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# Coalgebras and Bialgebras

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# Preface

The notion of a coalgebra is dual to the notion of an algebra. Nevertheless both of them are related to similar objects in linear algebra. Coalgebras over fields are well-studied in the literature, e.g. in the texts of Sweedler [9], Abe [1], Montgomery [8], and Dăscălescu, Năstăsescu and Raianu [4].

To understand coalgebras over commutative rings is not only of interest for the sake of generalisation but also to provide techniques for the study of such structures over noncommutative rings - the corings. In these lectures we concentrate on coalgebras over commutative rings but the methods provided will readily apply to the more general case.

There are parts of module theory over algebras $A$ that provide a useful framework for the theory of comodules. Given any left $A$-module $M$, denote by $\sigma[M]$ the full subcategory of the category ${}_A\mathbf{M}$ of left $A$-modules that is subgenerated by $M$. This is the smallest Grothendieck subcategory of ${}_A\mathbf{M}$ containing $M$. Internal properties of $\sigma[M]$ strongly depend on the module properties of $M$ and this can be used as key for a homological classification of modules, that is, a characterisation of the structural properties of $M$ by the properties of the category $\sigma[M]$.

On the other hand, by definition, $\sigma[M]$ is closed under direct sums, submodules and factor modules in ${}_A\mathbf{M}$, and so it is a *hereditary pretorsion class* in ${}_A\mathbf{M}$. Hence torsion theory provides a setting for studying the *outer properties* of $\sigma[M]$, that is, the behaviour of $\sigma[M]$ as a subclass of ${}_A\mathbf{M}$.

Both the inner and outer properties of the categories of type $\sigma[M]$ are important in the study of coalgebras and comodules and it is the purpose of these lectures to make this clear. To begin with we present material from algebra and module theory in a form that is immediately applicable to comodules. Then a solid introduction to coalgebras and their comodules is given.

Portions of the notes are taken from the joint book with Tomasz Brzeziński [3] and I would like to use this opportunity to express my appreciation for his significant contributions to the subject.

# Chapter 1

# Modules and Algebras

## 1 Tensor product, tensor functor

The tensor product of modules is an important notion both for the theory of algebras and coalgebras. To ensure the generality desired for our purposes we give an account of the construction of the tensor product over non-commutative rings $R$. Notice that for some notions and results $R$ need not have a unit.

**1.1 Definition.** Let $M_R$ be a right module, ${}_RN$ a left module over the ring $R$ and $G$ an abelian group.

A $\mathbb{Z}$-bilinear map $\beta : M \times N \to G$ is called *$R$-balanced* if, for all $m \in M$, $n \in N$ and $r \in R$, we have: $\beta(mr, n) = \beta(m, rn)$.

An abelian group $T$ with an $R$-balanced map $\tau : M \times N \to T$ is called the *tensor product of $M$ and $N$* if every $R$-balanced map

$$\beta : M \times N \to G,\ G \text{ an abelian group },$$

can be uniquely factorized over $\tau$, i.e., there is a unique $\mathbb{Z}$-homomorphism $\gamma : T \to G$ which renders the following diagram commutative:

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[d, "\tau"'] & G\,. \\ T \arrow[ru, "\gamma"'] & \end{tikzcd}$$

With standard arguments applied for universal constructions it is easily seen that the tensor product $(T, \tau)$ for a pair of modules $M_R$, ${}_RN$ is uniquely determined up to isomorphism (of $\mathbb{Z}$-modules).

**1.2. Existence of tensor products.** For the $R$-modules $M_R$, ${}_RN$, we form the direct sum of the family of $\mathbb{Z}$-modules $\{\mathbb{Z}_{(m,n)}\}_{M\times N}$ with $\mathbb{Z}_{(m,n)} \simeq \mathbb{Z}$, the free $\mathbb{Z}$-module over $M \times N$,

$$F = \bigoplus_{M\times N} \mathbb{Z}_{(m,n)} \simeq \mathbb{Z}^{(M\times N)}.$$

By construction, there is a (canonical) basis $\{f_{(m,n)}\}_{M\times N}$ in $F$. We simply write $f_{(m,n)} = [m, n]$. Let $K$ denote the submodule of $F$ generated by elements of the form

$$[m_1 + m_2, n] - [m_1, n] - [m_2, n], \quad [m, n_1 + n_2] - [m, n_1] - [m, n_2],$$

$$[mr, n] - [m, rn], \text{ with } m, m_i \in M, \; n, n_i \in N, \; r \in R.$$

Putting $M \otimes_R N := F/K$ we define the map

$$\tau : M \times N \to M \otimes_R N, \quad (m, n) \mapsto m \otimes n := [m, n] + K \; .$$

By definition of $K$, the map $\tau$ is $R$-balanced. Observe that $\tau$ is not surjective but the image of $\tau$, Im $\tau = \{m \otimes n \mid m \in M, \; n \in N\}$, is a generating set of $M \otimes_R N$ as a $\mathbb{Z}$-module.

If $\beta : M \times N \to G$ is an $R$-balanced map we obtain a $\mathbb{Z}$-homomorphism $\tilde{\gamma} : F \to G$, $[m, n] \mapsto \beta(m, n)$, and obviously $K \subset Ke\,\tilde{\gamma}$. Hence $\tilde{\gamma}$ factorizes over $\tau$ and we have the commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[d, "\tau"'] & G \, . \\ M \otimes_R N \arrow[ur, "\gamma"'] & \end{tikzcd}$$

$\gamma$ is unique since its values on the generating set Im $\tau$ of $T$ are uniquely determined.

Observe that every element in $M \otimes_R N$ can be written as a finite sum

$$m_1 \otimes n_1 + \cdots + m_k \otimes n_k.$$

However this presentation is not unique. $m \otimes n$ only represents a coset and $m, n$ are not uniquely determined. Also a presentation of zero in $M \otimes_R N$ is not unique. We may even have that $M \otimes_R N$ is zero for non-zero $M$ and $N$, e.g. $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = 0$.

**1.3. Tensor product of homomorphisms.** *Consider two $R$-homomorphisms $f : M_R \to M'_R$ and $g : {}_RN \to {}_RN'$.*

(1) *There is a unique $\mathbb{Z}$-linear map*

$$f \otimes g : M \otimes_R N \to M' \otimes_R N',$$

$$\textit{with } \; f \otimes g(m \otimes n) = f(m) \otimes g(n), \; m \in M, \, n \in N.$$

*$f \otimes g$ is called the* tensor product of the homomorphisms *$f$ and $g$.*

(2) *If $f$ and $g$ are surjective, then $f \otimes g$ is surjective and*

$$\mathrm{Ke}\, f \otimes g = \mathrm{Ke}\, f \otimes' N + M \otimes' \mathrm{Ke}\, g,$$

*where $\mathrm{Ke}\, f \otimes' N$ denotes the submodule of $M \otimes_R N$ generated by the elements $u \otimes n$ with $u \in \mathrm{Ke}\, f$ and $n \in N$, etc.*

**Proof.** (1) Define a map

$$f \times g : M \times N \to M' \otimes_R N', \quad (m,n) \mapsto f(m) \otimes g(n).$$

It is $\mathbb{Z}$-bilinear and $R$-balanced since $f(mr) \otimes g(n) = f(m) \otimes g(rn)$. Hence the map $f \times g$ can be factorized over $M \otimes_R N$ and we obtain the desired map $f \otimes g$.

(2) If $f$ and $g$ are surjective, then obviously $M' \otimes_R N'$ is generated as $\mathbb{Z}$-module by the elements $\{f(m) \otimes g(n) \,|\, m \in M,\, n \in N\}$ and hence $f \otimes g$ is surjective.

It is clear that the subgroup $H = \operatorname{Ke} f \otimes' N + M \otimes' \operatorname{Ke} g$ lies in $\operatorname{Ke} f \otimes g$ and hence, with the canonical projection $p$, the map $f \otimes g$ factors into

$$M \otimes_R N \xrightarrow{p} M \otimes_R N/H \xrightarrow{\nu} M' \otimes_R N'.$$

Obvioulsy $\nu$ is surjective. To show that $\nu$ is an isomorphism we first consider a map

$$\alpha : M' \times N' \to M \otimes_R N/H,\ (m',n') \mapsto m \otimes n + H,\ \text{where } m' = f(m),\ n' = g(n).$$

To see that the map is well-defined, i.e., that it is independent of the choice of $m \in M$ with $f(m) = m'$, and similarly of $n \in N$, observe that for $f(m_1) = m'$ and $g(n_1) = n'$ we have

$$m_1 \otimes n_1 = m \otimes n + [(m_1 - m) \otimes n + m \otimes (n_1 - n) + (m_1 - m) \otimes (n_1 - n)],$$

where the right summand $[\ldots]$ lies in $H$. Clearly $\nu$ is $\mathbb{Z}$-linear and $R$-balanced and hence induces a map $\bar{\alpha} : M' \otimes_R N' \to M \otimes_R N/H$ with $\bar{\alpha} \circ \nu = I$, proving that $\nu$ is injective. $\square$

**1.4. Tensor product and direct sums.** *Let $M_R$ be an $R$-module and ${}_RN = \bigoplus_\Lambda N_\lambda$, with the canonical injections $\varepsilon_\lambda : {}_RN_\lambda \to {}_RN$ and projections $\pi_\lambda : {}_RN \to {}_RN_\lambda$.*

*Then $(M \otimes_R N, I_M \otimes \varepsilon_\lambda)$ is a direct sum of $\{M \otimes_R N_\lambda\}_\Lambda$, i.e.,*

$$M \otimes_R (\bigoplus\nolimits_\Lambda N_\lambda) \simeq \bigoplus\nolimits_\Lambda (M \otimes_R N_\lambda).$$

*We say the tensor product* commutes with direct sums*.*

**Proof.** For the maps $I_M \otimes \pi_\lambda : M \otimes_R N \to M \otimes_R N_\lambda$, we derive from properties of tensor products of homomorphisms

$$(I_M \otimes \varepsilon_\lambda)(I_M \otimes \pi_\mu) = \delta_{\lambda\mu} I_{M \otimes_R N_\lambda}.$$

For a family $\{f_\lambda : M \otimes_R N_\lambda \to X\}_\Lambda$ of $\mathbb{Z}$-linear maps, we define

$$f : M \otimes_R N \to X, \quad m \otimes n \mapsto \sum\nolimits_{\lambda \in \Lambda} f_\lambda(I_M \otimes \pi_\lambda(m \otimes n)),$$

where the sum is always finite.

Obviously, $f \circ (I_M \otimes \varepsilon_\lambda) = f_\lambda$ and $(M \otimes_R N, I_M \otimes \varepsilon_\lambda)$ is a direct sum of the $\{M \otimes_R N_\lambda\}_\Lambda$.

By symmetry, we obtain, for $M_R = \bigoplus_{\Lambda'} M_\mu$,

$$(\textstyle\bigoplus_{\Lambda'} M_\mu) \otimes_R N \simeq \bigoplus_{\Lambda'} (M_\mu \otimes_R N),$$
$$(\textstyle\bigoplus_{\Lambda'} M_\mu) \otimes_R (\bigoplus_\Lambda N_\lambda) \simeq \bigoplus_{\Lambda' \times \Lambda} (M_\mu \otimes N_\lambda).$$

□

**1.5. Module structure of tensor products.** By construction, the tensor product $M \otimes_R N$ of $M_R$ and ${}_RN$ is only an abelian group. However, if ${}_TM_R$ or ${}_RN_S$ are bimodules, then we may define module structures on $M \otimes_R N$:

If ${}_TM_R$ is a $(T, R)$-bimodule, then the elements of $T$ may be regarded as $R$-endomorphisms of $M$ and the tensor product with $I_N$ yields a map

$$T \to \mathrm{End}_{\mathbb{Z}}(M \otimes_R N), \quad t \mapsto t \otimes I_N.$$

From the properties of this construction noted in 1.3 we see that this is a ring homomorphism. Hence ${}_TM \otimes_R N$ becomes a *left $T$-module* and the action of $t \in T$ on $\sum m_i \otimes n_i \in M \otimes_R N$ is given by

$$t(\sum m_i \otimes n_i) = \sum (tm_i) \otimes n_i.$$

For an $(R, S)$-bimodule ${}_RN_S$, we obtain in the same way that $M \otimes_R N_S$ is a right $S$-module.

If ${}_TM_R$ and ${}_RN_S$ are bimodules, the structures defined above turn ${}_TM \otimes_R N_S$ into a $(T, S)$-bimodule since we have, for all $t \in T$, $s \in S$ and $m \otimes n \in M \otimes_R N$, that $(t(m \otimes n))s = (tm) \otimes (ns) = t((m \otimes n)s)$.

**1.6. Tensor product with *R*.** Regarding $R$ as an $(R, R)$-bimodule, for every $R$-module ${}_RN$, there is an $R$-isomorphism

$$\mu_R : R \otimes_R N \to RN, \ \sum r_i \otimes n_i \mapsto \sum r_i n_i.$$

The map exists since the map $R \times N \to RN$, $(r, n) \mapsto rn$ is balanced, and obviously has the given properties.

Since the tensor product commutes with direct sums (see 1.4), we obtain, for every free right $R$-module $F_R \simeq R_R^{(\Lambda)}$, $\Lambda$ an index set, a $\mathbb{Z}$-isomorphism $F \otimes_R N \simeq N^{(\Lambda)}$.

**1.7. Associativity of the tensor product.** *Assume three modules $M_R$, ${}_RN_S$ and ${}_SL$ to be given. Then $(M \otimes_R N) \otimes_S L$ and $M \otimes_R (N \otimes_S L)$ can be formed and there is an isomorphism*

$$\sigma : (M \otimes_R N) \otimes_S L \to M \otimes_R (N \otimes_S L), \ \ (m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)\,.$$

**Proof.** We only have to show the existence of such a map $\sigma$. Then, by symmetry, we obtain a corresponding map in the other direction which is inverse to $\sigma$:
We first define, for $l \in L$, a morphism $f_l : N \to N \otimes_S L,\ n \mapsto n \otimes l$, then form the tensor product $I_M \otimes f_l : M \otimes_R N \to M \otimes_R (N \otimes_S L)$ and obtain

$$\beta : (M \otimes_R N) \times L \to M \otimes_R (N \otimes_S L),\ \ (m \otimes n, l) \mapsto I_M \otimes f_l\,(m \otimes n).$$

It only remains to verify that $\beta$ is balanced to obtain the desired map. □

**1.8. Tensor functors.** *For an (S,R)-bimodule ${}_SU_R$, the assignments*

$$\begin{array}{rrcll} {}_SU \otimes_R - : & \mathrm{Obj}(R\text{-Mod}) & \longrightarrow & \mathrm{Obj}(S\text{-Mod}), & {}_RM \mapsto {}_SU \otimes_R M, \\ & \mathrm{Mor}(R\text{-Mod}) & \longrightarrow & \mathrm{Mor}(S\text{-Mod}), & f \mapsto I_U \otimes f, \end{array}$$

*yield a covariant functor* ${}_SU \otimes_R - : R\text{-Mod} \to S\text{-Mod}$ *with the properties*

*(1)* ${}_SU \otimes_R -$ *is additive and right exact;*

*(2)* ${}_SU \otimes_R -$ *preserves direct sums.*

*Similarly we obtain a functor* $- \otimes_S U_R : \text{Mod-}S \to \text{Mod-}R$ *with the same properties.*

**Proof.** Applying 1.3 it is easily checked that the given assignments define an additive functor. In 1.4 we have seen that it preserves direct sums. It remains to show that it is right exact. An exact sequence $K \xrightarrow{f} L \xrightarrow{g} N \to 0$ in $R$-Mod yields the sequence

$$U \otimes_R K \xrightarrow{I\otimes f} U \otimes_R L \xrightarrow{I\otimes g} U \otimes_R N \longrightarrow 0\,,$$

where, by 1.3(2), $I \otimes g$ is surjective and

$$\mathrm{Ke}\ I \otimes g = U \otimes' \mathrm{Ke}\ g = U \otimes' \mathrm{Im}\ f = \mathrm{Im}\, I \otimes f,$$

proving the exactness of this sequence. □

Notice that the relation between tensor products and *direct products* is more complicated than that between tensor products and direct sums.

**1.9. Tensor product and direct products.** *Let $U_R$ be a right $R$-module and $\{L_\lambda\}_\Lambda$ a family of left $R$-modules. With the canonical projections we have the maps*

$$I_U \otimes \pi_\mu : U \otimes_R (\prod\nolimits_\Lambda L_\lambda) \to U \otimes_R L_\mu$$

*and, by the universal property of the product,*

$$\varphi_U : U \otimes_R (\prod\nolimits_\Lambda L_\lambda) \to \prod\nolimits_\Lambda U \otimes_R L_\lambda,\ \ u \otimes (l_\lambda)_\Lambda \mapsto (u \otimes l_\lambda)_\Lambda.$$

*It is easy to see that, for $U = R$, and hence also for $U = R^n$, $\varphi_U$ is an isomorphism.*

*(1) The following assertions are equivalent:*

*(a) $U$ is finitely generated;*

*(b) $\varphi_U$ is surjective for every family $\{L_\lambda\}_\Lambda$;*

*(c) $\tilde{\varphi}_U : U \otimes R^\Lambda \to (U \otimes R)^\Lambda \simeq U^\Lambda$ is surjective for any set $\Lambda$ (or $\Lambda = U$).*

*(2) The following assertions are also equivalent:*

*(a) $U$ is finitely presented in $R$-*Mod*;*

*(b) $\varphi_U$ is bijective for every family $\{L_\lambda\}_\Lambda$;*

*(c) $\tilde{\varphi}_U : U \otimes_R R^\Lambda \to U^\Lambda$ is bijective for every set $\Lambda$.*

**Proof.** (1) (a) $\Rightarrow$ (b) If $U$ is finitely generated and $R^{(A)} \xrightarrow{f} R^n \xrightarrow{g} U \to 0$ is exact, we can form the commutative diagram with exact rows:

$$\begin{tikzcd}
R^{(A)} \otimes \prod_\Lambda L_\lambda \arrow[r, "f\otimes I"] \arrow[d, "\varphi_{R^{(A)}}"] & R^n \otimes \prod_\Lambda L_\lambda \arrow[r, "g\otimes I"] \arrow[d, "\varphi_{R^n}"] & U \otimes \prod_\Lambda L_\lambda \arrow[r] \arrow[d, "\varphi_U"] & 0 \\
\prod_\Lambda (R^{(A)} \otimes L_\lambda) \arrow[r, "\prod(f\otimes I)"] & \prod_\Lambda (R^n \otimes L_\lambda) \arrow[r, "\prod(g\otimes I)"] & \prod_\Lambda (U \otimes L_\lambda) \arrow[r] & 0.
\end{tikzcd}$$

As pointed out above, $\varphi_{R^n}$ is bijective and hence $\varphi_U$ is surjective.

$(b) \Rightarrow (c)$ is obvious.

(c) $\Rightarrow$ (a) Assume (c). Then, for $\Lambda = U$, the map $\tilde{\varphi} : U \otimes R^U \to U^U$ is surjective. For the element $(u_u)_U$ ($= I_U$ in $\mathrm{Map}(U,U) = U^U$), we choose $\sum_{i\le k} m_i \otimes (r_u^i)$ as a preimage under $\tilde{\varphi}_U$, with $r_u^i \in R$, $m_i \in U$, i.e.

$$(u_u)_U = \sum_{i\le k} (m_i r_u^i)_U = (\sum_{i\le k} m_i r_u^i)_U.$$

Hence, for every $u \in U$, we get $u = \sum_{i\le k} m_i r_u^i$, i.e. $m_1, \ldots, m_k$ is a generating set of $U$.

(2) (a) $\Rightarrow$ (b) In the proof (1) (a) $\Rightarrow$ (b) we can choose a finite index set $A$. Then $\varphi_{R^{(A)}}$ is an isomorphism and hence also $\varphi_U$.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) From (1) we already know that $U$ is finitely generated. Hence there is an exact sequence $0 \to K \to R^n \to U \to 0$, $n \in \mathbb{N}$. From this we obtain – for any set $\Lambda$ – the following commutative diagram with exact rows

$$\begin{tikzcd}
 & K \otimes R^\Lambda \arrow[r] \arrow[d, "\tilde{\varphi}_K"] & R^n \otimes R^\Lambda \arrow[r] \arrow[d, "\varphi_{R^n}"] & U \otimes R^\Lambda \arrow[r] \arrow[d, "\varphi_U"] & 0 \\
0 \arrow[r] & K^\Lambda \arrow[r] & (R^n)^\Lambda \arrow[r] & U^\Lambda \arrow[r] & 0\,.
\end{tikzcd}$$

Here $\tilde{\varphi}_{R^n}$ is an isomorphism (see above) and $\tilde{\varphi}_U$ is an isomorphism by (c). According to the Kernel Cokernel Lemma, $\tilde{\varphi}_K$ is surjective and, by(1), $K$ is finitely generated. Therefore, for some $m \in \mathbb{N}$, we get an exact sequence $R^m \to K \to 0$, and $R^m \to R^n \to U \to 0$ is also exact. □

As a consequence of the right exactness of the tensor functor the following two results can be shown:

**1.10. Zero in the tensor product.** *Let $\{n_i\}_{i\in\Lambda}$ a generating set of the $R$-module ${}_RN$ and $\{m_i\}_{i\in\Lambda}$ a family of elements in the $R$-module $M_R$ with only finitely many $m_i \neq 0$.*

*Then $\sum_\Lambda m_i \otimes n_i = 0$ in $M \otimes_R N$ if and only if there are finitely many elements $\{a_j\}_{j\in\Lambda'}$ in $M$ and a family $\{r_{ji}\}_{\Lambda'\times\Lambda}$ of elements in $R$ with the properties*

*(i) $r_{ji} \neq 0$ for only finitely many pairs (j,i),*

*(ii) $\sum_{i\in\Lambda} r_{ji} n_i = 0$ for every $j \in \Lambda'$,*

*(iii) $m_i = \sum_{j\in\Lambda'} a_j r_{ji}$.*

**Proof.** For elements with these properties we see

$$\sum_\Lambda m_i \otimes n_i = \sum_\Lambda \sum_{\Lambda'} a_j r_{ji} \otimes n_i = \sum_{\Lambda'} (a_j \otimes \sum_\Lambda r_{ji} n_i) = 0.$$

Now assume $\sum_\Lambda m_i \otimes n_i = 0$. With the canonical basis $\{f_i\}_{i\in\Lambda}$ and the map $g : R^{(\Lambda)} \to {}_RN,\ \ f_i \mapsto n_i$, we obtain the exact sequence

$$0 \longrightarrow {}_RK \xrightarrow{\varepsilon} R^{(\Lambda)} \xrightarrow{g} {}_RN \longrightarrow 0.$$

Tensoring with $M \otimes_R -$ yields the exact sequence

$$M \otimes_R K \xrightarrow{I\otimes\varepsilon} M \otimes_R R^{(\Lambda)} \xrightarrow{I\otimes g} M \otimes_R N \longrightarrow 0\ .$$

By assumption, $I \otimes g(\sum_\Lambda m_i \otimes f_i) = \sum_\Lambda m_i \otimes n_i = 0$ and there is an element $\sum_{j\in\Lambda'} a_j \otimes k_j \in M \otimes_R K$ with $I \otimes \varepsilon(\sum_{j\in\Lambda'} a_j \otimes k_j) = \sum_{i\in\Lambda} m_i \otimes f_i$.

Every $k_j \in K \subset R^{(\Lambda)}$ can be written as $k_j = \sum_{i\in\Lambda} r_{ji} f_i$ with only finitely many $r_{ji} \neq 0$. This implies $0 = (k_j)\varepsilon g = \sum_{i\in\Lambda} r_{ji} n_i$ for all $j \in \Lambda'$, and in $M \otimes_R R^{(\Lambda)}$ we get

$$\sum_{i\in\Lambda} m_i \otimes f_i = \sum_{j\in\Lambda'} a_j \otimes k_j = \sum_{i\in\Lambda} (\sum_{j\in\Lambda'} a_j r_{ji}) \otimes f_i.$$

From this the projections onto the components yield the desired condition $m_i = \sum_{j\in\Lambda'} a_j r_{ji}$. □

**1.11. Tensor product with cyclic modules.** *Let $I$ be a right ideal of a ring $R$ with many idempotents and ${}_RM$ a left module. Then*

$$R/I \otimes_R M \simeq RM/IM \ \ (\simeq M/IM \text{ if } 1 \in R).$$

**Proof.** From the exact sequence $0 \to I \to R \to R/I \to 0$ we obtain the first row exact in the commutative diagram

$$\begin{array}{ccccccccc} & & I \otimes_R M & \longrightarrow & R \otimes_R M & \longrightarrow & R/I \otimes_R M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_I} & & \downarrow{\scriptstyle \mu_R} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & IM & \longrightarrow & RM & \longrightarrow & RM/IM & \longrightarrow & 0 \end{array}$$

with the map $\mu_I : I \otimes_R M \to IM,\ \ i \otimes m \mapsto im$. By 1.6, $\mu_R$ is an isomorphism and hence $\gamma$ is an isomorphism by the Kernel Cokernel Lemma. □

An interesting connection between Hom- and tensor functors is derived from the definition of the tensor product:

**1.12. Hom-tensor relation.** Let $U_R$ and ${}_RM$ be $R$-modules, $N$ a $\mathbb{Z}$-module and denote by $\mathrm{Ten}(U \times M, N)$ the set of the $R$-balanced maps from $U \times M$ into $N$. By the definition of $U \otimes_R M$ (see 12.1), the canonical map $\tau : U \times M \to U \otimes_R M$ yields a $\mathbb{Z}$-isomorphism

$$\psi_1 : \mathrm{Hom}_{\mathbb{Z}}(U \otimes_R M, N) \to \mathrm{Ten}(U \times M, N),\ \ \alpha \mapsto \tau\alpha.$$

On the other hand, every $\beta \in \mathrm{Ten}(U \times M, N)$ defines an $R$-homomorphism

$$h_\beta : M \to \mathrm{Hom}_{\mathbb{Z}}(U, N),\ \ m \mapsto \beta(-, m),$$

where $\mathrm{Hom}_{\mathbb{Z}}(U, N)$ is regarded as a left $R$-module in the usual way. From this we obtain a $\mathbb{Z}$-isomorphism

$$\psi_2 : \mathrm{Ten}(U \times M, N) \to \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(U, N)),\ \ \beta \mapsto h_\beta.$$

Now every $\varphi \in \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(U, N))$ determines an $R$-balanced map

$$\tilde{\varphi} : U \times M \to N,\ \ (u, m) \mapsto \varphi(m)(u),$$

and the assignment $\varphi \mapsto \tilde{\varphi}$ is a map inverse to $\psi_2$. The composition of $\psi_1$ and $\psi_2$ leads to the $\mathbb{Z}$-*isomorphism*

$$\psi_M : \mathrm{Hom}(U \otimes_R M, N) \to \mathrm{Hom}_R(M, \mathrm{Hom}(U, N)),\ \ \delta \mapsto [m \mapsto \delta(- \otimes m)],$$

with inverse map $\psi_M^{-1} : \varphi \mapsto [u \otimes m \mapsto \varphi(m)(u)]$.

If ${}_SU_R$ is an $(S, R)$-bimodule and ${}_SN$ an $S$-module, then ${}_SU \otimes_R M$ is also a left $S$-module and with respect to this structure $\psi_M$ becomes a $\mathbb{Z}$-isomorphism

$$\psi_M : \mathrm{Hom}_S(U \otimes_R M, N) \to \mathrm{Hom}_R(M, \mathrm{Hom}_S(U, N)).$$

It is readily verified that, for every $R$-homomorphism $g : {}_RM \to {}_RM'$, the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Hom}_S(U \otimes_R M', N) & \xrightarrow{\mathrm{Hom}(I \otimes g, N)} & \mathrm{Hom}_S(U \otimes_R M, N) \\
\downarrow{\scriptstyle \psi_{M'}} & & \downarrow{\scriptstyle \psi_M} \\
\mathrm{Hom}_R(M', \mathrm{Hom}_S(U, N)) & \xrightarrow{\mathrm{Hom}(g, \mathrm{Hom}_S(U,N))} & \mathrm{Hom}_R(M, \mathrm{Hom}_S(U, N))\,.
\end{array}$$

Similarly we obtain, for modules ${}_RU_S$, $M_R$ and $N_S$, a $\mathbb{Z}$-isomorpism

$$\psi'_M : \mathrm{Hom}_S(M \otimes_R U, N) \to \mathrm{Hom}_R(M, \mathrm{Hom}_S(U, N))$$

and a corresponding commutative diagram.

