(x_1, \ldots, x_k)\psi = \sum_{i=1}^{k} x_i t_i \ \text{for all}\ (x_1, \ldots, x_k) \in M^k.$$

Put $h = \sum_{j=1}^{k} r_j \otimes t_j$. Then

$$hm_1 = \textstyle\sum_{j=1}^{k} r_j m_1 t_j = (r_1 m_1, r_2 m_1, \ldots, r_k m_1)\psi = me, \ \text{and}$$
$$hm_l = \textstyle\sum_{j=1}^{k} r_j m_l t_j = (r_1 m_l, r_2 m_l, \ldots, r_k m_l)\psi = 0 \ \text{for}\ l = 2, \ldots, n.$$

□

For modules M with $End_R(\widehat{M})$ commutative we are now able to characterize the density property of $\widehat{M}$ as bimodule.

7.7 Density property of the self-injective hull.
Let M be a polyform R-module, assume $T = End_R(\widehat{M})$ to be commutative and put $\Lambda = R \otimes_{\mathbb{Z}} T$. Then the following are equivalent:

(a) For any $a_1, \ldots, a_n \in \widehat{M}$ and $f \in End_T(\widehat{M})$, there exists $h \in \Lambda$ with $ha_i = fa_i$ for $i = 1, \ldots, n$.

(b) For any $m_1, \ldots, m_n, m \in M$ there exist $r_1, \ldots, r_k \in R$ such that for $s_1, \ldots, s_n \in R$ the relations

$$\sum_{l=1}^{n} s_l r_j m_l = 0 \ \textit{for}\ j = 1, \ldots, k,$$

imply $s_1 m \in \mathcal{T}_{Um_1}(M)$ for $U = An_R(m_2, \ldots, m_n)$.

(c) M is a PSP module and for any $m_1, \ldots, m_n \in M$ there exist $r_1, \ldots, r_k \in R$ such that for $s_1, \ldots, s_n \in R$ the relations

$$\sum_{l=1}^{n} s_l r_j m_l = 0 \ \textit{for}\ j = 1, \ldots, k,$$

imply $s_1 m_1 \in \mathcal{T}_{Um_1}(M)$ for $U = An_R(m_2, \ldots, m_n)$.

If $\widehat{M}$ is finitely presented in $\sigma[{}_\Lambda\widehat{M}]$ the above are equivalent to:

(d) $\widehat{M}$ is a PSP R-module.

Proof. Put $U = An_R(m_2, \ldots, m_n)$.

$(a) \Rightarrow (b)$ Put $N = \sum_{i=1}^{n} m_i T$. By 3.3, $\widehat{M}$ is a non-singular and N is an injective T-module. Since in a non-singular module the intersection of injective submodules is again injective, $K = m_1T \cap N$ is T-injective. Hence $m_1T = K \oplus L$ for some submodule $L \subset m_1T$. By 3.3, $An_T(m_1) = (1 - \varepsilon(m_1))T$. This means that the map

$$m_1T \to \varepsilon(m_1)T, \quad m_1t \mapsto \varepsilon(m_1)t,$$

is an isomorphism. So there exist idempotents $u, v \in T$ with the properties

$$(*) \quad uv = 0,\ u + v = \varepsilon(m_1),\ K = m_1uT \simeq uT \text{ and } L = m_1vT \simeq vT.$$

Since $m_1uT = K \subset N$ and $UN = 0$, $Um_1u = 0$ and

$$Um_1 = U[m_1\varepsilon(m_1)] = U[m_1(u+v)] = Um_1v.$$

By 3.4, $Um_1v \trianglelefteq Rm_1v$ and by 3.3, $\varepsilon(Um_1v) = \varepsilon(Rm_1v) = \varepsilon(m_1v)$.
The isomorphism $m_1vT \simeq vT$ implies $\varepsilon(m_1v) = v$ and hence we have

$$(**) \qquad \varepsilon(Um_1) = v\ .$$

Since $m \in M$ and $\mathcal{T}_{Um_1}(M) = M \cap \mathcal{T}_{Um_1}(\widehat{M})$, by 7.6,

$$(\mathcal{T}_{Um_1}(M) : m)_R = (\mathcal{T}_{Um_1}(\widehat{M}) : m)_R = An_R(mv).$$

As an injective submodule, $m_1vT \oplus N$ is a direct summand in $\widehat{M}$. Hence there exists a T-endomorphism ψ of $\widehat{M}$ satisfying $\psi N = 0$ and $\psi m_1 v = mv$ (recall $m_1vT \simeq vT$). Since $m_1u \in N$, we have $\psi m_1 u = 0$ and $\psi m_1 = \psi[m(u+v)] = mv$.

By assumption, there exists $h \in \Lambda$ satisfying

$$hm_1 = \psi m_1 = mv, \text{ and } hm_i = \psi m_i \text{ for } i = 2, \ldots, n\ .$$

Now the assertion follows from 7.6.

$(b) \Rightarrow (c)$ Putting $m_1 = \cdots = m_n = 0$ we see that M is a PSP module. The second part of the conditions in (c) follows from (b) for $m = m_1$.

$(c) \Rightarrow (a)$ Since $\widehat{M} = MT$ it suffices to show that for any $m_1, \ldots, m_n \in M$ and $f \in End_T(\widehat{M})$ there exists $h \in \Lambda$ such that $fm_i = hm_i$ for all $i = 1, \ldots, n$.

We prove this by induction on the cardinality $|I|$ of minimal subsets $I \subset \{1, \ldots, n\}$ satisfying $\sum_{i \in I} m_iT = \sum_{i=1}^n m_iT$.

Consider the case $|I| = 1$, i.e., $I = \{1\}$. By 7.3, $\widehat{M}\varepsilon(m_1) = \Lambda m_1$. Since

$$fm_1 = f[m_1\varepsilon(m_1)] = (fm_1)\varepsilon(m_1) \in \widehat{M}\varepsilon(m_1) = \Lambda m_1\ ,$$

$fm_1 = hm_1$ for some $h \in \Lambda$. By assumption $\sum_{i=1}^n m_iT = m_1T$, implying $fm_i = hm_i$ for all $i = 1, \ldots, n$.

Now assume $|I| = n$ and consider $N = \sum_{i=2}^n m_iT$. As shown above there exist idempotents $u, v \in T$ satisfying $(*)$ and $(**)$. By 7.6, there exists $h_1 \in \Lambda$ with

$$h_1m_1 = m_1v \text{ and } h_1m_i = 0 \text{ for } i = 2, \ldots, n\ .$$

Notice that $h_1m_1u = (h_1m_1)u = (m_1v)u = m_1(vu) = 0$.

By induction hypothesis, there exists $h_2 \in \Lambda$ with

$$h_2m_i = fm_i \text{ for } i = 2, \ldots, n\ .$$

Obviously, $h_2x = fx$ for any $x \in N$. From $(*)$ we obtain $h_2m_1u = fm_1u$.
Consider the element $m = fm_1v - h_2m_1v$. Clearly $m = mv$ and so $m \in \widehat{M}v$. By $(*)$, $v\varepsilon(m_1) = v$. Now it follows from 7.3 that

$$\Lambda m_1v = (\Lambda m_1)v = \widehat{M}\varepsilon(m_1)v = \widehat{M}v\ .$$

Hence there exists $h_3 \in \Lambda$ with

$$h_3m_1v = m = fm_1v - h_2m_1v\ .$$

Putting $h = h_3h_1 + h_2$ we have

$$\begin{aligned} hm_1 &= h_3(h_1m_1u) + h_3(h_1m_1v) + h_2m_1u + h_2m_1v \\ &= h_3m_1v + fm_1u + h_2m_1v \\ &= (fm_1v - h_2m_1v) + fm_1u + h_2m_1v \\ &= fm_1v + fm_1u = fm_1\ , \\ hm_i &= h_3(h_1m_i) + h_2m_i = fm_i \text{ for } i = 2, \ldots, n\ . \end{aligned}$$

$(a) \Leftrightarrow (d)$ This is clear by 7.5. □

Now we turn to the question which additional conditons on an SSP or PSP module M imply that M is polyform. It is interesting to observe that this is achieved by commutativity conditions on the endomorphism rings as well as by projectivity of the modules. Recall that a ring is said to be *(left and right) duo* if all its one-sided ideals are two-sided.

7.8 Duo endomorphism rings.

Let M be an R-module, $T = End_R(\widehat{M})$ and assume for every $N \trianglelefteq M$, $M \in \sigma[N]$.

(1) $Jac(T) \cap center(T) = 0$.

(2) Suppose T is a duo ring. Then M is polyform and SSP.

Proof. (2) Assume for $f \in T$, $N = Ke\,f \trianglelefteq \widehat{M}$. Then $M \in \sigma[N]$ which is equivalent to $NT = \widehat{M}$. This implies $\widehat{M}f = (NT)f = (Nf)T = 0$ and hence $f = 0$. So the Jacobson radical of T is zero, i.e. M is polyform. By 7.1, M is SSP.

(1) A similar argument also implies this assertion. □

Projectivity makes any PSP module polyform. This applies in particular for the left module structure of the ring itself.

7.9 Projective PSP modules.
Let M be a PSP module which is projective in $\sigma[M]$. Then

(1) For any submodules $K, N \subset M$, $\mathcal{T}^N(K) + \mathcal{T}_N(K) \trianglelefteq K$.

(2) M is polyform.

Proof. M is projective in $\sigma[M]$ if and only if $M^{(\Lambda)}$ is self-projective for any set Λ. From this it is obvious that M/X is projective in $\sigma[M/X]$ for every fully invariant submodule $X \subset M$.

(1) First we show $\mathcal{T}^N(K) \neq 0$ for any finitely generated submodules $K, N \subset M$ with $Hom(K, N) \neq 0$. For this we may assume that there is an epimorphism $f : K \to N$. Then $N \subset \mathcal{T}^K(M)$ and

$$\mathcal{T}_K(M) \cap N = 0 = \mathcal{T}_K(M) \cap K.$$

Put $L = \mathcal{T}_K(M)$ and $\overline{M} = M/L$. There are canonical inclusions $N \subset \overline{M}$ and $K \subset \overline{M}$. Since M is PSP, $\overline{M} \in \sigma[K]$ and so $\sigma[\overline{M}] = \sigma[K]$.

As outlined above, $\overline{M}$ is projective in $\sigma[K]$ and hence is a direct summand of $K^{(\Lambda)}$, for some set Λ. Therefore the compostion of the inclusion $N \subset \overline{M}$ with a suitable map $\overline{M} \to K$ yields a non-zero morphism $N \to K$. This means $\mathcal{T}^N(K) \neq 0$.

Now we prove $\mathcal{T}^N(K) + \mathcal{T}_N(K) \trianglelefteq K$ for any submodules $K, N \subset M$.