**1.13. Definitions.** Let $M$ be a left $R$-module. A right $R$-module $U_R$ is called *$M$-flat* if, for every monomorphism $f : K \to M$ in $R$-Mod, the map $I_U \otimes f : U \otimes_R K \to U \otimes_R M$ is also a monomorphism.

$U_R$ is said to be *flat (with respect to* $R$-Mod*)* if $U$ is $M$-flat for every $M \in R$-Mod.

Since $U \otimes_R -$ is always right exact, $U_R$ is flat (with respect to $R$-Mod) if and only if the functor $U \otimes_R - : R\text{-Mod} \to \mathbb{Z}\text{-Mod}$ is exact.

**1.14. Direct sum of $M$-flat modules.** *Let $\{U_\lambda\}_\Lambda$ be a family of right $R$-modules and ${}_RM \in R$-Mod. The direct sum $\bigoplus_\Lambda U_\lambda$ is $M$-flat if and only if $U_\lambda$ is $M$-flat for every $\lambda \in \Lambda$.*

**Proof.** From the exact sequence $0 \to K \xrightarrow{f} M$ we form the commutative diagram

$$\begin{array}{ccc}
(\bigoplus_\Lambda U_\lambda) \otimes_R K & \xrightarrow{I \otimes f} & (\bigoplus_\Lambda U_\lambda) \otimes_R M \\
\downarrow & & \downarrow \\
\bigoplus_\Lambda (U_\lambda \otimes_R K) & \xrightarrow{\oplus(I_\lambda \otimes f)} & \bigoplus_\Lambda (U_\lambda \otimes_R M),
\end{array}$$

in which the vertical maps are the canonical isomorphisms (see 1.4). Hence $I \otimes f$ is monic if and only if all $I_\lambda \otimes f$ are monic. $\square$

**1.15. Properties of $M$-flat modules.** *Let $U_R$ be a right $R$-module. Then:*

(1) *$U_R$ is $M$-flat if and only if $U \otimes_R -$ is exact with respect to every exact sequence $0 \to K' \to M$ with $K'$ finitely generated.*

(2) *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $R$-Mod. If $U_R$ is $M$-flat, then $U_R$ is also $M'$- and $M''$-flat.*

(3) *Let $\{M_\lambda\}_\Lambda$ be a family of left $R$-modules. If $U_R$ is $M_\lambda$-flat for every $\lambda \in \Lambda$, then $U_R$ is also $\bigoplus_\Lambda M_\lambda$-flat.*

**Proof.** (1) Let $0 \to K \xrightarrow{\varepsilon} M$ be exact and $\sum_{i\le n} u_i \otimes k_i \in U \otimes_R K$ with $(\sum_{i\le n} u_i \otimes k_i) I \otimes \varepsilon = 0 \in U \otimes_R M$. Let $K'$ denote the submodule of $K$ generated by $k_1, \ldots, k_n$. Since the map

$$I \otimes \varepsilon' : U \otimes_R K' \to U \otimes_R K \to U \otimes_R M$$

is monic by assumption, we get $\sum_{i\le n} u_i \otimes k_i = 0$ in $U \otimes_R K'$. Then it also has to be zero in $U \otimes_R K$, i.e., $I \otimes \varepsilon$ is monic.

(2) Let $U_R$ be $M$-flat. If $0 \to K \xrightarrow{\varepsilon'} M'$ is exact, the canonical map $U \otimes_R K \xrightarrow{I\otimes\varepsilon'} U \otimes_R M' \longrightarrow U \otimes_R M$ is monic and $U_R$ is $M'$-flat.

If $0 \to L \xrightarrow{f} M''$ is exact, we obtain, by a pullback, the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \ .
\end{array}$$

Tensoring with $U_R$ yields the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & U \otimes_R M' & \longrightarrow & U \otimes_R P & \longrightarrow & U \otimes_R L & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow I \otimes f & & \\
0 & \longrightarrow & U \otimes_R M' & \longrightarrow & U \otimes_R M & \longrightarrow & U \otimes_R M'' & \longrightarrow & 0 \ .
\end{array}$$

By the Kernel Cokernel Lemma, $I \otimes f$ has to be monic, i.e., $U$ is $M''$-flat.

(3) We show that $U_R$ is $M_1 \oplus M_2$-flat if it is both $M_1$- and $M_2$-flat. Then we get assertion (3) for finite index sets $\Lambda$ by induction. For arbitrary sets $\Lambda$ we use (1): A finitely generated submodule $K' \subset \bigoplus_\Lambda M_\lambda$ is contained in a finite partial sum. Since the tensor product preserves direct summands, the assertion follows from the finite case.

Let $U_R$ be $M_1$- and $M_2$-flat and $0 \to K \xrightarrow{f} M_1 \oplus M_2$ exact. Forming a pullback we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & K & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_2 & \longrightarrow & 0 \ .
\end{array}$$

Tensoring with $U_R$ yields the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & U \otimes_R P & \longrightarrow & U \otimes_R K & \longrightarrow & U \otimes_R L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle I \otimes f} & & \downarrow & & \\
0 & \longrightarrow & U \otimes_R M_1 & \longrightarrow & U \otimes_R (M_1 \oplus M_2) & \longrightarrow & U \otimes_R M_2 & \longrightarrow & 0 \quad .
\end{array}$$

By the Kernel Cokernel Lemma, $I \otimes f$ has to be monic. □

**1.16. Flat modules. Characterizations.** *For a right $R$-module $U_R$, the following assertions are equivalent:*

*(a) $U_R$ is flat (with respect to $R$-*Mod*);*

*(b) $U \otimes_R -$ is exact with respect to all exact sequences $0 \to {}_RI \to {}_RR$ (with ${}_RI$ finitely generated);*

*(c) for every (finitely generated) left ideal ${}_RI \subset R$, the canonical map $\mu_I : U \otimes_R I \to UI$ is monic (and hence an isomorphism).*

**Proof.** The equivalence of (a) and (b) follows from 1.13.

(b) $\Leftrightarrow$ (c) For every (finitely generated) left ideal $I \subset R$, we have the commutative diagram with exact rows (see 1.11)

$$\begin{array}{ccccccccc}
 & & U \otimes_R I & \xrightarrow{I \otimes \varepsilon} & U \otimes_R R & \longrightarrow & U \otimes_R R/I & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_I} & & \| & & \| & & \\
0 & \longrightarrow & UI & \longrightarrow & U & \longrightarrow & U/UI & \longrightarrow & 0 \quad .
\end{array}$$

Hence $\mu_I$ is monic (an isomorphism) if and only if $I \otimes \varepsilon$ is monic. □

In a ring $R$ with unit, for every left ideal $I \subset R$, we have $R \otimes_R I \simeq I$. Hence $R_R$ is a flat module (with respect to $R$-Mod). Then, by 1.14, all free $R$-modules and their direct summands (= projective modules) are flat (with respect to $R$-Mod).

An $R$-module $U_R$ is called *faithfully flat (with respect to $R$-*Mod*)* if $U_R$ is flat (w.r. to $R$-Mod) and, for $N \in R$-Mod, the relation $U \otimes_R N = 0$ implies $N = 0$.

**1.17. Faithfully flat modules. Characterizations.** *For a right $R$-module $U_R$ the following assertions are equivalent:*

*(a) $U_R$ is faithfully flat;*

*(b) $U_R$ is flat and, for every (maximal) left ideal $I \subset R$, $I \neq R$, we have $U \otimes_R R/I \neq 0$ (i.e., $UI \neq U$).*

**Proof.** (a) $\Rightarrow$ (b) Because of the isomorphism $U \otimes_R R/I \simeq U/UI$ (see 1.11), $U \otimes_R R/I \neq 0$ is equivalent to $UI \neq U$. By (a), $U \otimes_R R/I = 0$ would imply $I = R$.

(b) $\Rightarrow$ (a) If $UI \neq U$ for every maximal left ideal $I \subset R$, then this is also true for every proper left ideal $I \subset R$. Hence $U \otimes_R K \neq 0$ for every cyclic $R$-module $K$. Since every $R$-module $N$ contains a cyclic submodule and $U_R$ is flat, we have $U \otimes_R N \neq 0$. $\square$

**1.18. Pure morphisms.** Related to any morphism $f : M \to M'$ in $R-$Mod, there is an exact sequence

$$0 \longrightarrow \operatorname{Ke} f \longrightarrow M \xrightarrow{f} M' \longrightarrow \operatorname{Coke} f \longrightarrow 0\,.$$

Given $L \in$ Mod$-R$, we say the morphism $f$ is *$L$-pure* if tensoring this sequence with $L \otimes_R -$ yields an exact sequence. The morphism $f$ is said to be *pure* if it is $L$-pure for every $L \in$ Mod$-R$. Since the tensor functor is right exact, the following are equivalent:

*(a)* $f$ *is $L$-pure;*

*(b)* $0 \longrightarrow L \otimes_R \operatorname{Ke} f \longrightarrow L \otimes_R M \xrightarrow{I_L \otimes f} L \otimes_R M'$ *is exact;*

*(c)* $\operatorname{Ke} f \to M$ *and* $\operatorname{Im} f \to M'$ *are $L$-pure (mono) morphisms.*

For any inclusion $i : N \to M$, the image of the map

$$I_L \otimes i : L \otimes_R N \to L \otimes_R M$$

is called the *canonical image* of $L \otimes_R N$ in $L \otimes_R M$. If $I_L \otimes i$ is injective (i.e., $i$ is an $L$-pure morphism), then $N$ is said to be an *$L$-pure submodule* and we identify the canonical image of $I_L \otimes i$ with $L \otimes_R N$.

Obviously, any direct summand is a pure submodule, and if $L$ is a flat right $R$-module, then every morphism $f : M \to M'$ in $R-$Mod is $L$-pure.

**1.19. Tensor product over commutative rings.** Let $M$, $N$ and $L$ be modules over a commutative ring $R$.

An $R$-balanced map $\beta : M \times N \to L$ is called *$R$-bilinear* if

$$\beta(rm, n) = r\beta(m, n) \;\text{ for all }\; r \in R, m \in M, n \in N.$$

By 1.5, ${}_RM \otimes_R N$ is a left $R$-module with $r(m \otimes n) = (rm) \otimes n$ and we see that the balanced map

$$\tau : M \times N \to M \otimes_R N, \;\; (m, n) \mapsto m \otimes n,$$

is bilinear: $\tau(rm, n) = (rm) \otimes n = r\tau(m, n)$. Hence we have, for commutative rings $R$:

*A map $\beta : M \times N \to L$ is $R$-bilinear if and only if there is an $R$-linear map $\bar{\beta} : M \otimes_R N \to L$ with $\beta = \tau\bar{\beta}$.*

With the notation $\mathrm{Bil}_R(M \times N, L) = \{\beta : M \times N \to L \mid \beta\, R\text{-}bilinear\}$ we have an isomorphism of $R$-modules

$$\mathrm{Hom}_R(M \otimes_R N, L) \simeq \mathrm{Bil}_R(M \times N, L).$$

If $M$ and $N$ are vector spaces over a field $K$, then, by the above considerations, $M \otimes_K N$ is also a $K$-vector space and since the tensor product commutes with direct sums we find

$$dim_K(M \otimes_K N) = dim_K M \cdot dim_K N.$$

Every free $R$-module ${}_R F$ is isomorphic to $R^{(\Lambda)}$ for a suitable index set $\Lambda$. Over non-commutative rings the cardinality of $\Lambda$ need not be uniquely determined. However, over a commutative ring with unit we have:

*If $R^{(\Lambda)} \simeq R^{(\Lambda')}$, then $\Lambda$ and $\Lambda'$ have the same cardinality.*

**Proof.** For a maximal ideal $m$ of $R$, tensoring with $- \otimes_R R/m$ yields $(R/m)^{(\Lambda)} \simeq (R/m)^{(\Lambda')}$. For vector spaces over a field ($= R/m$) it is known that the cardinality of a basis is uniquely determined. $\square$

**1.20. Twist map.** For modules $M, N$ over a commutative ring $R$, there is an R-isomorphism

$$\mathsf{tw} : M \otimes_R N \to N \otimes_R M, \quad m \otimes n \mapsto n \otimes m,$$

called the *twist map.*

The purity condition on the submodules imply the important

**1.21. Intersection property.**

*(1) Let $M' \subset M$ and $K' \subset K$ be pure $R$-submodules, or assume $K$ and $K/K'$ to be ($M$-) flat.*

*Then the canonical image of $M' \otimes_R K'$ in $M \otimes_R K$ is equal to the intersection of the canonical images of $M' \otimes_R K$ and $M \otimes_R K'$ in $M \otimes_R K$, i.e.,*

$$M' \otimes_R K' = (M' \otimes_R K) \cap (M \otimes_R K').$$

*(2) Let $U, V \subset M$ be $R$-submodules and $K$ a flat $R$-module. Then*

$$(U \otimes_R K) \cap (V \otimes_R K) = (U \cap V) \otimes_R K.$$

**Proof.** (1) Under the given conditions we have the exact commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M' \otimes_R K' & \to & M' \otimes_R K & \to & M' \otimes_R K/K' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M \otimes_R K' & \to & M \otimes_R K & \to & M \otimes_R K/K' & \to & 0\,,
\end{array}
$$

where the left square is a pullback (e.g., [10, 10.3]), and hence we can make the identification stated.

(2) This is shown with a similar argument. □

# 2 Algebras and modules

From now on $R$ will usually denote an associative commutative ring with unit. We recall basic definitions for algebras in a form which is suitable for dualising to coalgebras. The unadorned symbol $\otimes$ will always stand for $\otimes_R$.

**2.1. Algebras.** An $R$-module $A$ is said to be an $R$-*algebra* if there exists an $R$-linear map

$$\mu : A \otimes_R A \to A,$$

called the *multiplication* of $A$. For $a, b \in A$, we write $\mu(a \otimes b) = a \cdot b$ (or $ab$).

**2.2. Unit element.** An element $e \in A$ is called a *unit* if $a \cdot e = a = e \cdot a$, for all $a \in A$. This yields an $R$-homomorphism $\iota : R \to A,\ r \mapsto re$, with

$$\mu \circ (\iota \otimes I_A) = \mu \circ (I_A \otimes \iota) = I_A,$$

(putting $A \otimes_R R = A = R \otimes_R A$) which correponds to the commutativity of the diagram

$$\begin{tikzcd}
R \otimes_R A \arrow[r, "\iota\otimes I_A"] \arrow[dr, "\simeq"'] & A \otimes_R A \arrow[d, "\mu"] & A \otimes_R R \arrow[l, "I_A\otimes\iota"'] \arrow[dl, "\simeq"] \\
 & A &
\end{tikzcd}$$

It is easy to check that for any $R$-linear map $\iota : R \to A$ with these properties, $\iota(1)$ is an identity element in $A$.

**2.3. Associativity and commutativity** An $R$-algebra $A$ is said to be *associative,* if $a(bc) = (ab)c$, for all $a, b, c \in A$. For the defining map $\mu : A \otimes_R A \to A$, this corresponds to the commutativity of the diagram

$$\begin{tikzcd}
A \otimes_R A \otimes_R A \arrow[r, "I_A\otimes\mu"] \arrow[d, "\mu\otimes I_A"'] & A \otimes_R A \arrow[d, "\mu"] \\
A \otimes_R A \arrow[r, "\mu"] & A\,.
\end{tikzcd}$$

The *commutativity* of $A$ is expressed by the commutativity of the following diagram, where $\mathsf{tw}$ denotes the twist map,

$$\begin{tikzcd}
A \otimes_R A \arrow[rr, "\mathsf{tw}"] \arrow[dr, "\mu"'] & & A \otimes_R A \arrow[dl, "\mu"] \\
 & A & 
\end{tikzcd} .$$

**2.4. Algebra morphisms.** Given two $R$-algebras $\mu_A : A \otimes A \to A$ and $\mu_B : B \otimes_R B \to B$, an $R$-linear map $f : A \to B$ is said to be an *algebra morphism*, if $f(ab) = f(a)f(b)$, for all $a, b \in A$, i.e., we have the commutative diagram

$$\begin{tikzcd} A \otimes_R A \arrow[r, "\mu_A"] \arrow[d, "f\otimes f"'] & A \arrow[d, "f"] \\ B \otimes_R B \arrow[r, "\mu_B"] & B\,. \end{tikzcd}$$

Clearly $f$ preserves the units $\iota_A : R \to A, \iota_B : R \to B$, if and only if we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "="] \arrow[d, "\iota_A"'] & R \arrow[d, "\iota_B"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

Obviously the identity $I_A : A \to A$ is an algebra morphism and the composition of algebra morphisms is again an algebra morphism.

**2.5. Tensor product of algebras.** Let $\mu_A : A \otimes_R A \to A$ and $\mu_B : B \otimes_R B \to B$ define $R$-algebras. Then $A \otimes_R B$ is an $R$-algebra by the $R$-linear map

$$\mu_{A\otimes B} : (A \otimes_R B) \otimes_R (A \otimes B) \xrightarrow{I\otimes \mathsf{tw} \otimes I} (A \otimes_R A) \otimes_R (B \otimes_R B) \xrightarrow{\mu_A \otimes \mu_B} A \otimes_R B\,.$$

It is straightforward to show that the algebra $A \otimes_R B$ is associative (commutative) if both $A$ and $B$ are associative (commutative).

If $e_A$ and $e_B$ are the units in $A$ and $B$, then $e_A \otimes e_B$ is the unit in $A \otimes_R B$.

In particular, if $B$ is an associative and commutative algebra with unit, then it is easy to verify that $\mu_{A\otimes B}$ is in fact $B$-linear, yielding a map

$$\mu_{A\otimes B} : (A \otimes_R B) \otimes_B (A \otimes_R B) \to A \otimes_R B,$$

showing that $A \otimes_R B$ is a $B$-algebra (*scalar extension* of $A$ by $B$).

**2.6. Tensor product of algebra morphisms.** *Let $f : A \to A_1$ and $g : B \to B_1$ be $R$-algebra morphisms. Then:*

(1) *There is an algebra morphism*

$$h : A \otimes_R B \to A_1 \otimes_R B_1,\ a \otimes b \mapsto f(a) \otimes g(b).$$

(2) *If $f$ and $g$ are surjective, then $h$ is also surjective and (see 1.3)*

$$A_1 \otimes_R B_1 \simeq (A \otimes_R B)/(\mathrm{Ke}\, f \otimes' B + A \otimes' \mathrm{Ke}\, g).$$

**Proof.** (1) Put $h = f \otimes g$, the tensor product of $f$ and $g$ as $R$-module morphism. It remains to show that $h$ is a ring morphism:

$$\begin{aligned} h((a_1 \otimes b_1)(a_2 \otimes b_2)) &= f(a_1a_2) \otimes g(b_1b_2) \\ &= (f(a_1) \otimes g(b_1))(f(a_2) \otimes g(b_2)) \\ &= h(a_1 \otimes b_1)h(a_2 \otimes b_2). \end{aligned}$$

(2) This is shown in 1.3. □

**2.7. Universal property of the tensor product.** *Let $f : A \to C$ and $g : B \to C$ be unital algebra morphisms such that*

$$[f(A), g(B)] = 0,$$

*where $[-,-]$ denotes the commutator. Then there exists a unique algebra morphism $h : A \otimes_R B \to C$, satisfying*

$$h(a \otimes b) = f(a)g(b)\,, \text{ for all } a \in A,\, b \in B.$$

**Proof.** Since $f$ and $g$ are $R$-module morphisms, there is an $R$-module morphism

$$h = f \otimes g : A \otimes_R B \to C, \quad (a, b) \mapsto f(a)g(b).$$

It remains to verify that $h$ is an algebra morphism. By our assumptions on the commutators of $f(A)$ and $g(B)$, we have

$$\begin{aligned} h((a_1 \otimes b_1)(a_2 \otimes b_2)) &= f(a_1a_2)g(b_1b_2) \\ &= f(a_1)f(a_2)g(b_1)g(b_2) \\ &= (f(a_1)g(b_1))(f(a_2)g(b_2)) \\ &= h(a_1 \otimes b_1)h(a_2 \otimes b_2). \end{aligned}$$

□

As a special case, 2.7 implies that in the category of commutative associative unital $R$-algebras, the tensor product yields the coproduct of two algebras.

**2.8. $A$-modules and homomorphisms.** Let $\mu : A \otimes_R A \to A$ define an associative $R$-algebra with unit $\iota : R \to A$. Then an $R$-module $M$ with an $R$-linear map $\varrho_M : A \otimes_R M \to M$ is called a (unital) *left $A$-module* if the following diagrams are commutative:

$$\begin{tikzcd} A \otimes_R A \otimes_R M \arrow[r, "I_A \otimes \varrho_M"] \arrow[d, "\mu \otimes I"'] & A \otimes_R M\,, \arrow[d, "\varrho_M"] \\ A \otimes_R M \arrow[r, "\varrho"', "M"] & M \end{tikzcd} \qquad \begin{tikzcd} R \otimes_R M \arrow[r, "\iota \otimes I_M"] \arrow[dr, "\simeq"'] & A \otimes_R M \arrow[d, "\varrho_M"] \\ & M\,. \end{tikzcd}$$

Again we will often write $\varrho_M(a \otimes m) = am$ and then the commutativity of the diagram can be rephrased by the familiar conditions

$$(ab)m = a(bm),\ 1_A m = m,\ \text{for all } a, b \in A,\ m \in M.$$

An $R$-linear map $g : M \to N$ between $A$-modules is called an $A$-*morphism* or $A$-*homomorphism* provided the following diagram commutes:

$$\begin{array}{ccc} A \otimes_R M & \xrightarrow{I_A \otimes g} & A \otimes_R N \\ {\scriptstyle \varrho_M}\downarrow & & \downarrow{\scriptstyle \varrho_N} \\ M & \xrightarrow{\quad g \quad} & N\,. \end{array}$$

Using the standard notation mentioned above, $g : M \to N$ is an $A$-morphism if and only if

$$g(am) = ag(m),\ \text{for all } a \in A,\ m \in M.$$

Notice that the $A$-morphisms between two $A$-modules $M, N$ are characterized by an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_R(M, N) \xrightarrow{\beta} \mathrm{Hom}_R(A \otimes_R M, N),$$

where $\beta(f) = \varrho_N \circ (id \otimes f) - f \circ \varrho_M$.

The $A$-modules together with the $A$-morphisms form a category which we denote by $A$-Mod.

It is easily checked that here the defining map $\varrho_M : A \otimes_R M \to M$ is an $A$-morphism and this implies that $A$ is a generator in $A$-Mod, i.e., every left $A$-module is a homomorphic image of a direct sum of copies of $A$ $(= A^{(\Lambda)})$.

There are well-known relations between $R$-morphisms and $A$-morphisms which are readily derived from the basic relation 1.12.

**2.9. Hom-tensor relations.** *For any $R$-module $X$, consider the $R$-linear map $\alpha : X \to A \otimes_R X$, $x \mapsto 1_A \otimes x$.*

*(1) For each $A$-module $\varrho_M : A \otimes_R M \to M$, the map*

$$\mathrm{Hom}_A(A \otimes_R X, M) \to \mathrm{Hom}_R(X, M),\ f \mapsto f \circ \alpha,$$

*is an $R$-isomorphism with inverse map $h \mapsto \varrho_M \circ (id \otimes h)$. So the functor*

$$A \otimes_R - : R\text{–Mod} \to A\text{–Mod}, \quad X \mapsto A \otimes_R X,$$

*is left adjoint to the forgetful functor $A$–Mod $\to$ $R$–Mod.*

*(2) For any left $A$-module $N$, the $R$-linear map*

$$\mathrm{Hom}_A(M \otimes X, N) \to \mathrm{Hom}_R(X, \mathrm{Hom}_A(M, N)), \quad g \mapsto [x \mapsto g \circ (- \otimes x)],$$

*is an $R$-isomorphism with inverse map $h \mapsto [m \otimes x \mapsto h(x)(m)]$. So*

$$M \otimes_R - : R\text{−Mod} \to A\text{−Mod}, \quad X \mapsto M \otimes_R X,$$

*is left adjoint to the functor*

$$\mathrm{Hom}_A(M, -) : A\text{−Mod} \to R\text{−Mod}, \quad N \mapsto \mathrm{Hom}_A(M, N).$$

**2.10. Tensor product with modules.** *Let $M$ and $M'$ be left $A$-modules and $Q$ an $R$-module. Consider the $R$-linear map*

$$\nu_M : \mathrm{Hom}_A(M, M') \otimes_R Q \to \mathrm{Hom}_A(M, M' \otimes_R Q), \;\; [h \otimes q \mapsto (-)h \otimes q].$$

*(1) If $Q$ is a flat $R$-module and $M$ a finitely generated (finitely presented) $A$-module, then $\nu_M$ is injective (an isomorphism).*

*(2) $\nu_M$ is also an isomorphism in the following cases:*

*(i) $M$ is a finitely generated, $M'$-projective $A$-module, or*

*(ii) $M$ is $M'$-projective and $Q$ is a finitely presented $R$-module, or*

*(iii) $Q$ is a finitely generated projective $R$-module.*

**Proof.** (1) It is easy to check that $\nu_M$ is an isomorphism for $M = A$ and $M = A^k$, $k \in \mathbb{N}$. Since ${}_AM$ is finitely generated, there exists an exact sequence of $A$-modules $A^{(\Lambda)} \to A^n \to M \to 0$, with $\Lambda$ an index set, $n \in \mathbb{N}$.

The functors $\mathrm{Hom}_A(-, M') \otimes_R Q$ and $\mathrm{Hom}_A(-, M' \otimes_R Q)$ yield the exact commutative diagram

$$\begin{array}{ccccc} 0 \to \mathrm{Hom}_A(M, M') \otimes_R Q & \to & \mathrm{Hom}_A(A^n, M') \otimes_R Q & \to & \mathrm{Hom}_A(A^{(\Lambda)}, M') \otimes_R Q \\ \downarrow \nu_M & & \downarrow \nu_{A^n} & & \downarrow \nu_{A^{(\Lambda)}} \\ 0 \to \mathrm{Hom}_A(M, M' \otimes_R Q) & \to & \mathrm{Hom}_A(A^n, M' \otimes_R Q) & \to & \mathrm{Hom}_A(A^{(\Lambda)}, M' \otimes_R Q). \end{array}$$

Since $\nu_{A^n}$ is an isomorphism, $\nu_M$ has to be injective.

If $M$ is finitely presented we can choose $\Lambda$ to be finite. Then also $\nu_{A^{(\Lambda)}}$ and $\nu_M$ are isomorphisms.

(2)($i$) From the exact sequence of $R$-modules $0 \to K \to R^{(\Lambda)} \to Q \to 0$, we construct the commutative diagram with the upper line exact,

$$\begin{array}{ccccccc} \mathrm{Hom}_A(M, M') \otimes_R K & \to & \mathrm{Hom}_A(M, M') \otimes_R R^{(\Lambda)} & \to & \mathrm{Hom}_A(M, M') \otimes_R Q & \to 0 \\ \downarrow \nu & & \downarrow \simeq & & \downarrow \nu_M & \\ \mathrm{Hom}_A(M, M' \otimes_R K) & \to & \mathrm{Hom}_A(M, M' \otimes_R R^{(\Lambda)}) & \to & \mathrm{Hom}_A(M, M' \otimes_R Q) & \to 0 \, , \end{array}$$

where $\nu$ is defined as above replacing $Q$ by $K$.

Since $M$ is $M' \otimes_R R^{(\Lambda)}$-projective, the lower sequence is also exact and hence $\nu_M$ is surjective. By the same argument we obtain that $\nu$ is surjective. Now it follows from the Kernel Cokernel Lemma that $\nu_M$ is injective.

$(ii)$ This statement is obtained from the proof of (1), with $\Lambda$ a finite set and $K$ a finitely generated $R$-module.