Consider some $x \in K \setminus \mathcal{T}_N(K)$. Then there exists a non-zero $g : Rx \to I_M(N)$ and $0 \neq (y)g \in N$ for some $y \in Rx$. From the above we know $\mathcal{T}_N(Ry) \neq 0$ implying

$$\mathcal{T}_N(Ry) \subset \mathcal{T}^N(K) \cap Rx \neq 0 \text{ and } \mathcal{T}^N(K) + \mathcal{T}_N(K) \trianglelefteq K.$$

(2) Assume M is not polyform. Then there exist a cyclic submodule $K \subset M$ and a non-zero morphism $f : K \to M$ with $Ke\,f \trianglelefteq K$. Put $N = (K)f$ and $\overline{M} = M/\mathcal{T}_N(M)$. Then $\mathcal{T}^N(K) \cap Ke\,f \trianglelefteq \mathcal{T}^N(K)$.

The map $K \xrightarrow{f} M \to \overline{M}$ factorizes through $\bar{f} : K/\mathcal{T}_N(K) \to \overline{M}$. Since

$$\mathcal{T}_N(K) \cap Ke\,f \trianglelefteq \mathcal{T}^N(K) \text{ and } \mathcal{T}^N(K) \trianglelefteq K/\mathcal{T}_N(K),$$

we conclude $Ke\,\bar{f} \trianglelefteq K/\mathcal{T}_N(K)$. This means that N is an $\overline{M}$-singular module.

By assumption, $\overline{M} \in \sigma[N] = \sigma[\overline{M}]$. So, in particular, $\overline{M}$ is $\overline{M}$-singular.

However, as noted above, $\overline{M}$ is projective in $\sigma[\overline{M}]$ and hence cannot be $\overline{M}$-singular. Therefore M is polyform. □

7.10 Projective SSP modules.
Let M be projective in $\sigma[M]$ and $T = End_R(\widehat{M})$. Then the following are equivalent:

(a) M is an SSP-module;

(b) for every submodule $K \subset M$, $\widehat{M} = KT \oplus \mathcal{T}_K(\widehat{M})$.

(c) M is polyform and for any $N \trianglelefteq M$, $M \in \sigma[N]$.

Proof. $(a) \Rightarrow (b)$ By 7.9, M is polyform and the decomposition was given in 7.1.

$(b) \Rightarrow (a)$ This decomposition implies in particular that every fully invariant submodule is a direct summand in $\widehat{M}$ as (R,T)-submodule. Hence $\widehat{M}$ is a semisimple (R,T)-bimodule and M is SSP by 4.5.

$(c) \Leftrightarrow (c)$ By 7.9, M is polyform and the assertion follows from 7.1. □

8 Left SSP and PSP rings

We call a ring R *left PSP*, if ${}_RR$ is a PSP module and R is called *left SSP*, if ${}_RR$ is an SSP module.

Notice that these definitions also apply to rings without units, considering such rings as modules over rings with units in a canonical way.

Before characterizing PSP and SSP rings we want to describe the torsion modules related to semiprime rings:

8.1 Semiprime rings and torsion modules.
Let R be a semiprime ring and $N \subset R$ a left ideal. Then:

(1) $\mathcal{T}_N(R) = An_R(N)$.

(2) $R/\mathcal{T}_N(R) \in \sigma[N]$ if and only if there exists a finite subset $X \subset N$ with $An_R(X) = An_R(N)$.

Proof. (1) The relation $\mathcal{T}_N(R) \subset An_R(N)$ always holds. Clearly $NAn_R(N)$ is a nilpotent left ideal and hence is zero.

Consider $f \in Hom_R(An_R(N), I_R(N))$ and put $K = (N)f^{-1}$. Then $(Kf)^2 \subset N(Kf) = (NK)f \subset (NAn_R(N))f = 0$. Since R is semiprime, $Kf = 0$ and so $Im\, f \cap N = 0$, implying $f = 0$. Hence $\mathcal{T}_N(R) \supset An_R(N)$.

(2) Assume $R/\mathcal{T}_N(R) \in \sigma[N]$. By 4.1, there exists a finite subset $X \subset N$ with

$$An_R(X)1 \subset \mathcal{T}_N(R) = An_R(N) \subset An_R(X).$$

Now assume $An_R(X) = An_R(N)$ for some finite $X \subset N$. Then $An_R(X) = \mathcal{T}_N(R)$ by (1), and for any $b \in R$, $An_R(X)b \subset \mathcal{T}_N(R)$. Now apply 4.1. □

8.2 Left SSP rings.

For a ring R let $Q = Q(R)$ denote the maximal (complete) left ring of quotients. Then the following are equivalent:

(a) R is left SSP;

(b) for every essential left ideal $N \subset R$, $R \in \sigma[N]$;

(c) every essential left ideal $N \subset R$ contains a finite subset X with $An_R(X) = 0$;

(d) for every left ideal $I \subset R$, $Q = IQ \oplus \mathcal{T}_I(Q)$;

(e) R is semiprime and every left ideal $I \subset R$ contains a finite subset $X \subset I$ with $An_R(X) = An_R(I)$;

(f) Q is a semisimple (R,Q)-module.

If R satisfies these conditions, then Q is left self-injective, von Neumann regular, and a finite product of simple rings.

Left ideals with property (c) above are also called *insulated*. So the rings described here are exactly the *left strongly semiprime rings* of Handelman [6], Theorem 1. They generalize *left strongly prime* rings as considered in Handelman-Lawrence [7] and Viola-Prioli [11].

Proof. $(a) \Rightarrow (b)$ is shown in 4.4.