$(iii)$ The assertion is obvious for $Q = R$ and is easily extended to finitely generated free (projective) modules $Q$. □

Combining the preceding observations we get

**2.11. Tensor product with an algebra.** *Let $A$, $B$ be $R$-algebras and $M$, $M'$ left $A$-modules. Consider the map*

$$\mathrm{Hom}_A(M, M') \otimes_R B \to \mathrm{Hom}_{A \otimes B}(M \otimes_R B, M' \otimes_R B),\ f \otimes b \mapsto [m \otimes b' \mapsto (m)f \otimes b'b].$$

*(1) If $B$ is a flat $R$-module and $M$ is a finitely generated (finitely presented) $A$-module, then the map is injective (an isomorphism).*

*(2) The map is also an isomorphism if*

- *(i) $M$ is a finitely generated, $M'$-projective $A$-module, or*
- *(ii) $M$ is $M'$-projective and $B$ is a finitely presented $R$-module, or*
- *(iii) $B$ is a finitely generated projective $R$-module.*

**Proof.** The map is the composition of maps

$$\begin{array}{rcl} \mathrm{Hom}_A(M, M') \otimes_R B & \to & \mathrm{Hom}_A(M, M' \otimes_R B) \text{ and} \\ \mathrm{Hom}_A(M, M' \otimes_R B) & \to & \mathrm{Hom}_{A \otimes_R B}(M \otimes_R B, M' \otimes_R B). \end{array}$$

□

**2.12. Tensor product of morphisms of modules.** *Let $A$, $B$ be associative unital $R$-algebras, $M, M'$ left $A$-modules and $N, N'$ left $B$-modules.*

*(1) For $f \in \mathrm{Hom}_A(M, M')$ and $g \in Hom_B(N, N')$,*

$$f \otimes g \in \mathrm{Hom}_{A \otimes B}(M \otimes_R N, M' \otimes_R N').$$

*(2) The mapping $(f, g) \mapsto f \otimes g$ induces an $R$-module morphism*

$$\psi : \mathrm{Hom}_A(M, M') \otimes_R \mathrm{Hom}_B(N, N') \to \mathrm{Hom}_{A \otimes B}(M \otimes_R N, M' \otimes_R N').$$

*Assume $M$ and $N$ are finitely generated. Then $\psi$ is an isomorphism if*

- *(i) $M$ is $M'$-projective and $N$ is $N'$-projective, or*
- *(ii) $M$ and $N$ are projective as $A$-, resp. $B$-modules, or*
- *(iii) $M$ is a finitely presented $A$-module, $N$ and $N'$ are finitely generated, projective $B$-modules, and $B$ is a flat $R$-module.*

*(3)* $\psi : End_A(M) \otimes_R End_B(N) \to End_{A\otimes_R B}(M \otimes_R N)$ *is an algebra morphism.*

**Proof.** (1) Just verify that $f \otimes g$ is in fact an $A \otimes B$-module morphism.

(2) $\psi$ is well-defined since $(f, g) \mapsto f \otimes g$ yields an $R$-bilinear map.

$(i)$ We have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{A\otimes_R B}(M \otimes_R N, M' \otimes_R N') &\simeq \mathrm{Hom}_B(N, \mathrm{Hom}_A(M, M' \otimes_R N')) \\ &\simeq \mathrm{Hom}_B(N, \mathrm{Hom}_A(M, M') \otimes_R N') \\ &\simeq \mathrm{Hom}_A(M, M') \otimes \mathrm{Hom}_B(N, N')\,. \end{aligned}$$

$(ii)$ is a special case of $(i)$.

$(iii)$ Since $B \simeq \mathrm{Hom}_B(B, B)$ we know that $\psi$ is an isomorphism for $N = N' = B$. Similar to the above argument, this isomorphism can be extended to finitely generated free and projective modules $N$ and $N'$.

(3) is easily verified. $\square$

# 3 The category $\sigma[M]$

Throughout $A$ will denote an $R$-algebra and $M$ a left $A$-module. The module structure of $M$ is reflected by the smallest Grothendieck category of $A$-modules containing $M$, which we briefly describe in this section.

An $A$-module $N$ is called *$M$-generated* if there exists an epimorphism $M^{(\Lambda)} \to N$ for some set $\Lambda$.

**3.1. The category $\sigma[M]$.** An $A$-module $N$ is called *$M$-subgenerated* if it is (isomorphic to) a submodule of an $M$-generated module. By $\sigma[M]$ we denote the full subcategory of ${}_A\mathbf{M}$ whose objects are all $M$-subgenerated modules. Obviously the finitely generated (cyclic) submodules of $M^{(\mathbb{N})}$ form a set of generators in $\sigma[M]$.

The *trace functor* $\mathcal{T}^M : {}_A\mathbf{M} \to \sigma[M]$, which sends any $X \in {}_A\mathbf{M}$ to

$$\mathcal{T}^M(X) := \sum\{f(N) \mid N \in \sigma[M],\ f \in {}_A\mathrm{Hom}(N, X)\},$$

is right adjoint to the inclusion functor $\sigma[M] \to {}_A\mathbf{M}$. For any family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M]$, the product in $\sigma[M]$ is

$$\prod_\Lambda^M N_\lambda = \mathcal{T}^M(\prod_\Lambda N_\lambda),$$

where the unadorned $\prod$ denotes the usual (Cartesian) product of $A$-modules, since, for any $P \in \sigma[M]$,

$${}_A\mathrm{Hom}(P, \mathcal{T}^M(\prod_\Lambda^M N_\lambda)) \simeq \prod_\Lambda {}_A\mathrm{Hom}(P, N_\lambda).$$

Moreover, for any injective $A$-module $Q$, $\mathcal{T}^M(Q)$ is an injective object in the category $\sigma[M]$.

$N \in \sigma[M]$ is said to be a *generator in* $\sigma[M]$ if it generates all modules in $\sigma[M]$, and $M$ is called a *self-generator* if it generates all its own submodules.

**3.2. Injective modules.** Let $U$ and $M$ be $A$-modules. $U$ is said to be *$M$-injective* if every diagram in ${}_A\mathbf{M}$ with exact row

$$\begin{array}{ccccc} 0 & \to & K & \to & M \\ & & \downarrow & & \\ & & U & & \end{array}$$

can be extended commutatively by some morphism $M \to U$. This holds if ${}_A\mathrm{Hom}(-, U)$ is exact with respect to all exact sequences of the form $0 \to K \to M \to N \to 0$ (in $\sigma[M]$). $U$ is *injective in* $\sigma[M]$ *(in* ${}_A\mathbf{M}$*)* if it is $N$-injective, for every $N \in \sigma[M]$ ($N \in {}_A\mathbf{M}$, resp.).

**3.3. Injectives in $\sigma[M]$.** (Cf. [10, 16.3, 16.11, 17.9].)

*(1) For $Q \in \sigma[M]$ the following are equivalent:*

*(a) $Q$ is injective in $\sigma[M]$;*

*(b) the functor ${}_A\mathrm{Hom}(-, Q) : \sigma[M] \to \mathbf{M}_R$ is exact;*

*(c) $Q$ is $M$-injective;*

*(d) $Q$ is $N$-injective for every (finitely generated) submodule $N \subset M$;*

*(e) every exact sequence $0 \to Q \to N \to L \to 0$ in $\sigma[M]$ splits.*

*(2) Every $M$-injective object in $\sigma[M]$ is $M$-generated.*

*(3) Every object in $\sigma[M]$ has an injective hull.*

**3.4. Projectivity.** Let $M$ and $P$ be $A$-modules. $P$ is said to be *$M$-projective* if the functor ${}_A\mathrm{Hom}(P, -)$ is exact on all exact sequences of the form $0 \to K \to M \to N \to 0$ in ${}_A\mathbf{M}$. $P$ is called *projective in $\sigma[M]$ (in ${}_A\mathbf{M}$)* if it is $N$-projective, for every $N \in \sigma[M]$ ($N \in {}_A\mathbf{M}$, repectively).

**3.5. Projectives in $\sigma[M]$.** (Cf. [10, 18.3].)

*For $P \in \sigma[M]$ the following are equivalent:*

*(a) $P$ is projective in $\sigma[M]$;*

*(b) the functor ${}_A\mathrm{Hom}(P, -) : \sigma[M] \to \mathbf{M}_R$ is exact;*

*(c) $P$ is $M^{(\Lambda)}$-projective, for any index set $\Lambda$;*

*(d) every exact sequence $0 \to K \to N \to P \to 0$ in $\sigma[M]$ splits.*

*If $P$ is finitely generated, then (a)–(d) are equivalent to:*

*(e) $P$ is $M$-projective.*

A module $P \in \sigma[M]$ is called a *progenerator* in $\sigma[M]$ if it is finitely generated, projective and a generator in $\sigma[M]$. Notice that there may be no projective objects in $\sigma[M]$. A module $N \in \sigma[M]$ is a *subgenerator in $\sigma[M]$* if $\sigma[N] = \sigma[M]$.

**3.6. Subgenerators.** (Cf. [10].)

*(1) For an $A$-module $M$ the following are equivalent:*

*(a) $M$ is a subgenerator in ${}_A\mathbf{M}$ (that is, $\sigma[M] = {}_A\mathbf{M}$);*

*(b) $M$ generates all injective modules in ${}_A\mathbf{M}$;*

*(c) there is a monomorphism $A \to M^k$, for some $k \in \mathbb{N}$.*

*(2) A faithful module ${}_AM$ is a subgenerator in ${}_A\mathbf{M}$ provided*

*(i) ${}_AM$ is finitely generated over $\mathrm{End}_A(M)$, or*

*(ii) ${}_AA$ is finitely cogenerated, or*

*(iii) $\sigma[M]$ is closed under products in ${}_A\mathbf{M}$.*

**3.7. Semisimple modules.**

*(1) The following are equivalent:*

*(a) $M$ is a (direct) sum of simple modules;*

*(b) every submodule of $M$ is a direct summand;*

*(c) every module (in $\sigma[M]$) is $M$-projective (or $M$-injective);*

*(d) every simple module (in $\sigma[M]$) is $M$-projective;*

*(e) every cyclic module (in $\sigma[M]$) is $M$-injective.*

Modules $M$ with these properties are called *semisimple modules.*

*(2) Assume $M$ to be semisimple.*

*(i) There exists a fully invariant decomposition*

$$M = \bigoplus_{\Lambda} \mathrm{Tr}(E_\lambda, M),$$

*where $\{E_\lambda\}_\Lambda$ is a minimal representing set of simple submodules of $M$ and the $\mathrm{Tr}(E_\lambda, M)$ are minimal fully invariant submodules.*

*(ii) The ring $S = {}_A\mathrm{End}(M)$ is von Neumann regular and $M$ is semisimple as a right $S$-module.*

*(iii) If all simple submodules of ${}_AM$ are isomorphic, then all simple submodules of $M_S$ are isomorphic.*

**Proof.** The first parts are shown in [10, 20.2–20.6].

(2)(ii) Let $Am \subset M$ be a simple submodule. We show that $mS \subset M$ is a simple $S$-submodule. For any $t \in S$ with $mt \neq 0$, $Am \simeq Amt$. Since these are direct summands in $M$, there exists some $\phi \in S$ with $mt\phi = m$ and hence $mS = mtS$, implying that $mS$ has no nontrivial $S$-submodules. As a semisimple module, $M = \sum_\Lambda Am_\lambda$ with $Am_\lambda$ simple. Now $M = A(\sum_\Lambda m_\lambda S)$, showing that $M$ is a sum of simple $S$-modules $am_\lambda S$, where $a \in A$.

(2)(iii) It is straightforward to show that, for any $m, n \in M$, $Am \simeq An$ implies $mS \simeq nS$. □

**Definitions.** A module $M$ has *finite length* if it is Noetherian and Artinian. $M$ is called *locally Noetherian (Artinian, of finite length)* provided every finitely generated submodule of $M$ is Noetherian (Artinian, of finite length). $M$ is called *semi-Artinian* if every factor module of $M$ has a nonzero socle.

**3.8. Local finiteness conditions.** (Cf. [10, 27.5, 32.5].)

*(1) The following are equivalent for a left $A$-module $M$:*

*(a) $M$ is locally Noetherian;*

*(b) every finitely generated module in $\sigma[M]$ is Noetherian;*

*(c) any direct sum of $M$-injective modules is $M$-injective;*

*(d) every injective module in $\sigma[M]$ is a direct sum of uniform modules.*

*(2) The following are equivalent for a left $A$-module $M$:*

*(a) $M$ is locally of finite length;*

*(b) every finitely generated module in $\sigma[M]$ has finite length;*

*(c) every injective module in $\sigma[M]$ is a direct sum of $M$-injective hulls of simple modules.*

*(3) A module $M$ is locally Artinian if and only if every finitely generated module in $\sigma[M]$ is Artinian.*

*(4) A module $M$ is semi-Artinian if and only if every module in $\sigma[M]$ has a nonzero socle.*

**Definitions.** A submodule $K$ of $M$ is said to be *superfluous* or *small* in $M$ if, for every submodule $L \subset M$, $K + L = M$ implies $L = M$. A small submodule is denoted by $K \ll M$. An epimorphism $\pi : P \to N$ with $P$ projective in $\sigma[M]$ and $Ke\,\pi \ll P$ is said to be a *projective cover of $N$ in $\sigma[M]$*. A module is called *local* if it has a largest proper submodule.

**3.9. Local modules.** (Cf. [10, 19.7].)
*For a projective module $P \in \sigma[M]$, the following are equivalent:*

*(a) $P$ is local;*

*(b) $P$ is a projective cover of a simple module in $\sigma[M]$;*

*(c)* $\mathrm{End}({}_AP)$ *is a local ring.*

**Definitions.** Let $U$ be a submodule of the $A$-module $M$. A submodule $V \subset M$ is called a *supplement* of $U$ in $M$ if $V$ is minimal with the property $U + V = M$. It is easy to see that $V$ is a supplement of $U$ if and only if $U + V = M$ and $U \cap V \ll V$. Notice that supplements need not exist in general. $M$ is said to be *supplemented* provided each of its submodules has a supplement.

**Definitions.** A module $P \in \sigma[M]$ is said to be *semiperfect in $\sigma[M]$* if every factor module of $N$ has a projective cover in $\sigma[M]$. $P$ is *perfect in $\sigma[M]$* if any direct sum $P^{(\Lambda)}$ is semiperfect in $\sigma[M]$.

**3.10. Semiperfect modules.** (Cf. [10, 42.5, 42.12].)
*For a projective module $P$ in $\sigma[M]$, the following are equivalent:*

*(a) $P$ is semiperfect in $\sigma[M]$;*

*(b) $P$ is supplemented;*

*(c) every finitely $P$-generated module has a projective cover in $\sigma[M]$;*

*(d) (i) $P/Rad(P)$ is semisimple and* $\mathrm{Rad}(P) \ll P$*, and*

*(ii) decompositions of* $P/\mathrm{Rad}(P)$ *can be lifted to $P$;*

(e) *every proper submodule is contained in a maximal submodule of $P$, and every simple factor module of $P$ has a projective cover in $\sigma[M]$;*

(f) *$P$ is a direct sum of local modules and $Rad(P) \ll P$.*

**3.11. Perfect modules.** (Cf. [10, 43.2].)
*For a projective module $P$ in $\sigma[M]$, the following are equivalent:*

(a) *$P$ is perfect in $\sigma[M]$;*

(b) *$P$ is semiperfect and, for any set $\Lambda$, $\mathrm{Rad}(P^{(\Lambda)}) \ll P^{(\Lambda)}$;*

(c) *every $P$-generated module has a projective cover in $\sigma[M]$.*

**Definition.** We call $\sigma[M]$ a *(semi)perfect category* if every (simple) module in $\sigma[M]$ has a projective cover in $\sigma[M]$.

**3.12. Semiperfect and perfect categories.**

(1) *For an $A$-module $M$ the following are equivalent:*

   (a) *$\sigma[M]$ is semiperfect;*

   (b) *$\sigma[M]$ has a generating set of local projective modules;*

   (c) *in $\sigma[M]$ every finitely generated module has a projective cover.*

(2) *For $M$ the following are equivalent:*

   (a) *$\sigma[M]$ is perfect;*

   (b) *$\sigma[M]$ has a projective generator that is perfect in $\sigma[M]$.*

**Proof.** (1) (a) $\Rightarrow$ (b) The projective covers of all simple objects in $\sigma[M]$ are local and form a generating set of $\sigma[M]$ (by [10, 18.5]). Notice that local modules are supplemented.

(b) $\Rightarrow$ (c) Any finite direct sum of supplemented modules is supplemented. Hence, for every finitely generated $N \in \sigma[M]$, there exists an epimorphism $P \to N$ with some supplemented projective module $P \in \sigma[M]$. By 3.10, every factor module of $P$ has a projective cover in $\sigma[M]$, and so does $N$.

(c) $\Rightarrow$ (a) is trivial.

(2) (a) $\Rightarrow$ (b) Let $P$ be the direct sum of projective covers of a representative set of the simple modules in $\sigma[M]$. Then $P$ is a projective generator and every factor module of $P^{(\Lambda)}$ has a projective cover, and hence $P$ is perfect.

(b) $\Rightarrow$ (a) is obvious. □

**3.13. Left perfect rings.** (Cf. [10, 43.9].)
*For $A$ the following are equivalent:*

(a) *$A$ is a perfect module in ${}_A\mathbf{M}$;*

(b) *$A/\mathrm{Jac}(A)$ is left semisimple and $\mathrm{Jac}(A)$ is right t-nilpotent;*

(c) *every left $A$-module has a projective cover;*

*(d) $A$ satisfies the descending chain condition (dcc) on cyclic right ideals.*

**3.14. (f-)semiperfect rings.** A ring $A$ is said to be *semiperfect* if $A$ is semiperfect as a left $A$-module or – equivalently – as a right $A$-module. More generally, $A$ is called *f-semiperfect* (or *semiregular*) if $A/\mathrm{Jac}(A)$ is von Neumann regular and idempotents lift modulo $\mathrm{Jac}(A)$. Note that $A$ is semiperfect if and only if finitely generated left and right $A$-modules have projective covers, and $A$ is f-semiperfect if and only if every finitely presented left (and right) $A$-module has a projective cover (see [10, 42.11]). From [10, 42.12, 22.1] we recall:

**3.15. (f-)semiperfect endomorphism rings.** *Put $S = {}_A\mathrm{End}(M)$.*

*(1) Assume $M$ to be projective in $\sigma[M]$. Then:*

*(i) $S$ is semiperfect if and only if $M$ is finitely generated and semiperfect.*

*(ii) If $M$ is semiperfect, then $S$ is f-semiperfect.*

*(iii) If $S$ is f-semiperfect, then $\mathrm{Rad}(M) \ll M$ and $M$ is a direct sum of cyclic modules.*

*(2) If $M$ is self-injective, then $S$ is f-semiperfect.*

*(3) If $M$ is self-injective and $\mathrm{Soc}(M) \trianglelefteq M$, then*

$$Jac(S) = {}_A\mathrm{Hom}(M/\mathrm{Soc}(M), M).$$

**3.16. Weak QF modules.** *If ${}_AM$ is faithful, the following are equivalent:*

*(a) ${}_AM$ is a weak QF module;*

*(b) (i) ${}_AM$ is weakly ${}_AM$-injective and $M_S$ is weakly $M_S$-injective, and*

*(ii) $A$ is dense in $\mathrm{End}_S(M)$;*

*(c) $M_S$ is a weak QF module and $A$ is dense in $\mathrm{End}_S(M)$;*

*(d) ${}_AM$ and $M_S$ are weak cogenerators in $\sigma[{}_AM]$ and $\sigma[M_S]$, respectively.*

*For any weak QF module ${}_AM$, $\mathrm{Soc}\, {}_AM = \mathrm{Soc}\, M_S$.*

A locally Noetherian weak QF module $M$ is an injective cogenerator in $\sigma[M]$ (by [10, 16.5]). A Noetherian weak QF module is called a *quasi-Frobenius* or *QF module*. A ring is a *QF ring* if it is QF as a left (or right) module.

Caution: "quasi Frobenius" is used in different ways in the literature.

Let $\sigma_f[M]$ denote the full subcategory of $\sigma[M]$ whose objects are submodules of *finitely* $M$-generated modules. With this notation $\sigma_f[S_S]$ is the category of submodules of finitely generated right $S$-modules. This type of category is of particular interest in studying dualities.

**3.17. Morita dualities.**

(1) *The following are equivalent:*

  (a) ${}_A\mathrm{Hom}(-, M)$*:* $\sigma_f[M] \to \sigma_f[S_S]$ *is a duality;*

  (b) ${}_AM$ *is an injective cogenerator in* $\sigma[M]$*, and* $M_S$ *is an injective cogenerator in* $\mathbf{M}_S$*;*

  (c) ${}_AM$ *is linearly compact, finitely cogenerated, and an injective cogenerator in* $\sigma[M]$*.*

(2) *If* $M$ *is an injective cogenerator in* $\sigma[M]$*, the following are equivalent:*

  (a) ${}_AM$ *is Artinian;*

  (b) ${}_AM$ *is semi-Artinian and* $M_S$ *is* $S$*-injective;*

  (c) $M_S$ *is a* $\Sigma$*-injective cogenerator in* $\mathbf{M}_S$*;*

  (d) $S$ *is right Noetherian.*

Next we consider *relative properties* of $A$-modules related to a fixed ring morphism $\phi : B \to A$. In this case, any left $A$-module $M$ is naturally a left $B$-module and there is an interplay between the properties of $M$ as an $A$-module and those of $M$ as a $B$-module.

**3.18.** $(A, B)$**-finite modules.** The module $M$ is said to be $(A, B)$*-finite* if every finitely generated $A$-submodule of $M$ is finitely generated as a $B$-module. $\sigma[M]$ is said to be $(A, B)$*-finite* if every module in $\sigma[M]$ is $(A, B)$-finite.

*Let* $\sigma[M]$ *be* $(A, B)$*-finite.*

(i) *If* $B$ *is a right perfect ring, then every module in* $\sigma[M]$ *has the dcc on finitely generated* $A$*-submodules.*

(ii) *If* $B$ *is left Noetherian, then every module in* $\sigma[M]$ *is locally Noetherian.*

(iii) *If* $B$ *is left Artinian, then every module in* $\sigma[M]$ *has locally finite length.*

For the following observations we refer to [11, Section 20].

**3.19. Relative notions.** An exact sequence $K \xrightarrow{f} M \xrightarrow{g} N$ in ${}_A\mathbf{M}$ is called $(A, B)$-exact if $\mathrm{Im}\, f$ is a direct summand of $M$ as a left $B$-module.

Let $M, P, Q$ be left $A$-modules. $P$ is called $(M, B)$*-projective* if ${}_A\mathrm{Hom}(P,-)$ is exact with respect to all $(A, B)$-exact sequences in $\sigma[M]$. This is the case if and only if every $(A, B)$-exact sequence $L \to P \to 0$ in $\sigma[M]$ splits.

$Q$ is called $(M, B)$*-injective* if ${}_A\mathrm{Hom}(-, Q)$ is exact with respect to all $(A, B)$-exact sequences in $\sigma[M]$. This happens if and only if every $(A, B)$-exact sequence $0 \to Q \to L$ in $\sigma[M]$ splits.

Over a semisimple ring $B$, $(M, B)$-projective and $(M, B)$-injective are synonymous to projective and injective in $\sigma[M]$, respectively.

**3.20. $(A,B)$-projectives and $(A,B)$-injectives.**

*(1) For any $B$-module $X$, $A\otimes_B X$ is $(A,B)$-projective.*

*(2) $P \in {}_A\mathbf{M}$ is $(A,B)$-projective if and only if the map $A\otimes_B P \to P$, $a\otimes p \mapsto ap$, splits in ${}_A\mathbf{M}$.*

*(3) For any $B$-module $Y$, $\mathrm{Hom}_B(A,Y)$ is $(A,B)$-injective.*

*(4) $Q \in {}_A\mathbf{M}$ is $(A,B)$-injective if and only if the map $Q \to \mathrm{Hom}_B(A,Q)$, $q \mapsto [a \mapsto aq]$, splits in ${}_A\mathbf{M}$.*

The module $M$ is called $(A,B)$-*semisimple* if every $(A,B)$-exact sequence in $\sigma[M]$ splits. The ring $A$ is said to be *left* $(A,B)$-*semisimple* if $A$ is $(A,B)$-semisimple as a left $A$-module.

**3.21. $(A,B)$-semisimple modules.** *The following are equivalent:*

*(a) $M$ is $(A,B)$-semisimple;*

*(b) every $A$-module (in $\sigma[M]$) is $(M,B)$-projective;*

*(c) every $A$-module (in $\sigma[M]$) is $(M,B)$-injective.*

# 4 External properties of $\sigma[M]$

So far we dealt with the internal structure of the category $\sigma[M]$. It is also of interest to look at the properties of $\sigma[M]$ as a class of modules in ${}_A\mathbf{M}$.

Let $T$ be any associative ring (without a unit). A left $T$-module $N$ is called *s-unital* if $u \in Tu$ for every $u \in N$. $T$ itself is called *left s-unital* if it is s-unital as a left $T$-module. For an ideal $T \subset A$, every $A$-module is a $T$-module and we observe the elementary properties:

**4.1. s-unital $T$-modules.** *For any subring $T \subset A$ the following assertions are equivalent:*

*(a) $M$ is an s-unital $T$-module;*

*(b) for any $m_1, \ldots, m_k \in N$, there exists $t \in T$ with $m_i = tm_i$ for all $i \le k$;*

*(c) for any set $\Lambda$, $N^{(\Lambda)}$ is an s-unital $T$-module.*

**Proof.** (a) $\Rightarrow$ (b) We proceed by induction. Assume the assertion holds for $k-1$ elements. Choose $t_k \in T$ such that $t_k n_k = n_k$ and put $a_i = m_i - m_k n_i$, for all $i \le k$. By assumption there exists $t' \in T$ satisfying $a_i = t'a_i$, for all $i \le k-1$. Then $t := t' + t_k - t't_k \in T$ is an element satisfying the condition in (b). The remaining assertions are easily verified. □

**4.2. Flat factor rings.** *For an ideal $T \subset A$ the following are equivalent:*

*(a) $A/T$ is a flat right $A$-module;*

*(b) for every left ideal $I$ of $A$, $TI = T \cap I$;*

*(c) every injective left $A/T$-module is $A$-injective;*

*(d) for every $A$-module ${}_AL \subset {}_AN$, $TL = TN \cap L$;*

*(e) $T$ is left s-unital.*

Under these conditions $T$ is a flat right $A$-module, and, for any $N \in_A \mathbf{M}$, the canonical map $T \otimes_A N \to TN$ is an isomorphism.

**Proof.** The equivalence of (a) and (b) is shown in [10, 36.6].

(a) $\Rightarrow$ (c) Put $D := A/T$. Let $N$ be an injective $D$-module and $L \subset A$ a left ideal. By (a), the sequence $0 \to D \otimes_A L \to D \otimes_A A$ is exact in ${}_D\mathbf{M}$ and there is a commutative diagram with exact rows and canonical isomorphisms,

$$\begin{array}{ccccc}
{}_D\mathrm{Hom}(D \otimes_A A, N) & \longrightarrow & {}_D\mathrm{Hom}(D \otimes_A L, N) & \longrightarrow & 0 \\
\simeq \downarrow & & \downarrow \simeq & & \\
{}_A\mathrm{Hom}(A, {}_D\mathrm{Hom}(D, N)) & \longrightarrow & {}_A\mathrm{Hom}(L, {}_D\mathrm{Hom}(D, N)) & \longrightarrow & 0 \\
\simeq \downarrow & & \downarrow \simeq & & \\
{}_A\mathrm{Hom}(A, N) & \longrightarrow & {}_A\mathrm{Hom}(L, M) & \longrightarrow & 0\,.
\end{array}$$

Since $N$ is an injective $D$-module, the first row is exact and so are the others, that is, $N$ is injective as an $A$-module.

(c) $\Rightarrow$ (a) Let $N$ be a cogenerator in ${}_D\mathbf{M}$ that is $A$-injective. For a left ideal $L \subset A$ there is an exact sequence $0 \to K \to D \otimes_A L \to D \otimes_A A$ in ${}_D\mathbf{M}$, and we want to prove $K = 0$. Consider the exact sequence

$$ {}_D\mathrm{Hom}(D \otimes_A A, N) \to {}_D\mathrm{Hom}(D \otimes_A L, N) \to {}_D\mathrm{Hom}(K, N) \to 0. $$

Now in the above diagram the bottom row is exact ($N$ is $A$-injective). This implies that the top row is also exact, that is, ${}_D\mathrm{Hom}(K, N) = 0$. Since $N$ is a cogenerator in ${}_D\mathbf{M}$, we conclude $K = 0$.