$(b) \Rightarrow (a)$ Assume for every essential left ideal $N \subset R$, $R \in \sigma[N]$. Any such N is a faithful R-module.

First we show that R is semiprime. For this consider an ideal $I \subset R$ with $I^2 = 0$. Let J be the right annihilator of I and $L \subset R$ any non-zero left ideal. Obviously, $I \subset J$. Assume $L \cap J = 0$. Then $IL \neq 0$. However, $IL \subset L \cap J = 0$, a contradiction. This implies that J is an essential left ideal in R. By our assumption, J is a faithful left module and $IJ = 0$ means $I = 0$.

In view of 4.5 and 3.2 it remains to show that R is left non-singular (compare [9], Lemma 2.4 and [6], Proposition 6).

If the left singular ideal $Z(R) \subset R$ is non-zero, $Z(R) \oplus An_R(Z(R))$ is an essential left ideal in R. Hence there are $a_1, \dots, a_k$ with $An_R(a_1, \dots, a_k) = 0$. From this we see that there is a monomorphism $Z(R) \to Z(R)(b_1, \dots, b_r)$ with $b_1, \dots, b_r \in Z(R)$.

The kernel of this map is $Z(R) \cap An_R(b_1) \cap \dots \cap An_R(b_r)$. Since all the $An_R(b_i)$ are essential left ideals in R, this intersection could not be zero, a contradiction. Hence R is left non-singular.

$(b) \Leftrightarrow (c)$ This is obvious by 4.1.

$(a) \Rightarrow (d)$ By 7.9, R is semiprime and left non-singular. Therefore $Q = \widehat{R}$ and $End_R(Q) = Q$ (see Lambek [10], §4.3). Now the assertion follows from 7.1.

$(d) \Rightarrow (a)$ We show that R is left non-singular.

Assume for $a \in R$, $An_R(a)$ is an essential left ideal in R. Since $Q = RaQ \oplus \mathcal{T}_a(Q)$, we have $RaQ = eQ$ for some idempotent $e \in Q$. Write $e = \sum_{i=1}^n r_i a q_i$, with $r_i \in R, q_i \in Q$.

Obviously, $L = \bigcap_{i=1}^n (An_R(a) : r_i)_R$ is an essential left ideal in R and $Le = 0$. Therefore $L \cap Qe = 0$ and also $L \cap (Qe \cap R) = 0$. However, $L \subset R$ is an essential left ideal and $Qe \cap R \neq 0$, a contradiction. Hence R is left non-singular and so $Q = \widehat{R}$ and $End_R(Q) = Q$. Now apply 7.1.

$(a) \Leftrightarrow (e) \Leftrightarrow (f)$ follow from 7.10, 4.3 and 4.5. □

8.3 Semiprime and left PSP rings.

For a ring R the following conditions are equivalent:

(a) R is a left PSP-ring;

(b) R is semiprime and for every finitely generated left ideal $N \subset R$, $An_R(N) = An_R(X)$ for some finite subset $X \subset N$.

In this case R is left non-singular.

Proof. $(a) \Rightarrow (b)$ First we show that R is semiprime. Consider a cyclic left ideal $N \in R$ with $N^2 = 0$. By assumption, $R/\mathcal{T}_N(R) \in \sigma[N]$. Since $NK = 0$ for any $K \in \sigma[N]$, in particular $N(R/\mathcal{T}_N(R)) = 0$ and hence $NR \subset \mathcal{T}_N(R)$. This implies $NR \subset \mathcal{T}^N(R) \cap \mathcal{T}_N(R) = 0$ and $N = 0$. Now the assertion is clear by 8.1.

$(b) \Rightarrow (a)$ also follows from 8.1.

By 7.9, R is left non-singular. □

As mentioned in the introduction, a commutative ring is PSP if and only if it is semiprime, i.e. if it has no nilpotent elements. It is interesting to observe that the latter property suffices to make non-commutative rings PSP.

Recall that a ring without non-zero nilpotent elements is said to be *reduced*.

8.4 Reduced rings.
Any reduced ring R is left PSP.

Proof. Assume R is reduced. It is shown in Lemma 1 of chapter 4, §2 in Andrunakievic-Rjabuhin [1] that for any subset $U \subset R$, the left annihilator $An_R(U)$ of U coincides with the right annihilator of U in R. This implies $An_R(U) = An_R(RU)$. Now it is obvious from 8.3 that R is PSP. □

9 Bimodule structure of rings

Let A be a not necessarily associative ring. Left and right multiplication by any $a \in A$,

$$L_a : A \to A,\ x \mapsto ax, \quad R_a : A \to A,\ x \mapsto xa \ \text{ for } x \in A,$$

define $\mathbb{Z}$-endomorphisms of A, i.e. $L_a, R_a \in End(_{\mathbb{Z}}A)$.

The subring of $End(_{\mathbb{Z}}A)$ generated by all left multiplications in A and the identity map id_A is called the *left multiplication ring $L(A)$ of A*. Similarly the *right multiplication ring* $R(A)$ is defined.

The subring of $End(_{\mathbb{Z}}A)$ generated by all left and right multiplications and id_A is called the *multiplication ring* $M(A)$ of A, i.e.

$$M(A) =< \{L_a, R_a \mid a \in A\} \cup \{id_A\} > \quad \subset End(_{\mathbb{Z}}A).$$

Obviously, $L(A)$, $R(A)$ and $M(A)$ are associative rings with units and we consider A as a left module over these rings.