The remaining implications are straightforward to verify. □

**4.3. s-unital modules over ideals.** *Let ${}_AM$ be faithful. For an ideal $T \subset A$ the following are equivalent:*

*(a) $M$ is an s-unital $T$-module;*

*(b) for every $N \in \sigma[M]$, $N = TN$;*

*(c) for every $N \in \sigma[M]$, the canonical map $\varphi_N : T \otimes_A N \to N$ is an isomorphism.*

*If $T \in \sigma[M]$, then (a)–(c) are equivalent to:*

*(d) $T^2 = T$ and $T$ is a generator in $\sigma[M]$.*

**Proof.** The implications follow from 4.1 and 4.2. □

**4.4. Trace ideals.** The trace of $M$ in $A$, $\mathrm{Tr}(M, A)$, is called the *trace ideal of* $M$, and the trace of $\sigma[M]$ in $A$, $\mathcal{T}^M(A) = \mathrm{Tr}(\sigma[M], A)$, is called the *trace ideal of* $\sigma[M]$. Clearly $\mathrm{Tr}(M, A) \subset \mathcal{T}^M(A)$, and equality holds if $M$ is a generator in $\sigma[M]$, or else if $A$ is a left self-injective algebra.

**4.5. Trace ideals of $M$.** Denote ${}^*M = {}_A\mathrm{Hom}(M, A)$, and $T = \mathrm{Tr}(M, A) = {}^*M(M)$. Any $f \in {}^*M$ defines an $A$-linear map

$$ \phi_f : M \to S, \quad m \mapsto f(-)m. $$

$\Delta = \sum_{f \in {}^*M} \mathrm{Im}\, \phi_f$ is an ideal in $S$ and $M\Delta \subset TM$.

*The following are equivalent:*

*(a) $M = TM$;*

*(b) $M = M\Delta$;*

*(c) for any $L \in {}_A\mathbf{M}$, $\mathrm{Tr}(M, L) = TL$.*

*If this holds, $T$ and $\Delta$ are idempotent ideals and $\Delta = \mathrm{Tr}(M_S, S)$.*

**Proof.** (a) $\Leftrightarrow$ (b) are obvious from the definitions.

(a) $\Leftrightarrow$ (c) Clearly $T$ is $M$-generated, and $M = TM$ implies that $M$-generated $A$-modules are $T$-generated.

Assume the conditions hold. By definition, $\varDelta \subset \mathrm{Tr}(M_S, S)$. For any $S$-linear map $g : M \to S$ and $m \in M$, write $m = \sum_i m_i\delta_i$, where $m_i \in M$ and $\delta_i \in \Delta$, to obtain

$$g(m) = g(\sum\nolimits_i m_i\delta_i) = \sum\nolimits_i g(m_i)\delta_i \in \varDelta,$$

thus showing $\mathrm{Tr}(M_S, S) \subset \varDelta$. □

**4.6. Canonical map.** For any $N \in \mathbf{M}_A$ there is a map

$$\alpha_{N,M} : N \otimes_A M \to \mathrm{Hom}_A({}^*M, N),\ n \otimes m \mapsto [f \mapsto nf(m)],$$

which is injective if and only if

for any $u \in N \otimes_A M$, $(I_N \otimes f)(u) = 0$ for all $f \in {}^*M$, implies $u = 0$.

A module $M$ is said to be *locally projective* if, for any diagram of left $A$-modules with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "i"] & M \arrow[d, "g"] & \\ & L \arrow[r, "f"] & N \arrow[r] & 0, \end{tikzcd}$$

where $F$ is finitely generated, there exists $h : M \to L$ such that $g\circ i = f\circ h\circ i$.

**4.7. Locally projective modules.** *With the notation from 4.5, the following are equivalent:*

*(a) $M$ is locally projective;*

*(b) $\alpha_{N,M}$ is injective, for any (cyclic) right $A$-module $N$;*

*(c) for each $m \in M$, $m \in {}^*M(m)M$;*

*(d) for any $m_1, \ldots, m_k \in M$ there exist $x_1, \ldots, x_n \in M$, $f_1, \ldots, f_n \in {}^*M$, such that*

$$m_j = \sum\nolimits_i f_i(m_j)x_i, \quad \textit{for } j = 1, \ldots, k;$$

*(e) $M = TM$, and $M$ is an s-unital right $\varDelta$-module.*

**Proof.** (a) $\Rightarrow$ (d) Put $N = M$ and $L = A^{(\Lambda)}$ in the defining diagram.

(d) $\Rightarrow$ (a) follows by the fact that $A$ is projective as left $A$-module.

(b) $\Rightarrow$ (c) Assume $\alpha_{N,M}$ to be injective for cyclic right $A$-modules $N$. For any $m \in M$ put $J = {}^*M(m)$ and consider the monomorphism

$$\phi : M/JM \simeq A/J \otimes_A M \xrightarrow{\alpha_{N,M}} \mathrm{Hom}_A({}^*M, A/J).$$

For $x \in M$ and $f \in {}^*M$, $\phi(x+JM)(f) = f(x)+J$, and hence $\phi(m+JM) = 0$. By injectivity of $\phi$ this implies $m \in JM$.

(d) $\Rightarrow$ (b) Let $N \in \mathbf{M}_A$ and let $v = \sum_{j=1}^r n_j \otimes m_j \in N \otimes_A M$. Choose $x_1, \ldots, x_n \in M$ and $f_1, \ldots, f_n \in {}^*M$ such that $m_j = \sum_i f_i(m_j)x_i$, for $j = 1, \ldots, r$. Then

$$v = \sum_{i,j} n_j \otimes f_i(m_j)x_i = \sum_i \alpha_{N,M}(v)(f_i) \otimes x_i,$$

and hence $v = 0$ if $\alpha_{N,M}(v) = 0$, that is, $\alpha_{N,M}$ is injective.

(c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) For $m \in M$, $m \in {}^*M(m)M$ means that there are $x_1, \ldots, x_n \in M$ and $f_1, \ldots, f_n \in {}^*M$ such that

$$m = \sum_i f_i(m)x_i = m[\sum_i f_i(-)x_i] \in m\Delta,$$

showing that $M$ is an s-unital right $\Delta$-module (see 4.1). □

**4.8. $\mathcal{T}^M$ as an exact functor.** *Putting $\tilde{T} = \mathcal{T}^M(A)$, the following assertions are equivalent:*

*(a) the functor $\mathcal{T}^M : {}_A\mathbf{M} \to \sigma[M]$ is exact;*

*(b) $\sigma[M]$ is closed under extensions and the class $\{X \in {}_A\mathbf{M} \mid \mathcal{T}^M(X) = 0\}$ is closed under factor modules;*

*(c) for every $N \in \sigma[M]$, $\tilde{T}N = N$;*

*(d) $M$ is an s-unital $\tilde{T}$-module.*

**Proof.** (a) $\Rightarrow$ (b) Let $\mathcal{T}^M$ be exact. For any exact sequence in ${}_A\mathbf{M}$ as a bottom row, there is a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}^M(K) \arrow[r] \arrow[d] & \mathcal{T}^M(L) \arrow[r] \arrow[d] & \mathcal{T}^M(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & L \arrow[r] & N \arrow[r] & 0 .
\end{tikzcd}$$

In case $\mathcal{T}^M(K) = K$ and $\mathcal{T}^M(N) = N$ this implies $\mathcal{T}^M(L) = L$, showing that $\sigma[M]$ is closed under extensions. Moreover $\mathcal{T}^M(L) = 0$ implies $\mathcal{T}^M(N) = 0$ as required.

(b) $\Rightarrow$ (c) Since $\sigma[M]$ is closed under extensions, $\mathcal{T}^M(A/T) = 0$. For any $N \in \sigma[M]$, $N/\tilde{T}N$ is generated by $A/\tilde{T}$, and so by (b), $\mathcal{T}^M(N/\tilde{T}N) = 0$ and hence $N = \tilde{T}N$.

(c) $\Rightarrow$ (a) First observe that the hypothesis implies $\mathcal{T}^M(X) = TX$, for any $X \in {}_A\mathbf{M}$. Consider an exact sequence in ${}_A\mathbf{M}$, $0 \to K \to L \to N \to 0$. Since $T_A$ is flat (see 4.2), tensoring with $\tilde{T} \otimes_A -$ yields an exact sequence $0 \to \tilde{T}K \to \tilde{T}L \to \tilde{T}N \to 0$.

(c) $\Leftrightarrow$ (d) is shown in 4.3. □

We say $\sigma[M]$ is *closed under small epimorphisms* if, for any epimorphism $f : P \to N$ in ${}_A\mathbf{M}$, where $\mathrm{Ke}\, f \ll P$ and $N \in \sigma[M]$, we obtain $P \in \sigma[M]$.

**4.9. Corollary.** *Assume that the functor $\mathcal{T}^M : {}_A\mathbf{M} \to \sigma[M]$ is exact.*

*(1) $\sigma[M]$ is closed under small epimorphisms.*

*(2) If $P$ is finitely presented in $\sigma[M]$, then $P$ is finitely presented in ${}_A\mathbf{M}$.*

*(3) If $P$ is projective in $\sigma[M]$, then $P$ is projective in ${}_A\mathbf{M}$.*

**Proof.** (1) Put $\tilde{T} = \mathcal{T}^M(A)$. Consider an exact sequence $0 \to K \to P \to N \to 0$ in ${}_A\mathbf{M}$, where $K \ll P$ and $N \in \sigma[M]$. From this we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "="] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K+\tilde{T}P \arrow[r] & P \arrow[r] & P/(K+\tilde{T}P) \arrow[r] & 0.
\end{tikzcd}$$

Clearly $P/(K+\tilde{T}P) \in \sigma[M]$ and by condition 4.8(b), $\tilde{T}(P/(K+\tilde{T}P)) = 0$. This implies $P = K + \tilde{T}P$, that is, $P \in \sigma[M]$.

(2) It is enough to show this for any cyclic module $P \in \sigma[M]$ that is finitely presented in $\sigma[M]$. For this we construct the following commutative diagram with exact rows (applying $\mathcal{T}^M$):

$$\begin{tikzcd}
0 \arrow[r] & L_0 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & P \arrow[r] & 0 \\
 & I/L_0 \arrow[r, "\simeq"] & A/L_1, & &
\end{tikzcd}$$

where $L_0$ and $L_1$ are suitable finitely generated modules in $\sigma[M]$. So $I/L_0$ is finitely generated, and hence so is $I$ and $P$ is finitely presented in ${}_A\mathbf{M}$.

(3) This is shown with a similar diagram as in the proof of (2). □

**4.10. Corollary.** *Suppose that $\sigma[M]$ has a generator that is locally projective in ${}_A\mathbf{M}$. Then $\mathcal{T}^M : {}_A\mathbf{M} \to \sigma[M]$ is an exact functor.*

**Proof.** Let $P \in \sigma[M]$ be a locally projective generator. Then clearly $\sigma[M] = \sigma[P]$ and $\tilde{T} = \mathcal{T}^M(A) = \mathrm{Tr}(P, A)$. By 4.5 and 4.7, $\tilde{T}^2 = \tilde{T}$ and $\tilde{T}P = P$. So $\tilde{T}$ generates $P$ and 4.8 applies. □

**4.11. Projective covers in $\sigma[M]$.** *Let $\sigma[M]$ be locally Noetherian and suppose that $A$ is f-semiperfect. Then the following are equivalent:*

*(a) the functor $\mathcal{T}^M : {}_A\mathbf{M} \to \sigma[M]$ is exact;*

*(b) $\sigma[M]$ has a generator that is (locally) projective in ${}_A\mathbf{M}$;*

(c) *there are idempotents $\{e_\lambda\}_\Lambda$ in $A$ such that the $Ae_\lambda$ are in $\sigma[M]$ and form a generating set in $\sigma[M]$;*

(d) *$\sigma[M]$ is a semiperfect category.*

**Proof.** (a) ⇒ (c) Let $S$ be any simple module in $\sigma[M]$. $S$ is finitely presented in $\sigma[M]$ and hence in ${}_A\mathbf{M}$ (by 4.9(2)). Since $A$ is f-semiperfect, $S$ has a projective cover $P$ in ${}_A\mathbf{M}$ (see 3.14). By 4.9(1), $P \in \sigma[M]$ and clearly $P \simeq Ae$ for some idempotent $e \in A$. Now a representing set of simple modules in $\sigma[M]$ yields the required family of idempotents.

(c) ⇒ (b) is obvious and (b) ⇒ (a) follows from 4.10.

(c) ⇔ (d) This is clear by 3.12. □

A submodule $N \subset M$ is said to be *fully invariant* if it is invariant under endomorphisms of $M$, that is, $N$ is an $(A, S)$-submodule. The ring of $(A, S)$-endomorphisms of $M$ is the centre of $S$ (e.g., [11, 4.2]).

**4.12. Big cogenerators.** An $M$-injective module $Q \in \sigma[M]$ is said to be a *big injective cogenerator* in $\sigma[M]$ if every cyclic module in $\sigma[M]$ is isomorphic to a submodule of $Q^{(\mathbb{N})}$. Clearly every big injective cogenerator in $\sigma[M]$ is a cogenerator as well as a subgenerator in $\sigma[M]$. Such modules always exist:

*Let $\{N_\lambda\}_\Lambda$ be a representing set of the cyclic modules in $\sigma[M]$. Then the $M$-injective hull of $\bigoplus_\Lambda N_\lambda$ is a big injective cogenerator in $\sigma[M]$.*

*If $M$ is locally of finite length, every injective cogenerator in $\sigma[M]$ is big.*

**4.13. Correspondence relations.** *Let $Q$ be a big injective cogenerator in the category $\sigma[M]$.*

(1) *For every $N \in \sigma[M]$, $\sigma[N] = \sigma[\mathrm{Tr}(N, Q)]$.*

(2) *The assignment $\sigma[N] \mapsto \mathrm{Tr}(N, Q)$ yields a bijective correspondence between the subcategories of type $\sigma[N]$ of $\sigma[M]$ and the fully invariant submodules of $Q$.*

(3) *If $\sigma[N]$ is closed under essential extensions (injective hulls) in $\sigma[M]$, then $\mathrm{Tr}(N, Q)$ is an $A$-direct summand of $Q$.*

(4) *If $M$ is locally Noetherian and $\mathrm{Tr}(N, Q)$ is an $A$-direct summand of $Q$, then $\sigma[N]$ is closed under essential extensions in $\sigma[M]$.*

(5) *$N \in \sigma[M]$ is semisimple if and only if $\mathrm{Tr}(N, Q) \subset \mathrm{Soc}({}_AQ)$.*

**Proof.** Since $Q$ is $M$-injective, $\mathrm{Tr}(\sigma[N], Q) = \mathrm{Tr}(N, Q)$.

(1) $\mathrm{Tr}(N, Q)$ is a fully invariant submodule that, by definition, belongs to $\sigma[N]$. Consider any finitely generated $L \in \sigma[N]$. Then, by assumption, $L \subset Q^k$, for some $k \in \mathbb{N}$, and hence $L \subset \mathrm{Tr}(L, Q)^k \subset \mathrm{Tr}(N, Q)^k$. This implies $N \in \sigma[\mathrm{Tr}(N, Q)]$.

Parts (2) and (5) are immediate consequences of (1).

(3) If $\sigma[N]$ is closed under essential extensions in $\sigma[M]$, then $\mathrm{Tr}(N,Q)$ is an $A$-direct sumand in $Q$ (and hence is injective in $\sigma[M]$).

(4) Let $M$ be locally Noetherian and $\mathrm{Tr}(N,Q)$ a direct summand of $Q$. Consider any $N$-injective module $L$ in $\sigma[N]$. Then $L$ is a direct sum of $N$-injective uniform modules $U \in \sigma[M]$. Clearly $U$ is (isomorphic to) a direct summand of $\mathrm{Tr}(N,Q)$ and hence of $Q$; that is, $U$ is $M$-injective and so $L$ is $M$-injective, too. □

**4.14. Sum and decomposition of subcategories.** For any $K, L \in \sigma[M]$ we write $\sigma[K] \cap \sigma[L] = 0$, provided $\sigma[K]$ and $\sigma[L]$ have no nonzero module in common. Given a family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M]$, define

$$\sum\nolimits_\Lambda \sigma[N_\lambda] := \sigma[\bigoplus\nolimits_\Lambda N_\lambda].$$

Moreover, we write

$$\sigma[M] = \bigoplus\nolimits_\Lambda \sigma[N_\lambda],$$

provided, for every module $L \in \sigma[M]$, $L = \bigoplus_\Lambda \mathcal{T}^{N_\lambda}(L)$ (internal direct sum). This decomposition of $\sigma[M]$ is known as a *$\sigma$-decomposition*. The category $\sigma[M]$ is *$\sigma$-indecomposable* provided it has no nontrivial $\sigma$-decomposition.

**4.15. $\sigma$-decomposition of modules.** *For a decomposition $M = \bigoplus_\Lambda M_\lambda$, the following are equivalent (cf. [12]):*

*(a) for any distinct $\lambda, \mu \in \Lambda$, $M_\lambda$ and $M_\mu$ have no nonzero isomorphic subfactors;*

*(b) for any distinct $\lambda, \mu \in \Lambda$, $\sigma[M_\lambda] \cap \sigma[M_\mu] = 0$;*

*(c) for any $L \in \sigma[M]$, $L = \bigoplus_\Lambda \mathcal{T}^{N_\lambda}(L)$.*

*If these conditions hold, we call $M = \bigoplus_\Lambda M_\lambda$ a* $\sigma$-decomposition *and in this case*

$$\sigma[M] = \bigoplus\nolimits_\Lambda \sigma[M_\lambda].$$

**4.16. Corollary.** *Let $\sigma[M] = \bigoplus_\Lambda \sigma[N_\lambda]$ be a $\sigma$-decomposition of $\sigma[M]$. Then the trace functor $\mathcal{T}^M$ is exact if and only if the trace functors $\mathcal{T}^{N_\lambda}$ are exact, for all $\lambda \in \Lambda$.*

**4.17. Corollary.** *If $M$ is a projective generator or an injective cogenerator in $\sigma[M]$, then any fully invariant decomposition of $M$ is a $\sigma$-decomposition.*

**Proof.** Let $M = \bigoplus_\Lambda M_\lambda$ be a fully invariant decomposition. If $M$ is a projective generator in $\sigma[M]$, then every submodule of $M_\lambda$ is generated by $M_\lambda$. Since the $M_\lambda$ are projective in $\sigma[M]$, any nonzero (iso)morphism between (sub)factors of $M_\lambda$ and $M_\mu$ yields a nonzero morphism between $M_\lambda$ and $M_\mu$. So the assertion follows from 4.15.

Now suppose that $M$ is an injective cogenerator in $\sigma[M]$. Then every subfactor of $M_\lambda$ must be cogenerated by $M_\lambda$. From this it follows that for $\lambda \neq \mu$, there are no nonzero maps between subfactors of $M_\lambda$ and $M_\mu$ and so 4.15 applies. $\square$

As an example, consider the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p\,prime}\mathbb{Z}_{p^\infty}$ and the decomposition of the category of torsion Abelian groups as a direct sum of the categories of $p$-groups,

$$\sigma[\mathbb{Q}/\mathbb{Z}] = \bigoplus\nolimits_{p\,prime}\sigma[\mathbb{Z}_{p^\infty}].$$

Notice that, although $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in $\mathbf{M}_\mathbb{Z}$ with a nontrivial $\sigma$-decomposition, $\mathbf{M}_\mathbb{Z}$ is $\sigma$-indecomposable. This is possible since $\mathbb{Q}/\mathbb{Z}$ is not a subgenerator in $\mathbf{M}_\mathbb{Z}$.

# Chapter 2

# Coalgebras and comodules

Coalgebras and comodules are obtained by dualising the notions of algebras and modules. Throughout, $R$ denotes a commutative and associative ring with a unit.

## 5 Coalgebras

The main aim of this section is to introduce and give examples of coalgebras and explain the (dual) relationship between algebras and coalgebras.

**5.1. Coalgebras.** An *$R$-coalgebra* is an $R$-module $C$ with $R$-linear maps

$$\Delta : C \to C \otimes_R C \quad \text{and} \quad \varepsilon : C \to R,$$

called *(coassociative) coproduct* and *counit*, respectively, with the properties

$$(I_C \otimes \Delta) \circ \Delta = (\Delta \otimes I_C) \circ \Delta, \text{ and } (I_C \otimes \varepsilon) \circ \Delta = I_C = (\varepsilon \otimes I_C) \circ \Delta,$$

which can be expressed by commutativity of the diagrams

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_R C \arrow[d, "I_C \otimes \Delta"] \\
C \otimes_R C \arrow[r, "\Delta \otimes I_C"] & C \otimes_R C \otimes_R C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "I_C"] & C \otimes_R C \arrow[d, "\varepsilon \otimes I_C"] \\
C \otimes_R C \arrow[r, "I_C \otimes \varepsilon"'] & C \, .
\end{tikzcd}$$

A coalgebra $(C, \Delta, \varepsilon)$ is said to be *cocommutative* if $\Delta = \mathsf{tw} \circ \Delta$, where

$$\mathsf{tw} : C \otimes_R C \to C \otimes_R C, \quad a \otimes b \mapsto b \otimes a,$$

is the twist map.

**5.2. Sweedler's $\Sigma$-notation.** For an elementwise description of the maps we use the *$\Sigma$-notation*, writing for $c \in C$

$$\Delta(c) = \sum_{i=1}^{k} c_i \otimes \tilde{c}_i = \sum c_{\underline{1}} \otimes c_{\underline{2}}.$$

The first version is more precise; the second version, introduced by Sweedler, is very handy in explicit calculations. Notice that $c_{\underline{1}}$ and $c_{\underline{2}}$ do not represent single elements but families $c_1, \ldots, c_k$ and $\tilde{c}_1, \ldots, \tilde{c}_k$ of elements of $C$ that are by no means uniquely determined. Properties of $c_{\underline{1}}$ can only be considered in context with $c_{\underline{2}}$. With this notation, the coassociativity of $\Delta$ is expressed by

$$\sum \Delta(c_{\underline{1}}) \otimes c_{\underline{2}} = \sum c_{\underline{1}\,\underline{1}} \otimes c_{\underline{1}\,\underline{2}} \otimes c_{\underline{2}} = \sum c_{\underline{1}} \otimes c_{\underline{2}\,\underline{1}} \otimes c_{\underline{2}\,\underline{2}} = \sum c_{\underline{1}} \otimes \Delta(c_{\underline{2}}),$$

and, hence, it is possible and convenient to shorten the notation by writing

$$\begin{aligned} (\Delta \otimes I_C)\Delta(c) = (I_C \otimes \Delta)\Delta(c) &= \sum c_{\underline{1}} \otimes c_{\underline{2}} \otimes c_{\underline{3}}, \\ (I_C \otimes I_C \otimes \Delta)(I_C \otimes \Delta)\Delta(c) &= \sum c_{\underline{1}} \otimes c_{\underline{2}} \otimes c_{\underline{3}} \otimes c_{\underline{4}}, \end{aligned}$$

and so on. The conditions for the counit are described by

$$\sum \varepsilon(c_{\underline{1}})c_{\underline{2}} = c = \sum c_{\underline{1}}\varepsilon(c_{\underline{2}}).$$

Cocommutativity is equivalent to $\sum c_{\underline{1}} \otimes c_{\underline{2}} = \sum c_{\underline{2}} \otimes c_{\underline{1}}$.

$R$-coalgebras are closely related or dual to algebras. Indeed, the module of $R$-linear maps from a coalgebra $C$ to any $R$-algebra is an $R$-algebra.

**5.3. The algebra** $\mathrm{Hom}_R(C, A)$**.** *For any $R$-linear map $\Delta : C \to C \otimes_R C$ and an $R$-algebra $A$,* $\mathrm{Hom}_R(C, A)$ *is an $R$-algebra by the* convolution product

$$f * g = \mu \circ (f \otimes g) \circ \Delta, \quad i.e., \quad f * g(c) = \sum f(c_{\underline{1}})g(c_{\underline{2}}),$$

*for $f, g \in \mathrm{Hom}_R(C, A)$ and $c \in C$. Furthermore,*

*(1) $\Delta$ is coassociative if and only if $\mathrm{Hom}_R(C, A)$ is an associative $R$-algebra, for any $R$-algebra $A$.*

*(2) $C$ is cocommutative if and only if $\mathrm{Hom}_R(C, A)$ is a commutative $R$-algebra, for any commutative $R$-algebra $A$.*

*(3) $C$ has a counit if and only if $\mathrm{Hom}_R(C, A)$ has a unit, for all $R$-algebras $A$ with a unit.*

**Proof.** (1) Let $f, g, h \in \mathrm{Hom}_R(C, A)$ and consider the $R$-linear map

$$\tilde{\mu} : A \otimes_R A \otimes_R A \to A, \quad a_1 \otimes a_2 \otimes a_3 \mapsto a_1 a_2 a_3.$$

By definition, the products $(f * g) * h$ and $f * (g * h)$ in $\mathrm{Hom}_R(C, A)$ are the compositions of the maps

$$\begin{tikzcd} & C \otimes_R C \arrow[dr, "\Delta \otimes I_C"] & & & \\ C \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & C \otimes_R C \otimes_R C \arrow[r, "f \otimes g \otimes h"] & A \otimes_R A \otimes_R A \arrow[r, "\tilde{\mu}"] & A\,. \\ & C \otimes_R C \arrow[ur, "I_C \otimes \Delta"'] & & & \end{tikzcd}$$

It is obvious that coassociativity of $\Delta$ yields associativity of $\mathrm{Hom}_R(C, A)$.

To show the converse, we see from the above diagram that it suffices to prove that, (at least) for one associative algebra $A$ and suitable $f, g, h \in \mathrm{Hom}_R(C, A)$, the composition $\tilde{\mu} \circ (f \otimes g \otimes h)$ is a monomorphism. So let $A = T(C)$, the tensor algebra of the $R$-module $C$, and $f = g = h$, the canonical mapping $C \to T(C)$. Then $\tilde{\mu} \circ (f \otimes g \otimes h)$ is just the embedding $C \otimes C \otimes C = T_3(C) \to T(C)$.

(2) If $C$ is cocommutative and $A$ is commutative,

$$f * g\,(c) = \sum f(c_{\underline{1}}) g(c_{\underline{2}}) = \sum g(c_{\underline{1}}) f(c_{\underline{2}}) = g * f\,(c),$$

so that $\mathrm{Hom}_R(C, A)$ is commutative. Conversely, assume that $\mathrm{Hom}_R(C, A)$ is commutative for any commutative $A$. Then

$$\mu \circ (f \otimes g)(\Delta(c)) = \mu \circ (f \otimes g)(\mathsf{tw} \circ \Delta(c)).$$

This implies $\Delta = \mathsf{tw} \circ \Delta$ provided we can find a commutative algebra $A$ and $f, g \in \mathrm{Hom}_R(C, A)$ such that $\mu \circ (f \otimes g) : C \otimes_R C \to A$ is injective. For this take $A$ to be the symmetric algebra $\mathcal{S}(C \oplus C)$. For $f$ and $g$ we choose the mappings

$$C \to C \oplus C, \quad x \mapsto (x, 0), \qquad C \to C \oplus C, \quad x \mapsto (0, x),$$

composed with the canonical embedding $C \oplus C \to \mathcal{S}(C \oplus C)$.

With the canonical isomorphism $h : \mathcal{S}(C) \otimes \mathcal{S}(C) \to \mathcal{S}(C \oplus C)$ and the embedding $\lambda : C \to \mathcal{S}(C)$, we form $h^{-1} \circ \mu \circ (f \otimes g) = \lambda \otimes \lambda$. Since $\lambda(C)$ is a direct summand of $\mathcal{S}(C)$, we obtain that $\lambda \otimes \lambda$ is injective and so $\mu \circ (f \otimes g)$ is injective.

(3) It is easy to check that the unit in $\mathrm{Hom}_R(C, A)$ is

$$C \xrightarrow{\varepsilon} R \xrightarrow{\iota} A, \quad c \mapsto \varepsilon(c) 1_A.$$

For the converse, consider the $R$-module $A = R \oplus C$ and define a unital $R$-algebra

$$\mu : A \otimes_R A \to A, \quad (r, a) \otimes (s, b) \mapsto (rs, rb + as).$$

Suppose there is a unit element in $\mathrm{Hom}_R(C, A)$,

$$e : C \to A = R \oplus C, \quad c \mapsto (\varepsilon(c), \lambda(c)),$$

with $R$-linear maps $\varepsilon : C \to R$, $\lambda : C \to C$. Then, for $f : C \to A$, $c \mapsto (0, c)$, multiplication in $\mathrm{Hom}_R(C, A)$ yields

$$f * e : C \to A, \quad c \mapsto (0, (I_C \otimes \varepsilon) \circ \Delta(c)).$$

By assumption, $f = f * e$ and hence $I_C = (I_C \otimes \varepsilon) \circ \Delta$, one of the conditions for $\varepsilon$ to be a counit. Similarly, the other condition is derived from $f = e * f$.

Clearly $\varepsilon$ is the unit in $\mathrm{Hom}_R(C, R)$, showing the uniqueness of a counit for $C$. □

Note in particular that $C^* = \mathrm{Hom}_R(C, R)$ is an algebra with the convolution product known as the *dual* or *convolution algebra* of $C$.

**Notation.** From now on, $C$ (usually) will denote a coassociative $R$-coalgebra $(C, \Delta, \varepsilon)$, and $A$ will stand for an associative $R$-algebra with unit $(A, \mu, \iota)$.

Many properties of coalgebras depend on properties of the base ring $R$. The base ring can be changed in the following way.

**5.4. Scalar extension.** *Let $C$ be an $R$-coalgebra and $S$ an associative commutative $R$-algebra with unit. Then $C \otimes_R S$ is an $S$-coalgebra with the coproduct*

$$\tilde{\Delta} : C \otimes_R S \xrightarrow{\Delta \otimes I_S} (C \otimes_R C) \otimes_R S \xrightarrow{\simeq} (C \otimes_R S) \otimes_S (C \otimes_R S)$$

*and the counit $\varepsilon \otimes I_S : C \otimes_R S \to S$. If $C$ is cocommutative, then $C \otimes_R S$ is cocommutative.*

**Proof.** By definition, for any $c \otimes s \in C \otimes_R S$,

$$\tilde{\Delta}(c \otimes s) = \sum (c_{\underline{1}} \otimes 1_S) \otimes_S (c_{\underline{2}} \otimes s).$$

It is easily checked that $\tilde{\Delta}$ is coassociative. Moreover,

$$(\varepsilon \otimes I_S \otimes I_{C \otimes_R S}) \circ \tilde{\Delta}(c \otimes s) = \sum \varepsilon(c_{\underline{1}}) c_{\underline{2}} \otimes s = c \otimes s,$$

and similarly $(I_{C \otimes_R S} \otimes \varepsilon \otimes I_S) \circ \tilde{\Delta} = I_{C \otimes_R S}$ is shown. Obviously cocommutativity of $\Delta$ implies cocommutativity of $\tilde{\Delta}$. □

To illustrate the notions introduced above we consider some examples.

**5.5. $R$ as a coalgebra.** The ring $R$ is (trivially) a coassociative, cocommutative coalgebra with the canonical isomorphism $R \to R \otimes_R R$ as coproduct and the identity map $R \to R$ as counit.

**5.6. Free modules as coalgebras.** Let $F$ be a free $R$-module with basis $(f_\lambda)_\Lambda$, $\Lambda$ any set. Then there is a unique $R$-linear map

$$\Delta : F \to F \otimes_R F, \quad f_\lambda \mapsto f_\lambda \otimes f_\lambda,$$

defining a coassociative and cocommutative coproduct on $F$. The counit is provided by the linear map $\varepsilon : F \to R$, $f_\lambda \longmapsto 1$.

**5.7. Semigroup coalgebra.** Let $G$ be a semigroup. A coproduct and counit on the semigroup ring $R[G]$ can be defined by

$$\Delta_1 : R[G] \to R[G] \otimes_R R[G],\ g \mapsto g \otimes g, \quad \varepsilon_1 : R[G] \to R,\ g \mapsto 1.$$

If $G$ has a unit $e$, then another possibility is

$$\Delta_2 : R[G] \to R[G] \otimes_R R[G], \quad g \mapsto \begin{cases} e \otimes e & \text{if } g = e, \\ g \otimes e + e \otimes g & \text{if } g \neq e. \end{cases}$$

$$\varepsilon_2 : R[G] \to R, \quad g \mapsto \begin{cases} 1 & \text{if } g = e, \\ 0 & \text{if } g \neq e. \end{cases}$$

Both $\Delta_1$ and $\Delta_2$ are coassociative and cocommutative.

**5.8. Polynomial coalgebra.** A coproduct and counit on the polynomial ring $R[X]$ can be defined as algebra homomorphisms by

$$\begin{array}{ll} \Delta_1 : R[X] \to R[X] \otimes_R R[X], & X^i \mapsto X^i \otimes X^i, \\ \varepsilon_1 : R[X] \to R, & X^i \mapsto 1, \quad i = 0, 1, 2, \ldots. \end{array}$$

or else by

$$\begin{array}{ll} \Delta_2 : R[X] \to R[X] \otimes_R R[X], & 1 \mapsto 1,\ X^i \mapsto (X \otimes 1 + 1 \otimes X)^i, \\ \varepsilon_2 : R[X] \to R, & 1 \mapsto 1,\ X^i \mapsto 0, \quad i = 1, 2, \ldots. \end{array}$$

Again, both $\Delta_1$ and $\Delta_2$ are coassociative and cocommutative.

**5.9. Coalgebra of a projective module.** Let $P$ be a finitely generated projective $R$-module with dual basis $p_1, \ldots, p_n \in P$ and $\pi_1, \ldots, \pi_n \in P^*$. There is an isomorphism

$$P \otimes_R P^* \to \mathrm{End}_R(P), \quad p \otimes f \mapsto [a \mapsto f(a)p],$$

and on $P^* \otimes_R P$ the coproduct and counit are defined by

$$\Delta : P^* \otimes_R P \to (P^* \otimes_R P) \otimes_R (P^* \otimes_R P), \quad f \otimes p \mapsto \sum_i f \otimes p_i \otimes \pi_i \otimes p,$$

$$\varepsilon : P^* \otimes_R P \to R, \quad f \otimes p \mapsto f(p).$$

By properties of the dual basis,

$$(I_{P \otimes_R P^*} \otimes \varepsilon)\Delta(f \otimes p) = \sum_i f \otimes p_i \pi_i(p) = f \otimes p,$$

showing that $\varepsilon$ is a counit, and coassociativity of $\Delta$ is proved by the equality

$$(I_{P \otimes_R P^*} \otimes \Delta)\Delta(f \otimes p) = \sum\nolimits_{i,j} f \otimes p_i \otimes \pi_i \otimes p_j \otimes \pi_j \otimes p = (\Delta \otimes I_{P \otimes_R P^*})\Delta(f \otimes p).$$

The dual algebra of $P^* \otimes_R P$ is (anti)isomorphic to $\mathrm{End}_R(P)$ by the bijective maps

$$(P^* \otimes_R P)^* = \mathrm{Hom}_R(P^* \otimes_R P, R) \simeq \mathrm{Hom}_R(P, P^{**}) \simeq \mathrm{End}_R(P),$$

which yield a ring isomorphism or anti-isomorphism, depending from which side the morphisms are acting.

For $P = R$ we obtain $R = R^*$, and $R^* \otimes_R R \simeq R$ is the trivial coalgebra. As a more interesting special case we may consider $P = R^n$. Then $P^* \otimes_R P$ can be identified with the matrix ring $M_n(R)$, and this leads to the

**5.10. Matrix coalgebra.** Let $\{e_{ij}\}_{1 \leq i,j \leq n}$ be the canonical $R$-basis for $M_n(R)$, and define the coproduct and counit

$$\Delta : M_n(R) \to M_n(R) \otimes_R M_n(R), \quad e_{ij} \mapsto \sum\nolimits_k e_{ik} \otimes e_{kj},$$
$$\varepsilon : M_n(R) \to R, \quad e_{ij} \mapsto \delta_{ij} \,.$$

The resulting coalgebra is called the $(n, n)$-*matrix coalgebra over* $R$, and we denote it by $M_n^c(R)$.

Notice that the matrix coalgebra may also be considered as a special case of a semigroup coalgebra in 5.7.

From a given coalgebra one can construct the

**5.11. Opposite coalgebra.** Let $\Delta : C \to C \otimes_R C$ define a coalgebra. Then

$$\Delta^{\mathsf{tw}} : C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{\mathsf{tw}} C \otimes_R C, \quad c \mapsto \sum c_{\underline{2}} \otimes c_{\underline{1}},$$

where $\mathsf{tw}$ is the twist map, defines a new coalgebra structure on $C$ known as the *opposite* coalgebra with the same counit. The opposite coalgebra is denoted by $C^{cop}$. Note that a coalgebra $C$ is cocommutative if and only if $C$ coincides with its opposite coalgebra (i.e., $\Delta = \Delta^{\mathsf{tw}}$).

**5.12. Duals of algebras.** Let $(A, \mu, \iota)$ be an $R$-algebra and assume ${}_RA$ to be finitely generated and projective. Then there is an isomorphism

$$A^* \otimes_R A^* \to (A \otimes_R A)^*, \quad f \otimes g \mapsto [a \otimes b \mapsto f(a)g(b)],$$

and the functor $\mathrm{Hom}_R(-, R) = (-)^*$ yields a coproduct

$$\mu^* : A^* \to (A \otimes_R A)^* \simeq A^* \otimes_R A^*$$

and a counit (as the dual of the unit of $A$)

$$\varepsilon := \iota^* : A^* \to R, \quad f \mapsto f(1_A).$$

This makes $A^*$ an $R$-coalgebra that is cocommutative provided $\mu$ is commutative. If ${}_RA$ is not finitely generated and projective, the above construction does not work. However, under certain conditions the *finite dual* of $A$ has a coalgebra structure.

# 6 Coalgebra morphisms

Morphisms are defined as $R$-linear map between coalgebras that respect the coalgebra structures (coproducts and counits).

**6.1. Coalgebra morphisms.** Given $R$-coalgebras $C$ and $C'$, an $R$-linear map $f : C \to C'$ is said to be a *coalgebra morphism* provided the diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes_R C \arrow[r, "f\otimes f"] & C' \otimes_R C' , \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "\varepsilon"'] & C' \arrow[d, "\varepsilon'"] \\ & R \end{tikzcd}$$

are commutative. Explicitly, this means that

$$\Delta' \circ f = (f \otimes f) \circ \Delta, \quad \text{and} \quad \varepsilon' \circ f = \varepsilon,$$

that is, for all $c \in C$,

$$\sum f(c_{\underline{1}}) \otimes f(c_{\underline{2}}) = \sum f(c)_{\underline{1}} \otimes f(c)_{\underline{2}}, \quad \text{and} \quad \varepsilon'(f(c)) = \varepsilon(c).$$

Given an $R$-coalgebra $C$ and an $S$-coalgebra $D$, where $S$ is a commutative ring, a *coalgebra morphism* between $C$ and $D$ is defined as a pair $(\alpha, \gamma)$ consisting of a ring morphism $\alpha : R \to S$ and an $R$-linear map $\gamma : C \to D$ such that

$$\gamma' : C \otimes_R S \to D, \quad c \otimes s \mapsto \gamma(c)s,$$

is an $S$-coalgebra morphism. Here we consider $D$ as an $R$-module (induced by $\alpha$) and $C \otimes_R S$ is the scalar extension of $C$.

As shown in 5.3, for an $R$-algebra $A$, the contravariant functor $\mathrm{Hom}_R(-, A)$ turns coalgebras to algebras. It also turns coalgebra morphisms into algebra morphisms.

**6.2. Duals of coalgebra morphisms.** *For $R$-coalgebras $C$ and $C'$, an $R$-linear map $f : C \to C'$ is a coalgebra morphism if and only if*

$$\mathrm{Hom}(f, A) : \mathrm{Hom}_R(C', A) \to \mathrm{Hom}_R(C, A)$$

*is an algebra morphism, for any $R$-algebra $A$.*

**Proof.** Let $f$ be a coalgebra morphism. Putting $f^* = \mathrm{Hom}_R(f, A)$, we compute for $g, h \in \mathrm{Hom}_R(C', A)$

$$\begin{aligned} f^*(g * h) &= \mu \circ (g \otimes h) \circ \Delta' \circ f = \mu \circ (g \otimes h) \circ (f \otimes f) \circ \Delta \\ &= (g \circ f) * (h \circ f) \ = \ f^*(g) * f^*(h). \end{aligned}$$

To show the converse, assume that $f^*$ is an algebra morphism, that is,

$$\mu \circ (g \otimes h) \circ \Delta' \circ f = \mu \circ (g \otimes h) \circ (f \otimes f) \circ \Delta,$$

for any $R$-algebra $A$ and $g, h \in \mathrm{Hom}_R(C', A)$. Choose $A$ to be the tensor algebra $T(C)$ of the $R$-module $C$ and choose $g, h$ to be the canonical embedding $C \to T(C)$. Then $\mu \circ (g \otimes h)$ is just the embedding $C \otimes_R C \to T_2(C) \to T(C)$, and the above equality implies

$$\Delta' \circ f = (f \otimes f) \circ \Delta,$$

showing that $f$ is a coalgebra morphism. $\square$

**6.3. Coideals.** The problem of determining which $R$-submodules of $C$ are kernels of a coalgebra map $f : C \to C'$ is related to the problem of describing the kernel of $f \otimes f$ (in the category of $R$-modules $\mathbf{M}_R$). If $f$ is surjective, we know that $\mathrm{Ke}\,(f \otimes f)$ is the sum of the canonical images of $\mathrm{Ke}\, f \otimes_R C$ and $C \otimes_R \mathrm{Ke}\, f$ in $C \otimes_R C$. This suggests the following definition.

The kernel of a surjective coalgebra morphism $f : C \to C'$ is called a *coideal* of $C$.

**6.4. Properties of coideals.** *For an $R$-submodule $K \subset C$ and the canonical projection $p : C \to C/K$, the following are equivalent:*

*(a) $K$ is a coideal;*

*(b) $C/K$ is a coalgebra and $p$ is a coalgebra morphism;*

*(c) $\Delta(K) \subset \mathrm{Ke}\,(p \otimes p)$ and $\varepsilon(K) = 0$.*

*If $K \subset C$ is $C$-pure, then (c) is equivalent to:*

*(d) $\Delta(K) \subset C \otimes_R K + K \otimes_R C$ and $\varepsilon(K) = 0$.*

*If (a) holds, then $C/K$ is cocommutative provided $C$ is also.*

**Proof.** (a) $\Leftrightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c) There is a commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & C \arrow[r, "p"] \arrow[d, "\Delta"] & C/K \arrow[r] \arrow[d, "\bar{\Delta}"] & 0 \\
0 \arrow[r] & \mathrm{Ke}\,(p \otimes p) \arrow[r] & C \otimes_R C \arrow[r, "p\otimes p"] & C/K \otimes_R C/K \arrow[r] & 0,
\end{tikzcd}$$

where commutativity of the right square implies the existence of a morphism $K \to \mathrm{Ke}\,(p \otimes p)$, thus showing $\Delta(K) \subset \mathrm{Ke}\,(p \otimes p)$. For the counit $\bar{\varepsilon} : C/K \to R$ of $C/K$, $\bar{\varepsilon} \circ p = \varepsilon$ and hence $\varepsilon(K) = 0$

(c) $\Rightarrow$ (b) Under the given conditions, the left-hand square in the above diagram is commutative and the cokernel property of $p$ implies the existence of $\bar{\Delta}$. This makes $C/K$ a coalgebra with the properties required.

(c) $\Leftrightarrow$ (d) If $K \subset C$ is $C$-pure, $\mathrm{Ke}\,(p \otimes p) = C \otimes_R K + K \otimes_R C$. $\square$

**6.5. Factorisation theorem.** *Let $f : C \to C'$ be a morphism of $R$-coalgebras. If $K \subset C$ is a coideal and $K \subset \operatorname{Ke} f$, then there is a commutative diagram of coalgebra morphisms*

$$\begin{tikzcd} C \arrow[r, "p"] \arrow[dr, "f"'] & C/K \arrow[d, "\bar{f}"] \\ & C' . \end{tikzcd}$$

**Proof.** Denote by $\bar{f} : C/K \to C'$ the $R$-module factorisation of $f : C \to C'$. It is easy to show that the diagram

$$\begin{tikzcd} C/K \arrow[r, "\bar{f}"] \arrow[d, "\bar{\Delta}"'] & C' \arrow[d, "\Delta'"] \\ C/K \otimes_R C/K \arrow[r, "\bar{f}\otimes\bar{f}"] & C' \otimes_R C' \end{tikzcd}$$

is commutative. This means that $\bar{f}$ is a coalgebra morphism. □

**6.6. The counit as a coalgebra morphism.** *View $R$ as a trivial $R$-coalgebra as in 5.5. Then, for any $R$-coalgebra $C$,*

*(1) $\varepsilon$ is a coalgebra morphism;*

*(2) if $\varepsilon$ is surjective, then $\operatorname{Ke} \varepsilon$ is a coideal.*

**Proof.** (1) Consider the diagram

$$\begin{tikzcd} C \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & R \arrow[d, "\simeq"] \\ C \otimes_R C \arrow[r, "\varepsilon\otimes\varepsilon"] & R \otimes_R R \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, mapsto] \arrow[d, mapsto] & \varepsilon(c) \arrow[d, mapsto] \\ \sum c_{\underline{1}} \otimes c_{\underline{2}} \arrow[r, mapsto] & \sum \varepsilon(c_{\underline{1}}) \otimes \varepsilon(c_{\underline{2}}) . \end{tikzcd}$$

The properties of the counit yield

$$\sum \varepsilon(c_{\underline{1}}) \otimes \varepsilon(c_{\underline{2}}) = \sum \varepsilon(c_{\underline{1}})\varepsilon(c_{\underline{2}}) \otimes 1 = \varepsilon(\sum c_{\underline{1}}\varepsilon(c_{\underline{2}})) \otimes 1 = \varepsilon(c) \otimes 1,$$

so the above diagram is commutative and $\varepsilon$ is a coalgebra morphism.

(2)This is clear by (1) and the definition of coideals. □

**6.7. Subcoalgebras.** An $R$-submodule $D$ of a coalgebra $C$ is called a *subcoalgebra* provided $D$ has a coalgebra structure such that the inclusion map is a coalgebra morphism.

Notice that a pure $R$-submodule $D \subset C$ is a subcoalgebra provided $\Delta_D(D) \subset D \otimes_R D \subset C \otimes_R C$ and $\varepsilon|_D : D \to R$ is a counit for $D$. Indeed, since $D$ is a pure submodule of $C$, we obtain

$$\Delta_D(D) = D \otimes_R C \cap C \otimes_R D = D \otimes_R D \subset C \otimes_R C,$$

so that $D$ has a coalgebra structure for which the inclusion is a coalgebra morphism, as required.

From the above observations we obtain:

**6.8. Image of coalgebra morphisms.** *The image of any coalgebra map $f : C \to C'$ is a subcoalgebra of $C'$.*

**6.9. Coproduct of coalgebras.** For a family $\{C_\lambda\}_\Lambda$ of $R$-coalgebras, put $C = \bigoplus_\Lambda C_\lambda$, the coproduct in $\mathbf{M}_R$, $i_\lambda : C_\lambda \to C$ the canonical inclusions, and consider the $R$-linear maps

$$C_\lambda \xrightarrow{\Delta_\lambda} C_\lambda \otimes C_\lambda \subset C \otimes C, \quad \varepsilon : C_\lambda \to R.$$

By the properties of coproducts of $R$-modules there exist unique maps

$$\Delta : C \to C \otimes_R C \text{ with } \Delta \circ i_\lambda = \Delta_\lambda, \quad \varepsilon : C \to R \text{ with } \varepsilon \circ i_\lambda = \varepsilon_\lambda.$$

$(C, \Delta, \varepsilon)$ is called the *coproduct* (or *direct sum*) of the coalgebras $C_\lambda$. It is obvious that the $i_\lambda : C_\lambda \to C$ are coalgebra morphisms.

$C$ is coassociative (cocommutative) if and only if all the $C_\lambda$ have the corresponding property. This follows – by 5.3 – from the ring isomorphism

$$\mathrm{Hom}_R(C, A) = \mathrm{Hom}_R(\textstyle\bigoplus_\Lambda C_\lambda, A) \simeq \prod_\Lambda \mathrm{Hom}_R(C_\lambda, A),$$

for any $R$-algebra $A$, and the observation that the left-hand side is an associative (commutative) ring if and only if every component in the right-hand side has this property.

**Universal property of $C = \bigoplus_\Lambda C_\lambda$.** *For a family $\{f_\lambda : C_\lambda \to C'\}_\Lambda$ of coalgebra morphisms there exists a unique coalgebra morphism $f : C \to C'$ such that, for all $\lambda \in \Lambda$, there are commutative diagrams of coalgebra morphisms*

$$\begin{tikzcd} C_\lambda \arrow[r, "i_\lambda"] \arrow[dr, "f_\lambda"'] & C \arrow[d, "f"] \\ & C' . \end{tikzcd}$$

Recall that for the definition of the tensor product of $R$-algebras $A, B$, the twist map $\mathsf{tw} : A \otimes_R B \to B \otimes_R A$, $a \otimes b \mapsto b \otimes a$ is needed. It also helps to define the

**6.10. Tensor product of coalgebras.** Let $C$ and $D$ be two $R$-coalgebras. Then the composite map

$$C \otimes_R D \xrightarrow{\Delta_C \otimes \Delta_D} (C \otimes_R C) \otimes_R (D \otimes_R D) \xrightarrow{I_C \otimes \mathsf{tw} \otimes I_D} (C \otimes_R D) \otimes_R (C \otimes_R D)$$

defines a coassociative coproduct on $C \otimes_R D$, and with the counits $\varepsilon_C$ of $C$ and $\varepsilon_D$ of $D$ the map $\varepsilon_C \otimes \varepsilon_D : C \otimes_R D \to R$ is a counit of $C \otimes_R D$. With these maps, $C \otimes_R D$ is called the *tensor product coalgebra* of $C$ and $D$. Obviously $C \otimes_R D$ is cocommutative provided both $C$ and $D$ are cocommutative.

**6.11. Tensor product of coalgebra morphisms.** *Let $f : C \to C'$ and $g : D \to D'$ be morphisms of $R$-coalgebras. The tensor product of $f$ and $g$ yields a coalgebra morphism*

$$f \otimes g : C \otimes_R D \to C' \otimes_R D'.$$

**Proof.** The fact that $f$ and $g$ are coalgebra morphisms implies commutativity of the top square in the diagram

$$\begin{tikzcd}
C \otimes_R D \arrow[r, "f \otimes g"] \arrow[d, "\Delta_C \otimes \Delta_D"'] & C' \otimes_R D' \arrow[d, "\Delta_{C'} \otimes \Delta_{D'}"] \\
C \otimes_R C \otimes_R D \otimes_R D \arrow[r, "f \otimes f \otimes g \otimes g"] \arrow[d, "I_C \otimes \mathsf{tw} \otimes I_D"'] & C' \otimes_R C' \otimes_R D' \otimes_R D' \arrow[d, "I_{C'} \otimes \mathsf{tw} \otimes I_{D'}"] \\
C \otimes_R D \otimes_R C \otimes_R D \arrow[r, "f \otimes g \otimes f \otimes g"] & C' \otimes_R D' \otimes_R C' \otimes_R D' ,
\end{tikzcd}$$

while the bottom square obviously is commutative by the definitions. Commutativity of the outer rectangle means that $f \otimes g$ is a coalgebra morphism. $\square$

To define the comultiplication for the tensor product of two $R$-coalgebras $C, D$ in 6.10, the twist map $\mathsf{tw} : C \otimes_R D \to D \otimes_R C$ was used. Notice that any such map yields a formal comultiplication on $C \otimes_R D$, whose properties strongly depend on the properties of the map chosen (see [3, 2.14]).

# 7 Comodules

As before, $R$ denotes a commutative ring, $\mathbf{M}_R$ the category of $R$-modules, and $C$, more precisely $(C, \Delta, \varepsilon)$, stands for a (coassociative) $R$-coalgebra (with counit). We first introduce right comodules over $C$.

**7.1. Right $C$-comodules.** For $M \in \mathbf{M}_R$, an $R$-linear map $\varrho^M : M \to M \otimes_R C$ is called a *right coaction* of $C$ on $M$ or simply a *right $C$-coaction*. To denote the action of $\varrho^M$ on elements of $M$ we write $\varrho^M(m) = \sum m_{\underline{0}} \otimes m_{\underline{1}}$.

A $C$-coaction $\varrho^M$ is said to be *coassociative* and *counital* provided the diagrams

$$\begin{tikzcd} M \arrow[r, "\varrho^M"] \arrow[d, "\varrho^M"'] & M \otimes_R C \arrow[d, "I_M \otimes \Delta"] \\ M \otimes_R C \arrow[r, "\varrho^M \otimes I_C"] & M \otimes_R C \otimes_R C, \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\varrho^M"] \arrow[dr, "I_M"'] & M \otimes_R C \arrow[d, "I_M \otimes \varepsilon"] \\ & M \end{tikzcd}$$

are commutative. Explicitly, this means that, for all $m \in M$,

$$\sum \varrho^M(m_{\underline{0}}) \otimes m_{\underline{1}} = \sum m_{\underline{0}} \otimes \Delta(m_{\underline{1}}), \quad m = \sum m_{\underline{0}} \varepsilon(m_{\underline{1}}).$$

In view of the first of these equations we can shorten the notation and write

$$(I_M \otimes \Delta) \circ \varrho^M(m) = \sum m_{\underline{0}} \otimes m_{\underline{1}} \otimes m_{\underline{2}},$$

and so on, in a way similar to the notation for a coproduct. Note that the elements with subscript 0 are in $M$ while all the elements with positive subscripts are in $C$.

An $R$-module with a coassociative and counital right coaction is called a *right $C$-comodule.*

Recall that any semigroup induces a coalgebra $(R[G], \Delta_1, \varepsilon_1)$ (see 5.7) and for this the comodules have the following form.

**7.2. Graded modules.** *Let $G$ be a semigroup. Considering $R$ with the trivial grading, an $R$-module $M$ is $G$-graded if and only if it is an $R[G]$-comodule.*

**Proof.** Let $M = \bigoplus_G M_g$ be a $G$-graded module. Then a coaction of $(R[G], \Delta_1, \varepsilon_1)$ on $M$ is defined by

$$\varrho^M : M \longrightarrow M \otimes_R R[G], \quad m_g \mapsto m \otimes g.$$

It is easily seen that this coaction is coassociative and, for any $m \in M$,

$$(I_M \otimes \varepsilon_1)\varrho^M(m) = (I_M \otimes \varepsilon_1)(\sum_{g \in G} m_g \otimes g) = \sum_{g \in G} m_g = m\,.$$

Now assume that $M$ is a right $R[G]$-comodule and for all $m \in M$ write $\varrho^M(m) = \sum_{g\in G} m_g \otimes g$. By coassociativity,

$$\sum_{g\in G}(m_g)_h \otimes h \otimes g = \sum_{g\in G} m_g \otimes g \otimes g,$$

which implies $(m_g)_h = \delta_{g,h} m_g$ and also $\varrho^M(m_g) = m_g \otimes g$. Then $M_g = \{m_g \,|\, m \in M\}$ is an independent family of $R$-submodules of $M$. Now counitality of $M$ implies

$$m = (I_M \otimes \varepsilon_1)(\sum_{g\in G} m_g \otimes g) = \sum_{g\in G} m_g,$$

and hence $M = \bigoplus_G M_g$. □

**7.3. Comodule morphisms.** Let $M$, $N$ be right $C$-comodules. An $R$-linear map $f : M \to N$ is called a *comodule morphism* (or *(C-)colinear map*) if and only if the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle \varrho^M}\downarrow & & \downarrow{\scriptstyle \varrho^N} \\ M \otimes_R C & \xrightarrow{f\otimes I_C} & N \otimes_R C \end{array}$$

is commutative. Explicitly, this means that $\varrho^N \circ f = (f \otimes I_C) \circ \varrho^M$; that is, for all $m \in M$ we require

$$\sum f(m)_{\underline{0}} \otimes f(m)_{\underline{1}} = \sum f(m_{\underline{0}}) \otimes m_{\underline{1}}.$$

Clearly the sum of two $C$-morphisms is again a $C$-morphism and the set $\mathrm{Hom}^C(M,N)$ of $C$-morphisms from $M$ to $N$ is an $R$-module, which is determined by the exact sequence in $\mathbf{M}_R$,

$$0 \to \mathrm{Hom}^C(M,N) \to \mathrm{Hom}_R(M,N) \xrightarrow{\gamma} \mathrm{Hom}_R(M, N \otimes_R C),$$

where $\gamma(f) := \varrho^N \circ f - (f \otimes I_C) \circ \varrho^M$.