In particular we form the subcategory $\sigma[_{M(A)}A]$ of $M(A)$*-Mod* and denote it by $\sigma[A]$. $c(A) = End_{M(A)}(A)$ is the *centroid* of A. From [2] or [12], Theorem 4.2 we recall for semiprime rings:

9.1 Central closure of semiprime rings.
Let A be a semiprime ring, $\widehat{A}$ the injective hull of A in $\sigma[A]$ and $T = End_{M(A)}(\widehat{A})$. Then

(1) A is a polyform $M(A)$-module.

(2) T is a commutative, regular and self-injective ring.

(3) $\widehat{A} = AT$ is a semiprime ring with respect to the multiplication

$$(\sum a_i s_i)(\sum b_j t_j) = \sum (a_i b_j)(s_i t_j) \ \text{ for any finite } a_i, b_j \in A,\ s_i, t_j \in T$$

and its centroid is T.

T is called the *extended centroid* of A, and the ring $\widehat{A}$ the *central closure* of A.

Definition. A ring A is said to be *left PSP* if A is PSP a an $L(A)$-module. A is called a *PSP ring* if it is PSP a an $M(A)$-module.
Similarly we define SSP and *left (right) SSP* rings.

Obviously, every SSP ring is PSP.
Rings A which are strongly prime as $M(A)$-modules are exactly those prime rings whose central closures are simple rings (cf. [13], Theorem 3.2). In particular they are SSP rings.
Now let us see which PSP rings are semiprime. For this recall the definition of the
associator $(a, b, c) = (ab)c - a(bc)$ for $a, b, c \in A$ and the
right nucleus $n(A) = \{c \in A \mid (x, y, c) = 0 \text{ for all } x, y \in A\}$.
Clearly, $R_c \in End_{L(A)}(A)$ for all $c \in n(A)$.

9.2 *PSP* and semiprime rings.
Let A be any ring.

(1) Assume A is PSP and no non-zero ideal of A annihilates A. Then A is a semiprime ring.

(2) Assume A is left PSP and no non-zero left ideal of A annihilates the right nucleus $n(A)$. Then A is a semiprime ring.

(3) Assume A is (left) strongly prime and no ideal of A annihilates A. Then A is a prime ring.

Proof. (1) Assume $K \subset A$ is a non-zero finitely generated ideal with $K^2 = 0$. Consider $U = \{L_a \mid a \in K\}$. By assumption, $A/\mathcal{T}_K(A) \in \sigma[K]$. Now $UK = 0$ implies $U(A/\mathcal{T}_K(A)) = 0$ and $KA \subset \mathcal{T}_K(A)$. So $KA \subset \mathcal{T}_K(A) \cap K = 0$, contradicting the given condition.

(2) In the proof above, let K be a left ideal. Then $Kn(A) \subset \mathcal{T}_K(A) \cap \mathcal{T}^K(A) = 0$, contradicting the assumption in (2).

(3) Consider non-zero ideals $I, J \subset A$ with $IJ = 0$. Put $U = \{L_a \mid a \in I\}$. By assumption, $A \in \sigma[J]$ and hence $IA = UA = 0$, a contradiction. □

In general, a strongly prime ring need not be prime. For this consider the cyclic group $\mathbb{Z}_p$ of prime order p with the trivial multiplication $ab = 0$, for all $a, b \in \mathbb{Z}_p$. This is a strongly prime (simple) ring which is not prime.

For associative rings we notice a relationship between the one-sided and two-sided versions of the notions above:

9.3 Associative PSP rings.
Let A be an associative semiprime ring with unit.

(1) In $\sigma[A]$, $\mathcal{T}_U(A) = An_A(U)$ for any ideal $U \subset A$.

(2) If A is a PSP ring, then A is left PSP.

(3) If A is an SSP ring, then A is left SSP.

(4) If A is a strongly prime ring, then A is left strongly prime.

Proof. (1) $\mathcal{T}_U(A)$ is an ideal in A and $\mathcal{T}_U(A) \cap U = 0$, hence $\mathcal{T}_U(A) \subset An_A(U)$.
Consider $f \in Hom_{M(A)}(An_A(U), I_A(U))$ and put $K = (U)f^{-1}$. Then $(Kf)^2 \subset U(Kf) = (UK)f \subset (UAn_A(U))f = 0$. Since A is semiprime, $Kf = 0$ and so $Im\, f \cap U = 0$, implying $f = 0$. Hence $\mathcal{T}_U(A) \supset An_A(U)$ (compare 8.1).

(2) Consider $a, b \in A$. Since A is PSP, there exist $x_1, \dots, x_k \in M(A)$ such that

$$An_{M(A)}(x_1 a, \dots, x_k a)b \subset \mathcal{T}_{M(A)a}(A) = An_A(M(A)a) \subset An_A(L(A)a).$$

In particular,

$$An_{L(A)}(x_1 a, \dots, x_k a)b \subset An_A(L(A)a).$$

Now assume $x_i = \sum_{j=1}^{m_i} L_{y_{ij}} R_{z_{ij}}$. Then

$$An_{L(A)}(\{y_{ij}a \,|\, 1 \le j \le m_i, 1 \le i \le k\})b \subset An_A(L(A)a).$$

By 8.1, $\mathcal{T}_{L(A)a}(A) = An_A(L(A)a)$. Hence by 5.1, the above relation implies that A is left PSP.