The class of right comodules over $C$ together with the colinear maps form an additive category which we denote by $\mathbf{M}^C$.

**7.4. Left $C$-comodules** and their morphisms are defined symmetrically by $R$-linear maps ${}^M\!\varrho : M \to C \otimes_R M$. For $m \in M$ we write ${}^M\!\varrho(m) = \sum m_{\underline{-1}} \otimes m_{\underline{0}}$, and coassociativity is expressed as

$$\sum m_{\underline{-1}} \otimes {}^M\!\varrho(m_{\underline{0}}) = \sum \Delta(m_{\underline{-1}}) \otimes m_{\underline{0}} = \sum m_{\underline{-2}} \otimes m_{\underline{-1}} \otimes m_{\underline{0}},$$

where the final expression is a notation. The condition for the counit reads $m = \sum \varepsilon(m_{\underline{-1}}) m_{\underline{0}}$.

The $R$-module of left $C$-morphisms is denoted by ${}^C\mathrm{Hom}\,(M,N)$ and left $C$-comodules and their morphisms again form an additive category that is denoted by ${}^C\mathbf{M}$.

An example of a left and right $C$-comodule is provided by $C$ itself. In both cases coaction is given by $\Delta$ (the *regular coaction*).

**7.5. Kernels and cokernels in $\mathbf{M}^C$.** Let $f : M \to N$ be a morphism in $\mathbf{M}^C$. The cokernel $g$ of $f$ in $\mathbf{M}_R$ yields the exact commutative diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\varrho^M"'] & N \arrow[r, "g"] \arrow[d, "\varrho^N"] & L \arrow[r] & 0 \\
M \otimes_R C \arrow[r, "f\otimes I_C"] & N \otimes_R C \arrow[r, "g\otimes I_C"] & L \otimes_R C \arrow[r] & 0,
\end{tikzcd}$$

which can be completed commutatively in $\mathbf{M}_R$ by some $\varrho^L : L \to L \otimes_R C$ for which we obtain the diagram

$$\begin{tikzcd}
N \arrow[r, "\varrho^N"] \arrow[d, "g"'] & N \otimes_R C \arrow[r, shift left, "\varrho^N\otimes I_C"] \arrow[r, shift right, "I_N\otimes\Delta"'] \arrow[d, "g\otimes I_C"'] & N \otimes_R C \otimes_R C \arrow[d, "g\otimes I_C\otimes I_C"] \\
L \arrow[r, "\varrho^L"] & L \otimes_R C \arrow[r, shift left, "\varrho^L\otimes I_C"] \arrow[r, shift right, "I_L\otimes\Delta"'] & L \otimes_R C \otimes_R C\,.
\end{tikzcd}$$

The outer rectangle is commutative for the upper as well as for the lower morphisms, and hence

$$(\varrho^L \otimes I_C) \circ \varrho^L \circ g = (I_L \otimes \Delta) \circ \varrho^L \circ g\,.$$

Now, surjectivity of $g$ implies $(\varrho^L \otimes I_C) \circ \varrho^L = (I_L \otimes \Delta) \circ \varrho^L$, showing that $\varrho^L$ is coassociative. Moreover,

$$(I_L \otimes \varepsilon) \circ \varrho^L \circ g = (I_L \otimes \varepsilon) \circ (g \otimes I_C) \circ \varrho^N = g,$$

which shows that $(I_L \otimes \varepsilon) \circ \varrho^L = I_L$. Thus $\varrho^L$ is counital, and so it makes $L$ a comodule such that $g$ is a $C$-morphism. This shows that cokernels exist in the category $\mathbf{M}^C$.

Dually, for the kernel $h$ of $f$ in $\mathbf{M}_R$ there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "h"] & M \arrow[r, "f"] \arrow[d, "\varrho^M"'] & N \arrow[d, "\varrho^N"] \\
0 \arrow[r] & K \otimes_R C \arrow[r, "h\otimes I_C"] & M \otimes_R C \arrow[r, "f\otimes I_C"] & N \otimes_R C\,,
\end{tikzcd}$$

where the top sequence is always exact while the bottom sequence is exact under special conditions. If this is the case, the diagram can be extended commutatively by a coaction $\varrho^K : K \to K \otimes_R C$.

If moreover $f$ is $C\otimes_R C$-flat then - dual to the proof for cokernels - it can be shown that $\varrho^K$ is coassociative and counital. Thus kernels of $C$-morphisms are induced from kernels in $\mathbf{M}_R$ provided certain additional conditions are imposed.

**7.6. $C$-subcomodules.** Let $M$ be a right $C$-comodule. An $R$-submodule $K \subset M$ is called a *$C$-subcomodule of $M$* provided $K$ has a right comodule structure such that the inclusion is a comodule morphism.

If $K$ is a $C \otimes_R C$-pure submodule of $M$, then $K$ is a subcomodule of $M$ provided $\varrho^M(K) \subset K \otimes_R C \subset M \otimes_R C$.

**7.7. Coproducts in $\mathbf{M}^C$.** Let $\{M_\lambda, \varrho^M_\lambda\}_\Lambda$ be a family of $C$-comodules. Put $M = \bigoplus_\Lambda M_\lambda$, the coproduct in $\mathbf{M}_R$, $i_\lambda : M_\lambda \to M$ the canonical inclusions, and consider the linear maps

$$M_\lambda \xrightarrow{\varrho^M_\lambda} M_\lambda \otimes_R C \subset M \otimes_R C.$$

Note that the inclusions $i_\lambda$ are $R$-splittings, so that $M_\lambda \otimes_R C \subset M \otimes_R C$ is a pure submodule. By the properties of coproducts of $R$-modules there exists a unique coaction

$$\varrho^M : M \to M \otimes_R C, \text{ such that } \varrho^M \circ i_\lambda = \varrho^M_\lambda,$$

which is coassociative and counital since all the $\varrho^M_\lambda$ are, and thus it makes $M$ a $C$-comodule for which the $i_\lambda : M_\lambda \to M$ are $C$-morphisms with the following universal property:

*Let $\{f_\lambda : M_\lambda \to N\}_\Lambda$ be a family of morphisms in $\mathbf{M}^C$. Then there exists a unique $C$-morphism $f : M \to N$ such that, for each $\lambda \in \Lambda$, the following diagram of $C$-morphisms commutes:*

$$\begin{tikzcd} M_\lambda \arrow[r, "i_\lambda"] \arrow[dr, "f_\lambda"'] & M \arrow[d, "f"] \\ & N\,. \end{tikzcd}$$

Similarly to the coproduct, the direct limit of direct families of $C$-comodules is derived from the direct limit in $\mathbf{M}_R$.

**7.8. Comodules and tensor products.** *Let $M$ be in $\mathbf{M}^C$ and consider any morphism $f : X \to Y$ of $R$-modules. Then:*

*(1) $X \otimes_R M$ is a right $C$-comodule with the coaction*

$$I_X \otimes \varrho^M : X \otimes_R M \longrightarrow X \otimes_R M \otimes_R C,$$

*and the map $f \otimes I_M : X \otimes_R M \to Y \otimes_R M$ is a $C$-morphism.*

*(2) In particular, $X \otimes_R C$ has a right $C$-coaction*

$$I_X \otimes \Delta : X \otimes_R C \longrightarrow X \otimes_R C \otimes_R C,$$

*and the map $f \otimes I_C : X \otimes_R C \to Y \otimes_R C$ is a $C$-morphism.*

(3) *For any index set* $\Lambda$, $R^{(\Lambda)} \otimes_R C \simeq C^{(\Lambda)}$ *as comodules and there exists a surjective* $C$*-morphism*

$$C^{(\Lambda')} \to M \otimes_R C, \quad \textit{for some } \Lambda'.$$

(4) *The structure map* $\varrho^M : M \to M \otimes_R C$ *is a comodule morphism, and hence* $M$ *is a subcomodule of a* $C$*-generated comodule.*

**Proof.** (1) and (2) are easily verified from the definitions.

(3) Take a surjective $R$-linear map $h : R^{(\Lambda')} \to M$. Then, by (2),

$$h \otimes I_C : R^{(\Lambda')} \otimes_R C \to M \otimes_R C$$

is a surjective comodule morphism.

(4) By coassociativity, $\varrho^M$ is a comodule morphism (where $M \otimes_R C$ has the comodule structure from (1)). Note that $\rho^M$ is split by $I_M \otimes \varepsilon$ as an $R$-module; thus $M$ is a pure submodule of $M \otimes_R C$ and hence is a subcomodule. $\square$

Similarly to the classical Hom-tensor relations (see 1.12) we obtain

**7.9. Hom-tensor relations in $\mathbf{M}^C$.** *Let* $X$ *be any* $R$*-module.*

(1) *For any* $M \in \mathbf{M}^C$, *the* $R$*-linear map*

$$\varphi : \mathrm{Hom}^C(M, X \otimes_R C) \to \mathrm{Hom}_R(M, X), \quad f \mapsto (I_X \otimes \varepsilon) \circ f,$$

*is bijective, with inverse map* $h \mapsto (h \otimes I_C) \circ \varrho^M$.

(2) *For any* $M, N \in \mathbf{M}^C$, *the* $R$*-linear map*

$$\psi : \mathrm{Hom}^C(X \otimes_R M, N) \to \mathrm{Hom}_R(X, \mathrm{Hom}^C(M, N)), \; g \mapsto [x \mapsto g(x \otimes -)],$$

*is bijective, with inverse map* $h \mapsto [x \otimes m \mapsto h(x)(m)]$.

**Proof.** (1) For any $f \in \mathrm{Hom}^C(M, X \otimes_R C)$ the diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\varrho^M"'] & X \otimes_R C \arrow[d, "I_X \otimes \Delta"'] \arrow[dr, "="] & \\
M \otimes_R C \arrow[r, "f \otimes I_C"'] & X \otimes_R C \otimes_R C \arrow[r, "I_X \otimes \varepsilon \otimes I_C"'] & X \otimes_R C
\end{tikzcd}$$

is commutative, that is,

$$f = (I_X \otimes \varepsilon \otimes I_C) \circ (f \otimes I_C) \circ \varrho^M = (\varphi(f) \otimes I_C) \circ \varrho^M.$$

This implies that $\varphi$ is injective.

Since $\varrho^M$ is a $C$-morphism, so is $(h \otimes I_C) \circ \varrho^M$, for any $h \in \mathrm{Hom}_R(M, X)$. Therefore

$$\varphi((h \otimes I_C) \circ \rho^M) = (I_X \otimes \varepsilon) \circ (h \otimes I_C) \circ \varrho^M = h \circ (I_M \otimes \varepsilon) \circ \varrho^M = h,$$

implying that $\varphi$ is surjective.

(2) The Hom-tensor relations for modules provide one with an isomorphism of $R$-modules,

$$\psi : \mathrm{Hom}_R(X \otimes_R M, N) \to \mathrm{Hom}_R(X, \mathrm{Hom}_R(M, N)). \qquad (*)$$

For any $x \in X$, by commutativity of the diagram

$$\begin{tikzcd} M \arrow[r, "x\otimes -"] \arrow[d, "\varrho^M"'] & X \otimes_R M \arrow[d, "I_X \otimes \varrho^M"] \\ M \otimes_R C \arrow[r, "(x\otimes -)\otimes I_C"] & X \otimes_R M \otimes_R C, \end{tikzcd} \qquad \begin{tikzcd} m \arrow[r, mapsto] \arrow[d, mapsto] & x \otimes m \arrow[d, mapsto] \\ \varrho^M(m) \arrow[r, mapsto] & x \otimes \varrho^M(m)\,, \end{tikzcd}$$

the map $x \otimes -$ is a $C$-morphism. Hence, for any $g \in \mathrm{Hom}^C(X \otimes_R M, N)$, the composition $g \circ (x \otimes -)$ is a $C$-morphism. On the other hand, there is a commutative diagram, for all $h \in \mathrm{Hom}_R(X, \mathrm{Hom}^C(M, N))$,

$$\begin{tikzcd} X \otimes_R M \arrow[r] \arrow[d, "I_X \otimes \varrho^M"'] & N \arrow[d, "\varrho^N"] \\ X \otimes_R M \otimes_R C \arrow[r] & N \otimes_R C\,, \end{tikzcd} \qquad \begin{tikzcd} x\otimes m \arrow[r, mapsto] \arrow[d, mapsto] & h(x)(m) \arrow[d, mapsto] \\ x\otimes\varrho^M(m) \arrow[r, mapsto] & (h(x)\otimes I_C) \circ \varrho^M(m)\,. \end{tikzcd}$$

This shows that $\psi^{-1}(h)$ lies in $\mathrm{Hom}^C(X \otimes_R M, N)$ and therefore implies that $\psi$ in $(*)$ restricts to the bijective map $\psi : \mathrm{Hom}^C(X \otimes_R M, N) \to \mathrm{Hom}_R(X, \mathrm{Hom}^C(M, N))$, as required. $\square$

Unlike for $A$-modules, the $R$-dual of a right $C$-comodule need not be a left $C$-comodule unless additional conditions are imposed. To specify such sufficient conditions, first recall that, for a finitely presented $R$-module $M$ and a flat $R$-module $C$, there is an isomorphism (compare 2.9)

$$\nu_M : C \otimes_R \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(M, C), \quad c \otimes h \mapsto c \otimes h(-)\,.$$

**7.10. Comodules finitely presented as $R$-modules.** *Let ${}_RC$ be flat and $M \in \mathbf{M}^C$ such that ${}_RM$ is finitely presented. Then $M^* = \mathrm{Hom}_R(M, R)$ is a left $C$-comodule by the structure map*

$${}^{M^*}\!\varrho : M^* \to \mathrm{Hom}_R(M, C) \simeq C \otimes_R M^*, \quad g \mapsto (g \otimes I_C) \circ \varrho^M\,.$$

**Proof.** The comodule property of $M^*$ follows from the commutativity of the following diagram (with obvious maps), the central part of which arises from the coassociativity of $C$ (tensor over $R$):

$$\begin{tikzcd}
M^* \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}^C(M,C) \arrow[r] \arrow[d] & \mathrm{Hom}^C(M,C\otimes C) \arrow[r, "\simeq"] \arrow[d] & C\otimes M^* \arrow[d, "\Delta\otimes I_{M^*}"] \\
C\otimes M^* \arrow[r, "\simeq"] & \mathrm{Hom}^C(M,C\otimes C) \arrow[r] & \mathrm{Hom}^C(M,C\otimes C\otimes C) \arrow[r, "\simeq"] & C\otimes C\otimes M^* .
\end{tikzcd}$$

□

For $X = R$ and $M = C$, the isomorphism $\varphi$ describes the comodule endomorphisms of $C$.

**7.11. Comodule endomorphisms of $C$.**

*(1) There is an algebra anti-isomorphism* $\varphi : \mathrm{End}^C(C) \to C^*$, $f \mapsto \varepsilon \circ f$, *with the inverse map* $h \mapsto (h \otimes I_C) \circ \Delta$ *and so* $h \in C^*$ *acts on* $c \in C$ *from the right by*

$$c\leftharpoonup h = (h \otimes I_C)\Delta(c) = \sum h(c_{\underline{1}})c_{\underline{2}}.$$

*(2) There is an algebra isomorphism* $\varphi' : {}^C\mathrm{End}(C) \to C^*$, $f \mapsto \varepsilon \circ f$, *with the inverse map* $h \mapsto (I_C \otimes h) \circ \Delta$ *and so* $h \in C^*$ *acts on* $c \in C$ *from the left by*

$$h\rightharpoonup c = (I_C \otimes h)\Delta(c) = \sum c_{\underline{1}}h(c_{\underline{2}}).$$

*(3) For any* $f \in C^*$ *and* $c \in C$,

$$\begin{array}{rcl}
\Delta(f\rightharpoonup c) & = & \sum c_{\underline{1}} \otimes (f\rightharpoonup c_{\underline{2}}), \\
\Delta(c\leftharpoonup f) & = & \sum (c_{\underline{1}}\leftharpoonup f) \otimes c_{\underline{2}}, \\
\Delta(f\rightharpoonup c\leftharpoonup g) & = & \sum (c_{\underline{1}}\leftharpoonup g) \otimes (f\rightharpoonup c_{\underline{2}}), \\
\sum c_{\underline{1}} \otimes (c_{\underline{2}}\leftharpoonup f) & = & \sum (f\rightharpoonup c_{\underline{1}}) \otimes c_{\underline{2}}.
\end{array}$$

*(4) The coproduct* $\Delta$ *yields the embedding*

$$C^* \simeq \mathrm{Hom}^C(C,C) \to \mathrm{Hom}^C(C, C\otimes_R C) \simeq \mathrm{End}_R(C).$$

**Proof.** (1) By 7.9(1), $\varphi$ is $R$-linear and bijective. Take any $f$, $g \in \mathrm{End}^C(C)$, recall that $(f \otimes I_C) \circ \Delta = \Delta \circ f$, and consider the convolution product applied to any $c \in C$,

$$\begin{array}{rcl}
(\varepsilon \circ f) * (\varepsilon \circ g)(c) & = & \sum \varepsilon(f(c_{\underline{1}}))\,\varepsilon(g(c_{\underline{2}})) \\
& = & \varepsilon \circ g\,[(\varepsilon \otimes I_C) \circ (f \otimes I_C) \circ \Delta(c)] \\
& = & \varepsilon \circ g\,[(\varepsilon \otimes I_C) \circ \Delta \circ f(c)] \;=\; \varepsilon \circ (g \circ f)(c)\,.
\end{array}$$

This shows that $\varphi$ is an anti-isomorphism.

(2) For all $f, g \in {}^C\mathrm{End}\,(C)$, $(I_C \otimes g) \circ \Delta = \Delta \circ g$, and hence

$$\begin{aligned}(\varepsilon \circ f) * (\varepsilon \circ g)(c) &= \sum \varepsilon(f(c_{\underline{1}}))\,\varepsilon(g(c_{\underline{2}})) \\ &= \varepsilon \circ f\,[(I_C \otimes \varepsilon) \circ (I_C \otimes g) \circ \Delta(c)] \\ &= \varepsilon \circ g\,[(I_C \otimes \varepsilon) \circ \Delta \circ g(c)] = \varepsilon \circ (f \circ g)(c)\,.\end{aligned}$$

(3) By definition,

$$\begin{aligned}\Delta(f \rightharpoonup c) &= \Delta(\textstyle\sum c_{\underline{1}} f(c_{\underline{2}}) \;=\; \sum c_{\underline{11}} \otimes c_{\underline{12}} f(c_{\underline{2}}) \\ &= \textstyle\sum c_{\underline{1}} \otimes c_{\underline{21}} f(c_{\underline{22}}) \;=\; \sum c_{\underline{1}} \otimes (f \rightharpoonup c_{\underline{2}}).\end{aligned}$$

The remaining assertions are shown similarly.

(4) This follows from the Hom-tensor relations 7.9 for $M = C = X$ . □

Notice that in 7.11(1) the comodule morphisms are written on the left of the argument. By writing morphisms of right comodules on the right side, we obtain an isomorphism between $C^*$ and the comodule endomorphism ring.

The next theorem summarises observations on the category of comodules.

**7.12. The category $\mathbf{M}^C$.**

*(1) The category $\mathbf{M}^C$ has direct sums and cokernels, and $C$ is a subgenerator.*

*(2) $\mathbf{M}^C$ is a Grothendieck category provided that $C$ is a flat $R$-module.*

*(3) The functor $-\otimes_R C : \mathbf{M}_R \to \mathbf{M}^C$ is right adjoint to the forgetful functor $(-)_R : \mathbf{M}^C \to \mathbf{M}_R$.*

*(4) For any monomorphism $f : K \to L$ of $R$-modules,*

$$f \otimes I_C : K \otimes_R C \to L \otimes_R C$$

*is a monomorphism in $\mathbf{M}^C$.*

*(5) For any family $\{M_\lambda\}_\Lambda$ of $R$-modules, $(\prod_\Lambda M_\lambda) \otimes_R C$ is the product of the $M_\lambda \otimes_R C$ in $\mathbf{M}^C$.*

**Proof.** (1) The first assertions follow from 7.5 and 7.7. By 7.8(4), any comodule $M$ is a subcomodule of the $C$-generated comodule $M \otimes_R C$.

(2) By 7.5, $\mathbf{M}^C$ has kernels provided $C$ is a flat $R$-module. This implies that the intersection of two subcomodules and the preimage of a (sub)comodule is again a comodule. It remains to show that $\mathbf{M}^C$ has (a set of) generators. For any right $C$-comodule $M$, there exists a surjective comodule map $g : C^{(\Lambda)} \to M \otimes_R C$ (see 7.8). Then $L := g^{-1}(M) \subset C^{(\Lambda)}$ is a subcomodule. Furthermore, for any $m \in M$ there exist $k \in \mathbb{N}$ and an element $x$ in the comodule $C^k \cap L \subset C^k$ such that $g(x) = m$. Therefore $m \in g(C^k \cap L)$. This

shows that $M$ is generated by comodules of the form $C^k \cap L$, $k \in \mathbb{N}$. Hence the subcomodules of $C^k$, $k \in \mathbb{N}$, form a set of generators of $\mathbf{M}^C$.

(3) For all $M \in \mathbf{M}^C$ and $X \in \mathbf{M}_R$, let $\varphi_{M,X}$ denote the isomorphism constructed in 7.9(1). We need to show that $\varphi_{M,X}$ is natural in $M$ and $X$. First take any right $C$-comodule $N$ and any $g \in \mathrm{Hom}^C(M,N)$. Then, for all $f \in \mathrm{Hom}^C(N, X \otimes_R C)$,

$$\begin{aligned}(\varphi_{M,X} \circ \mathrm{Hom}^C(g, X \otimes_R C))(f) &= (I_X \otimes \varepsilon) \circ \mathrm{Hom}^C(g, X \otimes_R C)(f)\\ &= (I_X \otimes \varepsilon) \circ f \circ g\\ &= \mathrm{Hom}_R(g, X)((I_X \otimes \varepsilon) \circ f)\\ &= (\mathrm{Hom}_R(g, X) \circ \varphi_{N,X})(f).\end{aligned}$$

Similarly, take any $R$-module $Y$ and $g \in \mathrm{Hom}_R(X, Y)$. Then, for any map $f \in \mathrm{Hom}^C(M, X \otimes_R C)$,

$$\begin{aligned}(\varphi_{M,Y} \circ \mathrm{Hom}^C(M, g \otimes I_C))(f) &= (I_Y \otimes \varepsilon) \circ (\mathrm{Hom}^C(M, g \otimes I_C)(f))\\ &= (I_Y \otimes \varepsilon) \circ (g \otimes I_C) \circ f\\ &= (g \otimes \varepsilon) \circ f = g \circ (I_X \otimes \varepsilon) \circ f\\ &= (\mathrm{Hom}_R(M, g) \circ \varphi_{M,X})(f).\end{aligned}$$

This proves the naturality of $\varphi$ and thus the adjointness property. Note that the unit of this adjunction is provided by the coaction $\varrho^M : M \to M \otimes_R C$, while the counit is $I_X \otimes \varepsilon : X \otimes_R C \to X$.

(4) Any functor that has a left adjoint preserves monomorphisms (cf. [3, 38.21]). Note that monomorphisms in $\mathbf{M}^C$ need not be injective maps, unless ${}_RC$ is flat.

(5) By (3), for all $X \in \mathbf{M}^C$ there are isomorphisms

$$\begin{aligned}\mathrm{Hom}^C(X, (\textstyle\prod_\Lambda M_\lambda) \otimes_R C) &\simeq \mathrm{Hom}_R(X, \textstyle\prod_\Lambda M_\lambda)\\ &\simeq \textstyle\prod_\Lambda \mathrm{Hom}_R(X, M_\lambda)\\ &\simeq \textstyle\prod_\Lambda \mathrm{Hom}^C(X, M_\lambda \otimes_R C).\end{aligned}$$

These isomorphisms characterise $(\prod_\Lambda M_\lambda) \otimes_R C$ as product of the $M_\lambda \otimes_R C$ in $\mathbf{M}^C$. □

**7.13. $C$ as a flat $R$-module.** *The following are equivalent:*

(a) *$C$ is flat as an $R$-module;*

(b) *every monomorphism in $\mathbf{M}^C$ is injective;*

(c) *every monomorphism $U \to C$ in $\mathbf{M}^C$ is injective;*

(d) *the forgetful functor $\mathbf{M}^C \to \mathbf{M}_R$ respects monomorphisms.*

**Proof.** (a) $\Rightarrow$ (b) Consider a monomorphism $f : M \to N$. Since ${}_RC$ is flat, the inclusion $i : \mathrm{Ke}\, f \to M$ is a morphism in $\mathbf{M}^C$ (by 7.5) and $f \circ i = f \circ 0 = 0$ implies $i = 0$, that is, $\mathrm{Ke}\, f = 0$.

(b) $\Rightarrow$ (c) and (b) $\Leftrightarrow$ (d) are obvious.

(c) $\Rightarrow$ (a) For every ideal $J \subset R$, the canonical map $J \otimes_R C \to R \otimes_R C$ is a monomorphism in $\mathbf{M}^C$ by 7.12(4), and hence it is injective by assumption. This implies that ${}_RC$ is flat (see 1.16). $\square$

Recall that a monomorphism $i : N \to L$ in $\mathbf{M}_R$ is a *coretraction* provided there exists $p : L \to N$ in $\mathbf{M}_R$ with $p \circ i = I_N$.

**7.14. Relative injective comodules.** A right $C$-comodule $M$ is said to be *relative injective* or $(C, R)$*-injective* if, for every $C$-comodule map $i : N \to L$ that is an $R$-module coretraction, and for every morphism $f : N \to M$ in $\mathbf{M}^C$, there exists a right $C$-comodule map $g : L \to M$ such that $g \circ i = f$. In other words, we require that every diagram in $\mathbf{M}^C$

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[dr, "f"'] & L \\ & M \end{tikzcd}$$

can be completed commutatively by some $C$-morphism $g : L \to M$, provided there exists an $R$-module map $p : L \to N$ such that $p \circ i = I_N$.

**7.15.** $(C, R)$**-injectivity.** *Let $M$ be a right $C$-comodule.*

*(1) The following are equivalent:*

*(a) $M$ is $(C, R)$-injective;*

*(b) any $C$-comodule map $i : M \to L$ that is a coretraction in $\mathbf{M}_R$ is also a coretraction in $\mathbf{M}^C$;*

*(c) the coaction $\varrho^M : M \to M \otimes_R C$ is a coretraction in $\mathbf{M}^C$.*

*(2) For any $X \in \mathbf{M}_R$, $X \otimes_R C$ is $(C, R)$-injective.*

*(3) If $M$ is $(C, R)$-injective, then, for any $L \in \mathbf{M}^C$, the canonical sequence*

$$0 \longrightarrow \mathrm{Hom}^C(L, M) \xrightarrow{\ i\ } \mathrm{Hom}_R(L, M) \xrightarrow{\ \gamma\ } \mathrm{Hom}_R(L, M \otimes_R C)$$

*splits in $\mathbf{M}_B$, where $B = \mathrm{End}^C(L)$ and $\gamma(f) = \varrho^M \circ f - (f \otimes I_C) \circ \varrho^L$ (see 7.3).*

*In particular, $\mathrm{End}^C(C) \simeq C^*$ is a $C^*$-direct summand in $\mathrm{End}_R(C)$.*

**Proof.** (1) (a) $\Rightarrow$ (b) Suppose that $M$ is $(C, R)$-injective and take $N = M$ and $f = I_M$ in 7.14 to obtain the assertion.