(3) and (4) are shown in a similar way. □

9.4 Strongly semiprime rings.
Let A be a ring which is not annihilated by any non-zero ideal and $T = End_{M(A)}(\widehat{A})$. Then the following conditions are equivalent:

(a) A is an SSP ring;

(b) A is semiprime and for every essential ideal $U \subset A$, $A \in \sigma[U]$;

(c) for every ideal $I \subset A$, $\widehat{A} = IT \oplus \mathcal{T}_I(\widehat{A})$.

(d) A is semiprime and the central closure $\widehat{A}$ is a direct sum of simple ideals.

If A is associative, then (a)-(d) are equivalent to:

(e) A is semiprime and for every ideal $I \subset A$, $A/An_A(I) \in \sigma[I]$;

Proof. By 9.2, A being SSP implies that A is semiprime.

$(a) \Leftrightarrow (b)$ Since A is semiprime, A is a polyform $M(A)$-module by 9.1. Hence the assertion follows from 7.1.

$(a) \Leftrightarrow (c)$ This is also obtained from 7.1.

$(a) \Leftrightarrow (d)$ The central closure $\widehat{A}$ is a direct sum of simple ideals if and only if it is semisimple as an $(M(A), T)$-bimodule. Now apply 4.5.

$(a) \Leftrightarrow (e)$ By 9.2, R is a semiprime ring. Hence by 9.3, $\mathcal{T}_U(A) = An_A(U)$. Now the equivalence is clear by 4.3. □

9.5 Idempotent closure of semiprime rings.
Let A be a semiprime ring, $T = End_{M(A)}(\widehat{A})$, B the Boolean ring of idempotents of T and $\widetilde{A} = AB$ the idempotent closure of A as an $M(A)$-module. Then

(1) For every $a \in \widetilde{A}$, there exist $a_1, \dots, a_k \in A$ and pairwise orthogonal $e_1, \dots, e_k \in B$ such that
(i) $a = \sum_{i=1}^k a_i e_i$,
(ii) $e_i = \varepsilon(a_i)e_i$ for $i = 1, \dots, k$ and
(iii) $\varepsilon(a) = \sum_{i=1}^k e_i$.

(2) For every prime ideal $K \subset \widetilde{A}$, $P = K \cap A$ is a prime ideal in A and

$$\widetilde{A}/K = (A + K)/K \simeq A/P.$$

The set $x = \{e \in B | \widetilde{A}e \subset K\}$ is a maximal ideal in B and $K = PB + \widetilde{A}x$.

(3) For any prime ideal $P \subset A$, there exists a prime ideal $K \subset \widetilde{A}$ with $K \cap A = P$.

Proof. (1) Since a semiprime ring A is polyform as an $M(A)$-module, the assertion follows immediately from 3.5.

(2) Consider two ideals $I, J \subset A$ with $IJ \subset P$. Then IB and JB are ideals in $\widetilde{A}$ and $(IB)(JB) \subset K$. Hence $IB \subset K$ or $JB \subset K$. Assume $IB \subset K$. Then $I \subset IB \subset K \cap A \subset P$. So P is a prime ideal in A.

Consider $a \in A$ and $e \in B$. Since $(\widetilde{A}e)[\widetilde{A}(1-e)] = 0$, $\widetilde{A}e \subset K$ or $\widetilde{A}(1-e) \subset K$. Hence $ae \in K$ or $a(1-e) \in K$ which means $ae + K = K$ or $ae + K = a + K$.

Therefore for any $d \in \widetilde{A}$, there exists $x \in A$, with $d + K = x + K$. This implies $\widetilde{A}/K = (A+K)/K \simeq A/P$.

A straightforward argument shows that x is a maximal ideal in the Boolean ring B.

Put $U = PB + \widetilde{A}x$. Clearly $U \subset K$. For any $a \in K$, choose $a_1, \dots, a_k \in A$ and orthogonal idempotents $e_1, \dots, e_k \in B$ satisfying the conditions (*ii*) and (*iii*) of (1). In case all $e_1, \dots, e_k \in x$, then $a \in \widetilde{A}x \subset U$.

Assume, without restriction, $e_1 \notin x$. Since $e_ie_1 = 0$ for $i \neq 1$, we have $e_2, \dots, e_k \in K$. Also $1 - e_1 \in x$. Therefore

$$a_1 = a + a_1(1 - e_1) \in K + \tilde{A}x = K$$

and $a_1 \in P$. Consequently, $a_1e_1 \in PB$ and

$$a = a_1e_1 + \sum_{i=2}^{k} a_ie_i \in PB + \tilde{A}x.$$

So in any case $a \in U$, implying $K = U$.

(3) Put $S = A \setminus P$. Obviously, for any $a, b \in S$,

$$\emptyset \neq (M(A)a)(M(A)b) \cap S \subset (M(\widehat{A})a)(M(\widehat{A})b) \cap S.$$

Using this relationship we can show (as in the associative case) that an ideal $K \subset \tilde{A}$, which is maximal with respect to $K \cap S = \emptyset$, is a prime ideal.

Clearly $I = K \cap A \subset P$. As shown above, $\tilde{A}/K = (A + K)/K \simeq A/I$. Obviously, P/I is a prime ideal in A/I. Hence there exists a prime ideal $J \supset K$ of $\tilde{A}$, for which $J/K = (P + K)/K$. This means $J = P + K$, $J \cap A = P + K \cap A = P + I = P$ and $J \cap S = \emptyset$. By the choice of K we conclude $K = J$. □

9.6 Properly semiprime rings.

Let A be a semiprime ring, $T = End_{M(A)}(\widehat{A})$, B the Boolean ring of idempotents of T and $\tilde{A} = AB$ the idempotent closure of A. Then the following are equivalent:

(a) A is a PSP ring;

(b) for every $a \in A$, $M(A)aT = \widehat{A}\varepsilon(a)$;

(c) for every finitely generated ideal $K \subset A$, $KT = \widehat{A}\varepsilon(K)$;

(d) $\tilde{A}$ is a PSP ring.

Under the given conditions, for every $a \in A$, $T_a(\widehat{A}) = \widehat{A}(id - \varepsilon(a))$.

Proof. Since semiprime rings are polyform as bimodules, these equivalences essentially are obtained from 7.3.

We only have to show that (d), i.e. $\tilde{A}$ is PSP as an $M(\tilde{A})$-module, is equivalent to $\tilde{A}$ being PSP as an $M(A)$-module. This follows readily from $M(\tilde{A}) = M(A)B$. □

A ring A is said to be ***fully idempotent*** if, for any ideal $I \subset A$, $I = M(A)I^2$. Clearly such rings are semiprime. We show a module property of them:

9.7 Fully idempotent rings.

Let A be a fully idempotent ring. Then A is pseudo regular as $M(A)$-module.

Proof. Put $S = End_{M(A)}(A)$ and $T = End_{M(A)}(\widehat{A})$. For any subset $U \subset A$, define $L_o(U) = \{L_u \mid u \in U\}$. Consider $a \in A$ and $I = M(A)a$. By assumption, $I = M(A)I^2 = M(A)L_o(I)I$ and hence $a \in M(A)L_o(I)M(A)a$. Therefore there exists $\alpha \in M(A)L_o(I)M(A)$ with $\alpha a = a$. Obviously, $\alpha \in M(A)$ and $\alpha A \subset I \subset M(A)aT$. So A is a pseudo regular $M(A)$-module (see 6.1). □

A ring A is called *biregular* if every principal ideal of A is a direct summand (as an $M(A)$-module).

If A has a unit, then it is biregular if and only if every principal ideal is generated by an idempotent in the center of A. For simplicity here we only consider biregular rings with units, though our methods also apply to the general case.

The next result reveils the connection between biregular and PSP rings.

9.8 Biregular and PSP rings.

Let A be a semiprime ring with unit, $T = End_{M(A)}(\widehat{A})$, B the Boolean ring of idempotents of T and $\tilde{A} = AB$ the idempotent closure of A. Then the following are equivalent:

(a) A is biregular;

(b) A is a fully idempotent PSP ring;

(c) A is semiprime and for every $a \in A$, $M(A)a = A\varepsilon(a)$;

(d) A is semiprime and $\tilde{A}$ is biregular.

Proof. (a) ⇒ (b) Assume A is biregular. Then A is fully idempotent.

For any $a \in A$, $M(A)a = Ae$ for some idempotent $e \in C$. e extends to a unique idempotent $\bar{e} \in T$ and $M(A)aT = AT\bar{e} = \widehat{A}\bar{e}$. As easily checked, $\bar{e} = \varepsilon(a)$. So A is PSP by 9.6.

(b) ⇒ (a) By 9.7, fully idempotent rings are pseudo regular $M(A)$-modules. Hence the assertion follows from 6.3.

(b) ⇔ (c) Since A is polyform, these assertions follow from 6.3.

(b) ⇒ (d) Since A is polyform, $\tilde{A}$ is PSP by 7.3. We show that $\tilde{A}$ is fully idempotent. Any element in $\tilde{A}$ can be written as $a = \sum_{i=1}^{k} a_ie_i$, for some $a_i \in A$ and pairwise orthogonal $e_i \in B$. Put $K = M(\tilde{A})a$ and $K_i = M(A)a_i$. Clearly $K = \sum_{i=1}^{k} K_ie_iB$. Since A is fully idempotent, $K_i = M(A)K_i^2$. Hence

$$M(\tilde{A})K^2 = \sum_{i=1}^{k} M(A)K_i^2e_iB = \sum_{i=1}^{k} K_ie_iB = K,$$

and $\tilde{A}$ is fully idempotent. So $\tilde{A}$ is biregular by $(a) \Leftrightarrow (b)$.

$(d) \Rightarrow (a)$ Consider $a \in A$. By assumption, $M(\tilde{A})a = \tilde{A}\varepsilon(a)$. Obviously, $M(\tilde{A}) = M(A)B$ and $M(\tilde{A})a = M(A)aB$. Put $c = \varepsilon(a)$. Clearly $\varepsilon(c) = c$ and $c \in M(A)aB$. By 9.5, there exist $a_1, \ldots, a_k \in M(A)a$ and pairwise orthogonal $e_1, \ldots, e_k \in B$ such that

$$c = \sum_{i=1}^{k} a_i e_i \quad \text{and} \quad c = \varepsilon(c) = \sum_{i=1}^{k} e_i.$$

Therefore $ce_i = a_i e_i$, $ce_i = e_i$ and $(1 - a_i)e_i = 0$ for all $i = 1, \ldots, k$. Put

$$b = (...(((1 - a_1)(1 - a_2))(1 - a_3)...)(1 - a_k).$$

Clearly $be_i = 0$ for all $i = 1, \ldots, k$. Hence $bc = 0$. Since $a_i \in M(A)a$, $b = 1 + d$ where $d \in M(A)a$. Further since $ac = a\varepsilon(a) = a$ and $d \in M(A)a$, $dc = d$. So $0 = bc = (1+d)c = c + dc = c + d$ and $\varepsilon(a) = c = -d \in M(A)a$. Hence $M(A)a \subset A\varepsilon(a) \subset M(A)a$ and $M(A)a = A\varepsilon(a)$. □