(b) $\Rightarrow$ (c) View $M \otimes_R C$ as a right $C$-comodule with the coaction $I_M \otimes \Delta$, and note that $\varrho^M : M \to M \otimes_R C$ is a right $C$-comodule map that has an $R$-linear retraction $I_M \otimes \varepsilon$. Therefore $\varrho^M$ is a coretraction in $\mathbf{M}^C$.

(c) $\Rightarrow$ (a) Suppose there exists a right $C$-comodule map $h : M \otimes_R C \to M$ such that $h \circ \rho^M = I_M$, consider a diagram

$$\begin{tikzcd} N \arrow[rr, "i"] \arrow[dr, "f"'] & & L \\ & M & \end{tikzcd}$$

as in 7.14, and assume that there exists an $R$-module map $p : L \to N$ such that $p \circ i = I_N$. Define an $R$-linear map $g : L \to M$ as a composition

$$g : \; L \xrightarrow{\varrho^L} L \otimes_R C \xrightarrow{f \circ p \otimes I_C} M \otimes_R C \xrightarrow{h} M \, .$$

Clearly, $g$ is a right $C$-comodule map as a composition of $C$-comodule maps. Furthermore,

$$\begin{aligned} g \circ i &= h \circ (f \circ p \otimes I_C) \circ \varrho^L \circ i = h \circ (f \circ p \circ i \otimes I_C) \circ \varrho^N \\ &= h \circ (f \otimes I_C) \circ \varrho^N \; = \; h \circ \varrho^M \circ f \; = \; f, \end{aligned}$$

where we used that both $i$ and $f$ are $C$-colinear. Thus the above diagram can be completed to a commutative diagram in $\mathbf{M}^C$, and hence $M$ is $(C, R)$-injective.

(2) The coaction for $X \otimes_R C$ is given by $\varrho^{X \otimes_R C} = I_X \otimes \Delta$, and it is split by a right $C$-comodule map $I_X \otimes \varepsilon \otimes I_C$. Thus $X \otimes_R C$ is $(C, R)$-injective by part (1).

(3) Denote by $h : M \otimes_R C \to M$ the splitting map of $\varrho^M$ in $\mathbf{M}^C$. Then the map

$$\mathrm{Hom}_R(L, M) \simeq \mathrm{Hom}^C(L, M \otimes_R C) \to \mathrm{Hom}^C(L, M), \quad f \mapsto h \circ (f \otimes I_C) \circ \varrho^L,$$

splits the first inclusion in $\mathbf{M}_B$, and the map

$$\mathrm{Hom}_R(L, M \otimes_R C) \to \mathrm{Hom}_R(L, M), \; g \mapsto h \circ g$$

yields a splitting map $\mathrm{Hom}_R(L, M \otimes_R C) \to \mathrm{Hom}_R(L, M)/\mathrm{Hom}^C(L, M)$, since for any $f \in \mathrm{Hom}_R(L, M)$,

$$h \circ \gamma(f) = f - h \circ (f \otimes I_C) \circ \varrho^L \in f + \mathrm{Hom}^C(L, M).$$

$\square$

If ${}_R C$ is flat, $\mathbf{M}^C$ is a Grothendieck category by 7.12, so exact sequences are defined in $\mathbf{M}^C$ and we can describe

**7.16. Exactness of the $\mathrm{Hom}^C$-functors.** *Assume ${}_RC$ to be flat and let $M \in \mathbf{M}^C$. Then:*

*(1)* $\mathrm{Hom}^C(-, M) : \mathbf{M}^C \to \mathbf{M}_R$ *is a left exact functor.*

*(2)* $\mathrm{Hom}^C(M, -) : \mathbf{M}^C \to \mathbf{M}_R$ *is a left exact functor.*

**Proof.** (1) From any exact sequence $X \to Y \to Z \to 0$ in $\mathbf{M}^C$ we derive the commutative diagram (tensor over $R$)

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}^C(Z, M) & \longrightarrow & \mathrm{Hom}^C(Y, M) & \longrightarrow & \mathrm{Hom}^C(X, M) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_R(Z, M) & \longrightarrow & \mathrm{Hom}_R(Y, M) & \longrightarrow & \mathrm{Hom}_R(X, M) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_R(Z, M\otimes C) & \longrightarrow & \mathrm{Hom}_R(Y, M\otimes C) & \longrightarrow & \mathrm{Hom}_R(X, M\otimes C),
\end{array}$$

where the columns are exact by the characterisation of comodule morphisms (in 7.3). The second and third rows are exact by exactness properties of the functors $\mathrm{Hom}_R$. Now the diagram lemmata imply that the first row is exact, too.

Part (2) is shown with a similar diagram. $\square$

An object $Q \in \mathbf{M}^C$ is *injective in* $\mathbf{M}^C$ if, for any monomorphism $M \to N$ in $\mathbf{M}^C$, the canonical map $\mathrm{Hom}^C(N, Q) \to \mathrm{Hom}^C(M, Q)$ is surjective.

**7.17. Injectives in $\mathbf{M}^C$.** *Assume ${}_RC$ to be flat.*

*(1)* $Q \in \mathbf{M}^C$ *is injective if and only if* $\mathrm{Hom}^C(-, Q) : \mathbf{M}^C \to \mathbf{M}_R$ *is exact.*

*(2) If* $X \in \mathbf{M}_R$ *is injective in* $\mathbf{M}_R$*, then* $X \otimes_R C$ *is injective in* $\mathbf{M}^C$.

*(3) If* $M \in \mathbf{M}^C$ *is* $(C, R)$*-injective and injective in* $\mathbf{M}_R$*, then* $M$ *is injective in* $\mathbf{M}^C$.

*(4)* $C$ *is* $(C, R)$*-injective, and it is injective in* $\mathbf{M}^C$ *provided that* $R$ *is injective in* $\mathbf{M}_R$.

*(5) If* ${}_RM$ *is flat and* $N$ *is injective in* $\mathbf{M}^C$*, then* $\mathrm{Hom}^C(M, N)$ *is injective in* $\mathbf{M}_R$.

**Proof.** (1) The assertion follows by 7.16.

(2) This follows from the isomorphism in 7.9(1).

(3) Since $M$ is $R$-injective, assertion (2) implies that $M \otimes_R C$ is injective in $\mathbf{M}^C$. Moreover, by 7.15, $M$ is a direct summand of $M \otimes_R C$ as a comodule, and hence it is also injective in $\mathbf{M}^C$.

Part (4) is a special case of (2).

(5) This follows from the isomorphism in 7.9(2). □

An object $P \in \mathbf{M}^C$ is *projective in* $\mathbf{M}^C$ if, for any epimorphism $M \to N$ in $\mathbf{M}^C$, the canonical map $\mathrm{Hom}^C(P,M) \to \mathrm{Hom}^C(P,N)$ is surjective.

**7.18. Projectives in $\mathbf{M}^C$.** *Consider any $P \in \mathbf{M}^C$.*

*(1) If $P$ is projective in $\mathbf{M}^C$, then $P$ is projective in $\mathbf{M}_R$.*

*(2) If ${}_RC$ is flat, the following are equivalent:*

*(a) $P$ is projective in $\mathbf{M}^C$;*

*(b) $\mathrm{Hom}^C(P,-) : \mathbf{M}^C \to \mathbf{M}_R$ is exact.*

**Proof.** (1) For any epimorphism $f : K \to L$ in $\mathbf{M}_R$, $K \otimes_R C \xrightarrow{f \otimes I_C} L \otimes_R C$ is an epimorphism in $\mathbf{M}^C$ and the projectivity of $P$ implies the exactness of the top row in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^C(P, K \otimes_R C) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}^C(P, L \otimes_R C) \arrow[r] \arrow[d, "\simeq"] & 0 \\
\mathrm{Hom}_R(P,K) \arrow[r, "\mathrm{Hom}(P,f)"] & \mathrm{Hom}_R(P,L)\,, &
\end{tikzcd}$$

where the vertical maps are the functorial isomorphisms from 7.9(1). From this we see that $\mathrm{Hom}(P,f)$ is surjective, proving that $P$ is projective as an $R$-module.

(2) This follows from left exactness of $\mathrm{Hom}^C(P,-)$ described in 7.16. □

Note that, although there are enough injectives in $\mathbf{M}^C$, there are possibly no projective objects in $\mathbf{M}^C$. This remains true even if $R$ is a field.

**7.19. Tensor product and $\mathrm{Hom}^C$.** *Let ${}_RC$ be flat, and consider $M, N \in \mathbf{M}^C$ and $X \in \mathbf{M}_R$ such that*

*(i) $M_R$ is finitely generated and projective, and $N$ is $(C,R)$-injective; or*

*(ii) $M_R$ is finitely presented and $X$ is flat in $\mathbf{M}_R$.*

*Then there exists a canonical isomorphism*

$$\nu : X \otimes_R \mathrm{Hom}^C(M,N) \longrightarrow \mathrm{Hom}^C(M, X \otimes_R N), \quad x \otimes h \mapsto x \otimes h(-).$$

**Proof.** Consider the defining exact sequence for $\mathrm{Hom}^C$ (see 7.3),

$$(*) \quad 0 \longrightarrow \mathrm{Hom}^C(M,N) \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M, N \otimes_R C).$$

Tensoring with $X_R$ yields the commutative diagram (tensor over $R$)

$$\begin{tikzcd}
0 \arrow[r] & X \otimes \mathrm{Hom}^C(M,N) \arrow[r] \arrow[d, "\nu"] & X \otimes \mathrm{Hom}_R(M,N) \arrow[r] \arrow[d, "\simeq"] & X \otimes \mathrm{Hom}_R(M, N \otimes C) \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathrm{Hom}^C(M, X \otimes N) \arrow[r] & \mathrm{Hom}_R(M, X \otimes N) \arrow[r] & \mathrm{Hom}_R(M, X \otimes N \otimes C),
\end{tikzcd}$$

where the bottom row is exact (again by 7.3) and the vertical isomorphisms follow from the finiteness assumptions in (i) and (ii) (cf. 2.9).

If $X$ is flat, the top row is exact. On the other hand, if $N$ is $(C, R)$-injective, the sequence $(*)$ splits by 7.15, and hence the top row is exact, too. Therefore, in either case, the exactness of the diagram implies that $\nu$ is an isomorphism, as required. $\square$

# 8 $C$-comodules and $C^*$-modules

For any $R$-coalgebra $C$ the dual $R$-module $C^* = \mathrm{Hom}_R(C, R)$ is an associative algebra. In this section we study the relationship between $C$-comodules and $C^*$-modules.

**8.1. $C$-comodules and $C^*$-modules.**

*(1) Any $M \in \mathbf{M}^C$ is a (unital) left $C^*$-module by*

$$\rightharpoonup : C^* \otimes_R M \to M, \quad f \otimes m \mapsto (I_M \otimes f) \circ \varrho^M(m) = \sum m_{\underline{0}} f(m_{\underline{1}}).$$

*(2) Any morphism $h : M \to N$ in $\mathbf{M}^C$ is a left $C^*$-module morphism, that is,*

$$\mathrm{Hom}^C(M, N) \subset {}_{C^*}\mathrm{Hom}\,(M, N).$$

*(3) There is a faithful functor from $\mathbf{M}^C$ to $\sigma[{}_{C^*}C]$, the full subcategory of ${}_{C^*}\mathbf{M}$ consisting of all $C^*$-modules subgenerated by $C$ (cf. 3.1).*

**Proof.** (1) By definition, for all $f, g \in C^*$ and $m \in M$, the actions $f \rightharpoonup (g \rightharpoonup m)$ and $(f * g) \rightharpoonup m$ are the compositions of the maps in the top and bottom rows of the following commutative diagram:

$$\begin{tikzcd}
 & M \otimes_R C \arrow[dr, "\varrho^M \otimes I_C"] & & \\
M \arrow[ur, "\varrho^M"] \arrow[dr, "\varrho^M"'] & & M \otimes_R C \otimes_R C \arrow[r, "I_M \otimes f \otimes g"] & M\,. \\
 & M \otimes_R C \arrow[ur, "I_M \otimes \Delta"'] & &
\end{tikzcd}$$

Clearly, for each $m \in M$, $\varepsilon \rightharpoonup m = m$, and thus $M$ is a $C^*$-module.

(2) For any $h : M \to N$ in $\mathbf{M}^C$ and $f \in C^*$, $m \in M$, consider

$$\begin{aligned} h(f \rightharpoonup m) &= \sum h(m_{\underline{0}}\, f(m_{\underline{1}})) = (I_N \otimes f) \circ (h \otimes I_C) \circ \varrho^M(m) \\ &= (I_N \otimes f) \circ \varrho^N \circ h(m) = f \rightharpoonup h(m). \end{aligned}$$

This shows that $h$ is a $C^*$-linear map.

(3) By 7.12, $C$ is a subgenerator in $\mathbf{M}^C$ and hence all $C$-comodules are subgenerated by $C$ as $C^*$-modules (by (1),(2)); thus they are objects in $\sigma[{}_{C^*}C]$, and hence (1)–(2) define a faithful functor $\mathbf{M}^C \to \sigma[{}_{C^*}C]$. $\square$

Now, the question arises when $\mathbf{M}^C$ is a full subcategory of $\sigma[{}_{C^*}C]$ (or ${}_{C^*}\mathbf{M}$), that is, when $\mathrm{Hom}^C(M, N) = \mathrm{Hom}_{C^*}(M, N)$, for any $M, N \in \mathbf{M}^C$. In answering this question the following property plays a crucial role.

**8.2. The $\alpha$-condition.** $C$ is said to *satisfy the $\alpha$-condition* if the map

$$\alpha_N : N \otimes_R C \to \mathrm{Hom}_R(C^*, N), \quad n \otimes c \mapsto [f \mapsto f(c)n],$$

is injective, for every $N \in \mathbf{M}_R$. By 4.7, the following are equivalent:

(a) *$C$ satisfies the $\alpha$-condition;*

(b) *for any $N \in \mathbf{M}_R$ and $u \in N \otimes_R C$, $(I_N \otimes f)(u) = 0$ for all $f \in C^*$, implies $u = 0$;*

(c) *$C$ is locally projective as an $R$-module.*

*In particular, this implies that $C$ is a flat $R$-module, and that it is cogenerated by $R$.*

**8.3. $\mathbf{M}^C$ as a full subcategory of ${}_{C^*}\mathbf{M}$.** *The following are equivalent:*

(a) *$\mathbf{M}^C = \sigma[{}_{C^*}C]$;*

(b) *$\mathbf{M}^C$ is a full subcategory of ${}_{C^*}\mathbf{M}$;*

(c) *for all $M, N \in \mathbf{M}^C$, $\mathrm{Hom}^C(M, N) = {}_{C^*}\mathrm{Hom}(M, N)$;*

(d) *${}_RC$ is locally projective;*

(e) *every left $C^*$-submodule of $C^n$, $n \in \mathbb{N}$, is a subcomodule of $C^n$.*

*If these conditions are satisfied, the inclusion functor $\mathbf{M}^C \to {}_{C^*}\mathbf{M}$ has a right adjoint, and for any family $\{M_\lambda\}_\Lambda$ of $R$-modules,*

$$(\textstyle\prod_\Lambda M_\lambda) \otimes_R C \;\simeq\; \textstyle\prod^C_\Lambda (M_\lambda \otimes_R C) \subset \textstyle\prod_\Lambda (M_\lambda \otimes_R C),$$

*where $\prod^C$ denotes the product in $\mathbf{M}^C$.*

**Proof.** (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) follow by the fact that $C$ is always a subgenerator of ${}^C\mathbf{M}$ (see 7.12) and the definition of the category $\sigma[{}_{C^*}C]$ (cf. 3.1).

(a) $\Rightarrow$ (d) The equality obviously implies that monomorphisms in $\mathbf{M}^C$ are injective maps. Hence ${}_RC$ is flat by 7.12(4). For any $N \in \mathbf{M}_R$ we prove the injectivity of the map $\alpha_N : N \otimes_R C \to \mathrm{Hom}_R(C^*, N)$.

$\mathrm{Hom}_R(C^*, N)$ is a left $C^*$-module by

$$g \cdot \gamma(f) = \gamma(f * g), \quad \text{for } \gamma \in \mathrm{Hom}_R(C^*, N),\ f, g \in C^*,$$

and considering $N \otimes_R C$ as left $C^*$-module in the canonical way we have

$$\alpha_N(g \rightharpoonup (n \otimes c))(f) = \sum n\, f(c_{\underline{1}}) g(c_{\underline{2}}) = n\, f * g(c) = [g \cdot \alpha_N(n \otimes c)](f),$$

for all $f, g \in C^*$, $n \in N$, and $c \in C$. So $\alpha_N$ is $C^*$-linear, and for any right $C$-comodule $L$ there is a commutative diagram

$$\begin{array}{ccc}
{}_{C^*}\mathrm{Hom}(L, N \otimes_R C) & \xrightarrow{\mathrm{Hom}(L,\alpha_N)} & {}_{C^*}\mathrm{Hom}(L, \mathrm{Hom}_R(C^*, N)) \\
\simeq\downarrow & & \downarrow\simeq \\
\mathrm{Hom}_R(L, N) & \xrightarrow{=} & \mathrm{Hom}_R(L, N).
\end{array}$$

The first vertical isomorphism is obtained by assumption and the Hom-tensor relations 7.9, explicitly,

$$_{C^*}\mathrm{Hom}(L, N \otimes_R C) = \mathrm{Hom}^C(L, N \otimes_R C) \simeq \mathrm{Hom}_R(L, N).$$

The second vertical isomorphism results from the canonical isomorphisms

$$_{C^*}\mathrm{Hom}(L, \mathrm{Hom}_R(C^*, N)) \simeq \mathrm{Hom}_R(C^* \otimes_{C^*} L, N) \simeq \mathrm{Hom}_R(L, N).$$

This shows that $\mathrm{Hom}(L, \alpha_N)$ is injective for any $L \in \mathbf{M}^C$, and so (the corestriction of) $\alpha_N$ is a monomorphism in $\mathbf{M}^C$. Since $_RC$ is flat, this implies that $\alpha_N$ is injective (by 7.13).

(d) $\Rightarrow$ (e) We show that, for right $C$-comodules $M$, any $C^*$-submodule $N$ is a subcomodule. For this consider the map

$$\rho_N : N \to \mathrm{Hom}_R(C^*, N),\ n \mapsto [f \mapsto f \rightharpoonup n].$$

The inclusion $i : N \to M$ yields the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "i"] & M \arrow[r, "p"] \arrow[d, "\varrho^M"] & M/N \arrow[r] & 0 \\
0 \arrow[r] & N \otimes_R C \arrow[r, "i \otimes I"] \arrow[d, "\alpha_{N,C}"] & M \otimes_R C \arrow[r, "p \otimes I"] \arrow[d, "\alpha_{M,C}"] & M/N \otimes_R C \arrow[r] \arrow[d, "\alpha_{M/N,C}"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(C^*, N) \arrow[r, "\mathrm{Hom}(C^*, i)"] & \mathrm{Hom}_R(C^*, M) \arrow[r] & \mathrm{Hom}_R(C^*, M/N), &
\end{tikzcd}$$

where $\mathrm{Hom}(C^*, i) \circ \rho_N = \alpha_{M,C} \circ \varrho^M \circ i$. Injectivity of $\alpha_{M/N,C}$ implies $(p \otimes I) \circ \varrho^M \circ i = 0$, and by the kernel property $\varrho^M \circ i$ factors through $N \to N \otimes_R C$, thus yielding a $C$-coaction on $N$.

(e) $\Rightarrow$ (a) First we show that every finitely generated $C^*$-module $N \in \sigma[_{C^*}C]$ is a $C$-comodule. There exist a $C^*$-submodule $X \subset C^n$, $n \in \mathbb{N}$, and an epimorphism $h : X \to N$. By assumption, $X$ and the kernel of $h$ are comodules and hence $N$ is a comodule (see 7.5). So, for any $L \in \sigma[_{C^*}C]$, finitely generated submodules are comodules and this obviously implies that $L$ is a comodule.

It remains to prove that, for $M, N \in \mathbf{M}^C$, any $C^*$-morphism $f : M \to N$ is a comodule morphism. $\mathrm{Im}\, f \subset N$ and $\mathrm{Ke}\, f \subset M$ are $C^*$-submodules and hence – as just shown – are subcomodules of $N$ and $M$, respectively. Therefore the corestriction $M \to \mathrm{Im}\, f$ and the inclusion $\mathrm{Im}\, f \to N$ both are comodule morphisms and so is $f$ (as the composition of two comodule maps).

For the final assertions, recall that the inclusion $\sigma[_{C^*}C] \to {}_{C^*}\mathbf{M}$ has a right adjoint functor (trace functor, see 3.1) and this respects products. So the isomorphism follows from the characterisation of the products of the $M_\lambda \otimes_R C$ in $\mathbf{M}^C$ (see 7.12). $\square$

**8.4. Coaction and $C^*$-modules.** *Let $_RC$ be locally projective. For any $R$-module $M$, consider an $R$-linear map $\varrho : M \to M \otimes_R C$. Define a left $C^*$-action on $M$ by*

$$\rightharpoonup : C^* \otimes_R M \to M, \quad f \otimes m \mapsto (I_M \otimes f) \circ \varrho(m).$$

*Then the following are equivalent:*

(a) *$\varrho$ is coassociative and counital;*

(b) *$M$ is a unital $C^*$-module by $\rightharpoonup$.*

**Proof.** The implication (a) $\Rightarrow$ (b) is shown in 8.1. Conversely, suppose that $M$ is a unital $C^*$-module by $\rightharpoonup$, that is,

$$(f * g) \rightharpoonup m = f \rightharpoonup (g \rightharpoonup m), \text{ for all } f, g \in C^*, \, m \in M.$$

By the definition of the action $\rightharpoonup$, this means that

$$(I_M \otimes f \otimes g) \circ (I_M \otimes \Delta) \circ \varrho(m) = (I_M \otimes f \otimes g) \circ (\varrho \otimes I_C) \circ \varrho(m),$$

and from this $_RC$ locally projective implies $(I_M \otimes \Delta) \circ \varrho(m) = (\varrho \otimes I_C) \circ \varrho(m)$ (see 8.2), showing that $\varrho$ is coassociative. Moreover, for any $m \in M$, $m = \varepsilon \rightharpoonup m = (I_M \otimes \varepsilon) \circ \varrho(m)$. □

**8.5. Left $C$-comodules and right $C^*$-modules.**

(1) *Any $M \in {}^C\mathbf{M}$ is a (unital) right $C^*$-module by*

$$\leftharpoonup : M \otimes_R C^* \to M, \quad m \otimes f \mapsto (f \otimes I_M) \circ {}^M\varrho(m) = \sum f(m_{\underline{-1}}) m_{\underline{0}}.$$

(2) *Any morphism $h : M \to N$ in ${}^C\mathbf{M}$ is a right $C^*$-module morphism, so*

$${}^C\mathrm{Hom}\,(M, N) \subset \mathrm{Hom}_{C^*}(M, N)$$

*and there is a faithful functor ${}^C\mathbf{M} \to \sigma[C_{C^*}] \subset \mathbf{M}_{C^*}$.*

(3) *$_RC$ is locally projective if and only if ${}^C\mathbf{M} = \sigma[C_{C^*}]$.*

Since $C$ is a left and right $C$-comodule by the regular coaction, we can study the structure of $C$ as a $(C^*, C^*)$-bimodule.

**8.6. $C$ as a $(C^*, C^*)$-bimodule.** *$C$ is a $(C^*, C^*)$-bimodule by*

$$\begin{aligned} \rightharpoonup \; &: C^* \otimes C \to C, \quad f \otimes c \mapsto f \rightharpoonup c = (I_C \otimes f) \circ \Delta(c), \\ \leftharpoonup \; &: C \otimes C^* \to C, \quad c \otimes g \mapsto c \leftharpoonup g = (g \otimes I_C) \circ \Delta(c)\,. \end{aligned}$$

(1) *For any $f, g \in C^*$, $c \in C$,*

$$f * g\,(c) = f(g \rightharpoonup c) = g(c \leftharpoonup f).$$

*(2) $C$ is faithful as a left and right $C^*$-module.*

*(3) Assume $C$ to be cogenerated by $R$. Then for any central element $f \in C^*$ and any $c \in C$, $f \rightharpoonup c = c \leftharpoonup f$.*

*(4) If $C$ satisfies the $\alpha$-condition, it is a balanced $(C^*, C^*)$-bimodule, that is,*

$$ {}_{C^*}End(C) = \mathrm{End}^C(C) \simeq C^* \simeq {}^C\mathrm{End}(C) = \mathrm{End}_{C^*}(C) \text{ and} $$

$$ {}_{C^*}End_{C^*}(C) = {}^C\mathrm{End}^C(C) \simeq Z(C^*), $$

*where morphisms are written opposite to scalars and $Z(C^*)$ denotes the centre of $C^*$. In this case a pure $R$-submodule $D \subset C$ is a subcoalgebra if and only if $D$ is a left and right $C^*$-submodule.*

**Proof.** The bimodule property is shown by the equalities

$$ \begin{aligned} (f \rightharpoonup c) \leftharpoonup g &= (g \otimes I_C \otimes f) \circ ((\Delta \otimes I_C) \circ \Delta))(c) \\ &= (g \otimes I_C \otimes f) \circ ((I_C \otimes \Delta) \circ \Delta))(c) \quad = \quad f \rightharpoonup (c \leftharpoonup g). \end{aligned} $$

(1) From the definition it follows

$$ \begin{aligned} f * g\,(c) = (f \otimes g) \circ \Delta(c) \quad &= (f \otimes I_C) \circ (I_C \otimes g) \circ \Delta(c) = f(g \rightharpoonup c) \\ &= (I_C \otimes g) \circ (f \otimes I_C) \circ \Delta(c) = g(c \leftharpoonup f). \end{aligned} $$

(2) For $f \in C^*$, assume $f \rightharpoonup c = 0$ for each $c \in C$. Then applying (1) yields $f(c) = \varepsilon(f \rightharpoonup c) = 0$, and hence $f = 0$.

(3) For any central element $f \in C^*$, by (1),

$$ g(c \leftharpoonup f) = f * g(c) = g * f(c) = g(f \rightharpoonup c), $$

for all $c \in C$, $g \in C^*$. Since $C$ is cogenerated by $R$, this can only hold if, for all $c \in C$, $c \leftharpoonup f = f \rightharpoonup c$.

(4) The isomorphisms follow from 7.11, 8.3 and 8.5. Let $D \subset C$ be a pure $R$-submodule. If $D$ is a subcoalgebra of $C$, then it is a right and left subcomodule and hence a left and right $C^*$-submodule. Conversely, suppose that $D$ is a left and right $C^*$-submodule. Then the restriction of $\Delta$ yields a left and right $C$-coaction on $D$ and, by 1.21,

$$ \Delta(D) \subset D \otimes_R C \ \cap \ C \otimes D = D \otimes_R D\,, $$

proving that $D$ is a subcoalgebra. $\square$

**8.7. When is $\mathbf{M}^C = {}_{C^*}\mathbf{M}$?** *The following are equivalent:*

(a) $\mathbf{M}^C = {}_{C^*}\mathbf{M}$;

(b) *the functor* $-\otimes_R C : \mathbf{M}_R \to {}_{C^*}\mathbf{M}$ *has a left adjoint;*

(c) ${}_RC$ *is finitely generated and projective;*

(d) ${}_RC$ *is locally projective and $C$ is finitely generated as right $C^*$-module;*

(e) ${}^C\mathbf{M} = \mathbf{M}_{C^*}$.

**Proof.** (a) $\Rightarrow$ (b) is obvious (by 7.12(3)).

(b) $\Rightarrow$ (c) Since $-\otimes_R C$ is a right adjoint, it preserves monomorphisms (injective morphisms). Therefore, ${}_RC$ is flat. Moreover $-\otimes_R C$ preserves products, so for any family $\{M_\lambda\}_\Lambda$ in $\mathbf{M}_R$ there is an isomorphism

$$(\prod_\Lambda M_\lambda)\otimes_R C \simeq \prod_\Lambda (M_\lambda \otimes_R C),$$

which implies that ${}_RC$ is finitely presented and hence projective.