Combining our results 7.5, 7.7 and 3.2 in [14] we have:

9.9 Central closure as Azumaya ring.

Let A be a semiprime ring with unit, $T = End_{M(A)}(\widehat{A})$ and $\widehat{A}$ the central closure of A. Then the following are equivalent:

(a) $\widehat{A}$ is an Azumaya ring;

(b) $\widehat{A}$ is a PSP ring and the module ${}_{M(\widehat{A})}\widehat{A}$ is finitely presented in $\sigma[\widehat{A}]$;

(c) $\widehat{A}$ is a biregular ring and the module ${}_{M(\widehat{A})}\widehat{A}$ is projective in $\sigma[\widehat{A}]$;

(d) the module ${}_{M(\widehat{A})}\widehat{A}$ is a generator in $\sigma[\widehat{A}]$;

(e) $M(\widehat{A})$ is a dense subring of $End_T(\widehat{A})$;

(f) for any $m_1, \ldots, m_n, m \in A$ there exist $r_1, \ldots, r_k \in M(A)$ such that for $s_1, \ldots, s_n \in M(A)$ the relations

$$\sum_{l=1}^{n} s_l r_j m_l = 0 \quad \text{for } j = 1, \ldots, k,$$

imply $s_1 m \in T_{Um_1}(A)$ for $U = An_{M(A)}(m_2, \ldots, m_n)$.

(g) A is a PSP ring and for any $m_1, \ldots, m_n \in A$ there exist $r_1, \ldots, r_k \in M(A)$ such that for $s_1, \ldots, s_n \in M(A)$ the relations

$$\sum_{l=1}^{n} s_l r_j m_l = 0 \quad \text{for } j = 1, \ldots, k,$$

imply $s_1 m_1 \in T_{Um_1}(A)$ for $U = An_{M(A)}(m_2, \ldots, m_n)$.

9.10 Example

We construct a biregular PI-ring A with unit, whose central closure is not biregular. So A is PSP as an $M(A)$-module but $\widehat{A}$ is not PSP.

Let F be field of non-zero characteristic p, $K = F(X)$ the field of the rational functions and $G = F(X^p)$ a subfield of K. Consider the embedding

$$\alpha : K \to M_p(G) = End_G(K), \quad a \mapsto L_a.$$

Denote by $Q = M_p(G)^\omega$ a countable product of copies of $M_p(G)$, and $I = M_p(G)^{(\omega)}$. We have an embedding

$$\psi : K \to M_p(G)^\omega \subset Q, \quad \psi(a) = \{\alpha(a)\}_\omega \in M_p(G)^\omega \text{ for } a \in K.$$

Clearly I is an ideal in Q. Put $A = I + \psi(K)$. Obviously, A is a biregular ring with unit, whose maximal right ring of quotients is Q, and the center T of Q is the extended centroid of A.

Hence $R = AT$ is the central closure of A. I is an ideal in R and R/I is a commutative ring. Let $y \in Q$ denote an element all whose coefficients are equal to X and put $z = \psi(X)$. Then $y^p = z^p$. Therefore $y - z + I$ is a non-zero nilpotent element in the commutative ring R/I. Hence R cannot be biregular. □

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ENVELOPING ALGEBRAS OF INFINITE DIMENSIONAL LIE ALGEBRAS AND LIE SUPERALGEBRAS

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ABSTRACT. In this expository paper we examine the enveloping algebras of infinite dimensional Lie algebras and Lie superalgebras. Our goal is to reduce certain natural ring-theoretic questions from the arbitrary infinite dimensional situation to the more familiar finite-dimensional one. This is done by applying techniques analogous to the Δ-methods previously used to solve questions on group algebras.

1. Introduction

In this expository paper we examine the enveloping algebras of infinite dimensional Lie algebras and Lie superalgebras. Our goal is to reduce certain natural ring-theoretic questions from the arbitrary infinite dimensional situation to the more familiar finite-dimensional one. The proofs of the results in this paper appear in a series of fairly technical papers [3], [4], [5], and [6].

Throughout Sections 1, 2, and 3 of this paper L will be either a Lie algebra or a Lie superalgebra over a field K of characteristic 0 and $U(L)$ will denote its enveloping algebra. When L is a Lie algebra $U(L)$ is a Hopf algebra, whereas when L is a Lie superalgebra, there is a group G of order 2 such that the skew group ring $U(L)\#G$ is a Hopf algebra. Group algebras are a well known example of Hopf algebras and many questions on group algebras have been solved used Δ-methods [9]. Therefore, it was reasonable to try to find similar techniques in the Lie context. To this end, we considered

$$\Delta = \Delta(L) = \{l \in L \mid \dim_K[l, L] < \infty\},$$

which was first introduced in [1]. Δ can be viewed as the Lie analog of the finite-conjugate center of a group.

If we let Δ_L denote the join of all the finite-dimensional Lie ideals of L, then Δ_L is a characteristic ideal of L which is appreciably smaller than Δ. In [3] and [4], we reduce questions from L down to Δ and in [5] and [6] we sharpen these results by further reducing from Δ to Δ_L. Unfortunately, there does not appear to be any way to directly reduce from L to Δ_L without using Δ.

The reduction from L to Δ_L takes place by examining linear and derivation identities. Given any ring R, a linear identity for R is an equation of the form

$$\alpha_1 x\beta_1 + \alpha_2 x\beta_2 + \cdots + \alpha_n x\beta_n = 0$$