(c) $\Rightarrow$ (d) Clearly, projective modules are locally projective, and $C$ finitely generated as an $R$-module implies that $C$ is finitely generated as a right (and left) $C^*$-module.

(d) $\Rightarrow$ (a) By 8.6, $C$ is a faithful left $C^*$-module that is finitely generated as a module over its endomorphism ring $C^*$. This implies that $C$ is a subgenerator in ${}_{C^*}\mathbf{M}$, that is, $\mathbf{M}^C = \sigma[{}_{C^*}C] = {}_{C^*}\mathbf{M}$ (see 3.6). $\square$

The comodules of the coalgebra associated to any finitely generated projective $R$-module are of fundamental importance.

**8.8. Projective modules as comodules.** *Let $P$ be a finitely generated projective $R$-module with dual basis $p_1,\ldots,p_n \in P$ and $\pi_1,\ldots,\pi_n \in P^*$. Then $P$ is a right $P^*\otimes_R P$-comodule with the coaction*

$$\varrho^P : P \to P\otimes_R (P^*\otimes_R P), \quad p \mapsto \sum_i p_i \otimes \pi_i \otimes p.$$

*$P$ is a subgenerator in $\mathbf{M}^{P^*\otimes_R P}$, and there is a category isomorphism*

$$\mathbf{M}^{P^*\otimes_R P} \simeq \mathbf{M}_{\mathrm{End}_R(P)}.$$

*The dual $P^*$ is a left $P^*\otimes_R P$-comodule with the coaction*

$${}^P\varrho : P \to (P^*\otimes_R P)\otimes_R P, \quad f \mapsto \sum_i f\otimes p_i \otimes \pi_i.$$

**Proof.** Coassociativity of $\varrho^P$ follows from the equality

$$(I\otimes\Delta)\varrho^P(f\otimes p) = \sum_{i,j} f\otimes p_i\otimes\pi_i\otimes p_j\otimes\pi_j\otimes p = (\varrho^P\otimes I)\varrho^P(f\otimes p).$$

By properties of the dual basis, $(I_P\otimes\varepsilon)\varrho^P(p) = \sum_i p_i\pi_i(p) = p$, so that $P$ is indeed a right comodule over $P^*\otimes_R P$. There exists a surjective $R$-linear map

$R^n \to P^*$ that yields an epimorphism $P^n \simeq R^n \otimes P \to P^* \otimes_R P$ in $\mathbf{M}^{P^*\otimes_R P}$. So $P$ generates $P^* \otimes_R P$ as a right comodule and hence is a subgenerator in $\mathbf{M}^{P^*\otimes_R P}$. Since $P^* \otimes_R P$ is finitely generated and projective as an $R$-module, the category isomorphism follows by 8.7.

A simple computation shows that $P^*$ is a left comodule over $P^* \otimes_R P$. $\square$

As a special case, for any $n \in \mathbb{N}$, $R^n$ may be considered as a right comodule over the matrix coalgebra $M_n^c(R)$ (cf. 5.10).

For an algebra $A$, any two elements $a, b \in A$ define a subalgebra $aAb \subset A$, and for an idempotent $e \in A$, $eAe$ is a subalgebra with a unit. Dually, one considers

**8.9. Factor coalgebras.** *Let $f, g, e \in C^*$ with $e * e = e$. Then:*

*(1) $f \rightharpoonup C \leftharpoonup g$ is a coalgebra (without a counit) and there is a coalgebra morphism*

$$C \to f \rightharpoonup C \leftharpoonup g, \quad c \mapsto f \rightharpoonup c \leftharpoonup g.$$

*(2) $e \rightharpoonup C \leftharpoonup e$ is a coalgebra with counit $e$ and coproduct*

$$e \rightharpoonup c \leftharpoonup e \;\mapsto\; \sum e \rightharpoonup c_{\underline{1}} \leftharpoonup e \otimes e \rightharpoonup c_{\underline{2}} \leftharpoonup e.$$

*The kernel of $C \to e \rightharpoonup C \leftharpoonup e$ is equal to $(\varepsilon - e) \rightharpoonup C + C \leftharpoonup (\varepsilon - e)$.*

*(3) If $C$ is $R$-cogenerated, and $e$ is a central idempotent, then $e \rightharpoonup C$ is a subcoalgebra of $C$.*

**Proof.** (1) For any $f, g \in C^*$ consider the left, respectively right, comodule maps $L_f : C \to C$, $c \mapsto f \rightharpoonup c$, and $R_g : C \to C$, $c \mapsto c \leftharpoonup g$. Construct the commutative diagram

$$\begin{tikzcd}
C \arrow[r, "L_f"] \arrow[d, "\Delta"'] & C \arrow[r, "R_g"] \arrow[d, "\Delta"] & C \arrow[d, "\Delta"] \\
C \otimes_R C \arrow[r, "I_C \otimes L_f"] & C \otimes_R C \arrow[r, "R_g \otimes I_C"] & C \otimes_R C,
\end{tikzcd}$$

which leads to the identity $\Delta \circ R_g \circ L_f = (R_g \otimes L_f) \circ \Delta$. Putting $\delta := L_f \circ R_g = R_g \circ L_f$, we obtain the commutative diagram

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\delta"'] & C \otimes_R C \arrow[d, "R_g \otimes L_f"] & \\
\delta(C) \arrow[r, "\Delta"] & R_g(C) \otimes_R L_f(C) \arrow[r, "L_f \otimes R_g"] & \delta(C) \otimes_R \delta(C).
\end{tikzcd}$$

Thus $\Delta_\delta = (L_f \otimes R_g) \circ \Delta$ makes $\delta(C)$ a coalgebra. It is easily verified that

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\delta"'] & C \otimes_R C \arrow[d, "\delta \otimes \delta"] \\
\delta(C) \arrow[r, "\Delta_\delta"] & \delta(C) \otimes_R \delta(C)
\end{tikzcd}$$

is a commutative diagram, and hence $\delta$ is a coalgebra morphism.

(2) The form of the coproduct follows from (1). For $c \in C$, $\varepsilon(e \rightharpoonup c \leftharpoonup e) = \varepsilon * e(c \leftharpoonup e) = e(c)$ showing that $e$ is the counit of $e \rightharpoonup C \leftharpoonup e$.

For $x \in C$, $e \rightharpoonup x \leftharpoonup e = 0$ implies $x \leftharpoonup e = (\varepsilon - e) \rightharpoonup (x \leftharpoonup e) \in (\varepsilon - e) \rightharpoonup C$, and so

$$x = x \leftharpoonup e + x \leftharpoonup (\varepsilon - e) \in (\varepsilon - e) \rightharpoonup C + C \leftharpoonup (\varepsilon - e).$$

This proves the stated form of the kernel.

(3) By 8.6, for a central idempotent $e$ and $c \in C$, $e \rightharpoonup c \leftharpoonup e = e \rightharpoonup c$. Putting $f = g = e$ in (the proof of) (1) we obtain $\Delta_e(e \rightharpoonup C) \subset e \rightharpoonup C \otimes e \rightharpoonup C$. □

**8.10. Idempotents and comodules.** *Let $e \in C^*$ be an idempotent and consider the coalgebra $e \rightharpoonup C \leftharpoonup e$ (as in 8.9).*

*(1) For any $M \in \mathbf{M}^C$, $e \rightharpoonup M$ is a right $e \rightharpoonup C \leftharpoonup e$-comodule with the coaction*

$$e \rightharpoonup M \to e \rightharpoonup M \otimes_R e \rightharpoonup C \leftharpoonup e, \quad e \rightharpoonup m \mapsto \sum e \rightharpoonup m_{\underline{0}} \otimes e \rightharpoonup m_{\underline{1}} \leftharpoonup e.$$

*(2) For any $f : M \to N \in \mathbf{M}^C$, $f(e \rightharpoonup M) = e \rightharpoonup f(M)$, and so there is a covariant functor*

$$e \rightharpoonup - : \mathbf{M}^C \to \mathbf{M}^{e \rightharpoonup C \leftharpoonup e}, \quad M \mapsto e \rightharpoonup M.$$

*(3) For any $M \in \mathbf{M}^C$, $M^*$ is a right $C^*$-module canonically and*

$$\mathrm{Hom}_R(e \rightharpoonup M, R) = (e \rightharpoonup M)^* \simeq M^* \cdot e.$$

*(4) The map $- \leftharpoonup e : e \rightharpoonup C \to e \rightharpoonup C \leftharpoonup e$ is a surjective right $e \rightharpoonup C \leftharpoonup e$-comodule morphism, and so $e \rightharpoonup C$ is a subgenerator in $\mathbf{M}^{e \rightharpoonup C \leftharpoonup e}$.*

*(5) $(e \rightharpoonup C \leftharpoonup e)^* \simeq e * C^* * e$, and hence there is a faithful functor $\mathbf{M}^{e \rightharpoonup C \leftharpoonup e} \to {}_{e*C^**e}\mathbf{M}$.*

*(6) If ${}_RC$ is locally projective, then $e \rightharpoonup C \leftharpoonup e$ is a locally projective $R$-module and*

$$\mathbf{M}^{e \rightharpoonup C \leftharpoonup e} = \sigma[{}_{e*C^**e}\, e \rightharpoonup C] = \sigma[{}_{e*C^**e}\, e \rightharpoonup C \leftharpoonup e].$$

**Proof.** (1), (3) and (4) are easily verified.

(2) By 8.1, right comodule morphisms are left $C^*$-morphisms.

(5) The isomorphism in (3) holds similarly for the right action of $e$ on $C$ and from this the isomorphism in (5) follows.

(6) Clearly direct summands of locally projectives are locally projective, and hence the assertion follows from (3) and 8.3. □

**8.11. Finiteness Theorem.** *Assume $_RC$ to be locally projective.*

*(1) Let $M \in \mathbf{M}^C$. Every finite subset of $M$ is contained in a subcomodule of $M$ that is finitely generated as an $R$-module.*

*(2) Any finite subset of $C$ is contained in a $(C^*, C^*)$-sub-bimodule that is finitely generated as an $R$-module.*

*(3) Minimal $C^*$-submodules and minimal $(C^*, C^*)$-sub-bimodules of $C$ are finitely generated as $R$-modules.*

**Proof.** (1) Since any sum of subcomodules is again a subcomodule, it is enough to show that each $m \in M$ lies in a subcomodule that is finitely generated as an $R$-module. Moreover, by the correspondence of subcomodules and $C^*$-submodules, this amounts to proving that the submodule $C^* \rightharpoonup m$ is finitely generated as an $R$-module. Writing $\varrho^M(m) = \sum_{i=1}^k m_i \otimes c_i$, where $m_i \in C^* \rightharpoonup m$, $c_i \in C$, we compute for every $f \in C^*$

$$f \rightharpoonup m = (I_M \otimes f) \circ \varrho^M(m) = \textstyle\sum_{i=1}^k m_i\, f(c_i)\,.$$

Hence $C^* \rightharpoonup m$ is finitely generated by $m_1, \ldots, m_k$ as an $R$-module.

(2) It is enough to prove the assertion for single elements $c \in C$. By (1), $C^* \rightharpoonup c$ is generated as an $R$-module by some $c_1, \ldots, c_k \in C$. By symmetry, each $c_i \leftharpoonup C^*$ is a finitely generated $R$-module. Hence $C^* \rightharpoonup c \leftharpoonup C^*$ is a finitely generated $R$-module.

(3) This is an obvious consequence of (1) and (2). □

A right $C$-comodule $N$ is called *semisimple* (in $\mathbf{M}^C$) if every $C$-monomorphism $U \to N$ is a coretraction, and $N$ is called *simple* if all these monomorphisms are isomorphisms. Semisimplicity of $N$ is equivalent to the fact that every right $C$-comodule is $N$-injective. (Semi)simple left comodules and bicomodules are defined similarly.

The coalgebra $C$ is said to be *left (right) semisimple* if it is semisimple as a left (right) comodule. $C$ is called a *simple coalgebra* if it is simple as a $(C, C)$-bicomodule.

**8.12. Semisimple comodules.** *Assume that $_RC$ is flat.*

*(1) Any $N \in \mathbf{M}^C$ is simple if and only if $N$ has no nontrivial subcomodules.*

*(2) For $N \in \mathbf{M}^C$ the following are equivalent:*

*(a) $N$ is semisimple (as defined above);*

*(b) every subcomodule of $N$ is a direct summand;*

*(c) $N$ is a sum of simple subcomodules;*

*(d) $N$ is a direct sum of simple subcomodules.*

**Proof.** (1) By 7.13, any monomorphism $U \to N$ is injective, and hence it can be identified with a subcomodule. From this the assertion is clear.

(2) ${}_RC$ flat implies that the intersection of any two subcomodules is again a subcomodule. Hence in this case the proof for modules (e.g., [10, 20.2]) can be transferred to comodules. □

**8.13. Right semisimple coalgebras.** *For $C$ the following are equivalent:*

*(a) $C$ is a semisimple right $C$-comodule;*

*(b) ${}_RC$ is flat and every right subcomodule of $C$ is a direct summand;*

*(c) ${}_RC$ is flat and $C$ is a direct sum of simple right comodules;*

*(d) ${}_RC$ is flat and every comodule in $\mathbf{M}^C$ is semisimple;*

*(e) ${}_RC$ is flat and every short exact sequence in $\mathbf{M}^C$ splits;*

*(f) ${}_RC$ is projective and $C$ is a semisimple left $C^*$-module;*

*(g) every comodule in $\mathbf{M}^C$ is ($C$-)injective;*

*(h) every comodule in $\mathbf{M}^C$ is projective;*

*(i) $C$ is a direct sum of simple coalgebras that are right (left) semisimple;*

*(j) $C$ is a semisimple left $C$-comodule.*

**Proof.** (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e) Assume every monomorphism $i : U \to C$ to be a coretraction. Then $i$ is in particular an injective map, and hence, by 7.13, ${}_RC$ is flat. Now the assertions follow by 8.12.

The implications (e) ⇒ (g) and (e) ⇒ (h) are obvious.

(h) ⇒ (f) By 7.18, any projective comodule is projective as an $R$-module. In particular, $C$ is a projective $R$-module, and hence $\mathbf{M}^C = \sigma[{}_{C^*}C]$ and all modules in $\sigma[{}_{C^*}C]$ are projective. This characterises $C$ as a semisimple $C^*$-module (see 3.7).

The implication (f) ⇒ (a) is obvious since $\mathbf{M}^C = \sigma[{}_{C^*}C]$.

(g) ⇒ (a) This is obvious. Notice that, in view of (f), ${}_RC$ is projective, and hence the $C$-injectivity of any comodule $N$ implies that $N$ is injective in $\mathbf{M}^C = \sigma[{}_{C^*}C]$.

(f) ⇒ (i) Let $C$ be a left semisimple $C^*$-module. Let $\{E_i\}_I$ be a minimal representative set of simple $C^*$-submodules of $C$. Form the traces $D_i := \mathrm{Tr}_{C^*}(E_i, C)$. By the structure theorem for semisimple modules (see 3.7),

$$C \simeq \bigoplus_I D_i,$$

where the $D_i$ are minimal fully invariant $C^*$-submodules. Considering $C^*$ as an endomorphism ring acting from the right, this means that the $D_i$ are minimal $(C^*, C^*)$-submodules. By 8.6, each $D_i$ is a minimal subcoalgebra of $C$ and every subcoalgebra of $D_i$ is a subcoalgebra of $C$. So every $D_i$ is a right semisimple simple coalgebra.

(i) $\Rightarrow$ (f) It follows from the proof (a) $\Rightarrow$ (f) that all simple comodules of $C$ are projective as $R$-modules and hence ${}_RC$ is also projective. Now the assertion follows.

(f) $\Leftrightarrow$ (j) By 3.7, the semisimple module ${}_{C^*}C$ is semisimple over its endomorphism ring, that is, $C_{C^*}$ is also semisimple. Since ${}^C\mathbf{M} = \sigma[C_{C^*}]$, the assertion follows from the preceding proof by symmetry. $\square$

**8.14. Simple coalgebras.** *For $C$ the following are equivalent:*

*(a) $C$ is a simple coalgebra that is right (left) semisimple;*

*(b) ${}_RC$ is projective and $C$ is a simple $(C^*, C^*)$-bimodule containing a minimal left (right) $C^*$-submodule;*

*(c) $C$ is a simple coalgebra and a finite-dimensional vector space over $R/m$, for some maximal ideal $m \subset R$.*

**Proof.** (a) $\Rightarrow$ (b) We know from 8.13 that ${}_RC$ is projective. Clearly a simple right subcomodule is a simple left $C^*$-submodule. Let $D \subset C$ be a $(C^*, C^*)$-sub-bimodule. Then it is a direct summand as a left $C^*$-module, and hence it is a subcoalgebra of $C$ (by 8.6) and so $D = C$.

(b) $\Rightarrow$ (c) Let $D \subset C$ be a minimal left $C^*$-submodule. For any maximal ideal $m \subset R$, $mD \subset D$ is a $C^*$-submodule and hence $mD = 0$ or $mD = D$. Since $D$ is finitely generated as an $R$-module (by 8.11), $mD = 0$ for some maximal $m \subset R$. Moreover, $mC = mD \leftharpoonup C^* = 0$, and so $C$ is a finite-dimensional $R/m$-algebra.

(c) $\Rightarrow$ (a) is obvious. Notice that in this case $\mathbf{M}^C = {}_{C^*}\mathbf{M}$ (see 8.7). $\square$

The Finiteness Theorem 8.11 and the Hom-tensor relations 7.9 indicate that properties of $R$ have a strong influence on properties of $C$-comodules.

**8.15. Coalgebras over special rings.** *Let ${}_RC$ be locally projective.*

*(1) If $R$ is Noetherian, then $C$ is locally Noetherian as a right and left comodule, and in $\mathbf{M}^C$ and ${}^C\mathbf{M}$ direct sums of injectives are injective.*

*(2) If $R$ is perfect, then in $\mathbf{M}^C$ and ${}^C\mathbf{M}$ any comodule satisfies the descending chain condition on finitely generated subcomodules.*

*(3) If $R$ is Artinian, then in $\mathbf{M}^C$ and ${}^C\mathbf{M}$ every finitely generated comodule has finite length.*

**Proof.** All these assertions are special cases of 3.18. $\square$

Notice that, over Artinian (perfect) rings $R$, ${}_RC$ is locally projective if and only if ${}_RC$ is projective (any flat $R$-module is projective).

# 9 The rational functor

We know from Section 8 that there is a faithful functor from the category of right $C$-comodules to the category of left $C^*$-modules. Now we want to study an opposite problem provided the $\alpha$-condition is satisfied: Suppose $M$ is a left $C^*$-module; does there exist a (maximal) part of $M$ on which a right $C$-coaction can be defined?

Throughout we ${}_RC$ to be locally projective.

**9.1. Rational functor.** For any left $C^*$-module $M$, define the *rational submodule*

$$\mathrm{Rat}^C(M) = \mathcal{T}^C(M) = \sum\{\mathrm{Im}\, f \mid f \in {}_{C^*}\mathrm{Hom}(U, M),\, U \in \mathbf{M}^C\},$$

where $\mathcal{T}^C$ is the trace functor ${}_{C^*}\mathbf{M} \to \sigma[C]$ (cf. 3.1). Clearly $\mathrm{Rat}^C(M)$ is the largest submodule of $M$ that is subgenerated by $C$, and hence it is a right $C$-comodule. The induced functor (subfunctor of the identity)

$$\mathrm{Rat}^C : {}_{C^*}\mathbf{M} \to \mathbf{M}^C, \quad M \mapsto \mathrm{Rat}^C(M),$$

is called the *rational functor*. Since $\mathrm{Rat}^C$ is a trace functor, it is right adjoint to the inclusion $\mathbf{M}^C \to {}_{C^*}\mathbf{M}$ (see 3.1) and its properties depend on (torsion-theoretic) properties of the class $\mathbf{M}^C$ in ${}_{C^*}\mathbf{M}$. Of course $\mathrm{Rat}^C(M) = M$ for $M \in {}_{C^*}\mathbf{M}$ if and only if $M \in \mathbf{M}^C$, and $\mathbf{M}^C = {}_{C^*}\mathbf{M}$ if and only if ${}_RC$ is finitely generated and projective (see 8.7).

**9.2. Rational elements.** Let $M$ be a left $C^*$-module. An element $k \in M$ is said to be *rational* if there exists an element $\sum_i m_i \otimes c_i \in M \otimes_R C$, such that

$$fk = \sum_i m_i f(c_i), \text{ for all } f \in C^*.$$

This means that, from the diagram

$$\begin{array}{ccccc} & & M & & m \\ & & \downarrow{\scriptstyle \psi_M} & & \downarrow \\ M \otimes_R C & \xrightarrow{\alpha_M} & \mathrm{Hom}_R(C^*, M) & \qquad m \otimes c \longmapsto [f \mapsto mf(c)], \qquad & [f \mapsto fm]\,, \end{array}$$

we obtain $\psi_M(k) = \alpha_M(\sum_i m_i \otimes c_i)$ (see 8.2). Since it is assumed that $\alpha_M$ is injective, the element $\sum_i m_i \otimes c_i$ is uniquely determined.

**9.3. Rational submodule.** *Let $M$ be a left $C^*$-module.*

*(1) An element $k \in M$ is rational if and only if $C^*k$ is a right $C$-comodule with $fk = f\rightharpoonup k$, for all $f \in C^*$.*

*(2)* $\mathrm{Rat}^C(M) = \{k \in M \mid k \textit{ is rational}\}$.

**Proof.** (1) Let $k \in M$ be rational and $\sum_i m_i \otimes c_i \in M \otimes_R C$ such that $fk = \sum_i m_i f(c_i)$ for all $f \in C^*$. Put $K := C^*k$ and define a map

$$\varrho : K \to M \otimes_R C, \quad fk \mapsto \sum\nolimits_i m_i \otimes f \rightharpoonup c_i.$$

For $f, h \in C^*$,

$$\alpha_M(\sum\nolimits_i m_i \otimes f \rightharpoonup c_i)(h) = \sum\nolimits_i m_i h(f \rightharpoonup c_i) = h * f\, k = h \cdot fk\,.$$

So the map $\varrho$ is well defined since $fk = 0$ implies $\alpha_M(\sum_i m_i \otimes f \rightharpoonup c_i) = 0$, and hence $\sum_i m_i \otimes f \rightharpoonup c_i = 0$ by injectivity of $\alpha_M$. Moreover, it implies that $\alpha_M \circ \varrho(K) \subset \mathrm{Hom}_R(C^*, K)$, and we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
 & & K \arrow[d, "\varrho"] & & \\
0 \arrow[r] & K \otimes_R C \arrow[r] \arrow[d, "\alpha_K"] & M \otimes_R C \arrow[r] \arrow[d, "\alpha_M"] & M/K \otimes_R C \arrow[r] \arrow[d, "\alpha_{M/K}"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(C^*, K) \arrow[r] & \mathrm{Hom}_R(C^*, M) \arrow[r] & \mathrm{Hom}_R(C^*, M/K) \arrow[r] & 0,
\end{tikzcd}$$

where all the $\alpha$ are injective. By the kernel property we conclude that $\varrho$ factors through some $\varrho^K : K \to K \otimes_R C$, and it follows by 8.4 that $\varrho^K$ is coassociative and counital, thus making $K$ a comodule. □

As a first application we consider the rational submodule of $C^{**}$. The canonical map $\Phi_C : C \to C^{**}$ is a $C^*$-morphism, since, for all $c \in C$, $f, h \in C^*$,

$$\Phi_C(f \rightharpoonup c)(h) = h(f \rightharpoonup c) = \sum h(c_{\underline{1}}) f(c_{\underline{2}}) = \Phi_C(c)(h * f) = f\Phi_C(c)(h).$$

Hence the image of $\Phi_C$ is a rational module. The next lemma shows that this is equal to the rational submodule of $C^{**}$.

**9.4. Rational submodule of $C^{**}$.** $\Phi_C : C \to \mathrm{Rat}^C(C^{**})$ *is an isomorphism.*

**Proof.** Local projectivity of ${}_RC$ implies that $\Phi_C$ is injective. Let $\varrho : \mathrm{Rat}^C(C^{**}) \to \mathrm{Rat}^C(C^{**}) \otimes_R C$ denote the comodule structure map. For $\gamma \in \mathrm{Rat}^C(C^{**})$ write $\varrho(\gamma) = \sum_i \gamma_i \otimes c_i$. Then, for any $f \in C^*$,

$$\gamma(f) = f \cdot \gamma(\varepsilon) = \sum\nolimits_i f(c_i)\gamma_i(\varepsilon) = f(\sum\nolimits_i \gamma_i(\varepsilon) c_i),$$

where $\sum_i \gamma_i(\varepsilon) c_i \in C$. So $\gamma \in \mathrm{Im}\, \Phi_C$, proving that $\Phi_C$ is surjective. □

The rational submodule of ${}_{C^*}C^*$ is a two-sided ideal in $C^*$ and is called the *left trace ideal*. From the above observations and the Finiteness Theorem it is clear that $\mathrm{Rat}^C(C^*) = C^*$ if and only if ${}_RC$ is finitely generated.

*Right rational $C^*$-modules* are defined in a symmetric way, yielding the *right trace ideal* ${}^C\mathrm{Rat}(C^*)$, which in general is different from $\mathrm{Rat}^C(C^*)$.

**9.5. Characterisation of the trace ideal.** *Let $T = \mathrm{Rat}^C(C^*)$ be the left trace ideal.*

*(1) Let $f \in C^*$ and assume that $f \rightharpoonup C$ is a finitely presented $R$-module. Then $f \in T$.*

*(2) If $R$ is Noetherian, then $T$ can be described as*

$$\begin{aligned} T_1 &= \{f \in C^* \mid C^* * f \text{ is a finitely generated } R\text{-module}\}; \\ T_2 &= \{f \in C^* \mid \mathrm{Ke}\, f \text{ contains a right } C^*\text{-submodule } K, \text{ such that} \\ &\qquad\qquad C/K \text{ is a finitely generated } R\text{-module}\}; \\ T_3 &= \{f \in C^* \mid f \rightharpoonup C \text{ is a finitely generated } R\text{-module}\}. \end{aligned}$$

**Proof.** Assertion (1) and the inclusion $T \subset T_1$ in (2) follow from the Finiteness Theorem 8.11.

$[T_1 \subset T_2]$: For $f \in T_1$, let $C^* * f$ be finitely $R$-generated by $g_1, \ldots, g_k \in C^*$. Consider the kernel of $C^* * f$,

$$K := \bigcap \{\mathrm{Ke}\ h \mid h \in C^* * f\} = \bigcap_{i=1}^{k} \mathrm{Ke}\, g_i\,.$$

Clearly $K$ is a right $C^*$-submodule of $C$. Moreover, all the $C/\mathrm{Ke}\, g_i$ are finitely generated $R$-modules, and hence

$$C/K \subset \bigoplus_{i=1}^{k} C/\mathrm{Ke}\, g_i$$

is a finitely generated $R$-module. This proves the inclusion $T_1 \subset T_2$.

$[T_2 \subset T_3]$: Let $f \in T_2$. Since $\Delta(K) \subset C \otimes_R K$, $f \rightharpoonup K = 0$ and $f \rightharpoonup C = f \rightharpoonup C/K$ is a finitely generated $R$-module, that is, $f \in T_3$.

$[T_3 \subset T]$: For $f \in T_3$, the rational right $C^*$-module $f \rightharpoonup C$ is a finitely presented $R$-module. Then, by 7.10, $(f \rightharpoonup C)^*$ is a rational left $C^*$-module. Since $\varepsilon(f \rightharpoonup c) = f(c)$ for all $c \in C$, we conclude $f \in (f \rightharpoonup C)^*$ and hence $f \in T$. $\square$

Before concentrating on properties of the trace ideal we consider density for any subalgebras of $C^*$. From the Density Theorem we know that for any $C$-dense subalgebra $T \subset C^*$ the categories $\mathbf{M}^C$ and $\sigma[{}_TC]$ can be identified.

**9.6. Density in $C^*$.** *For an $R$-submodule $U \subset C^*$, the following assertions are equivalent:*

*(a) $U$ is dense in $C^*$ in the finite topology (of $R^C$);*

*(b) $U$ is a $C$-dense subset of $C^*$ (in the finite topology of $\mathrm{End}_R(C)$).*

*If $C$ is cogenerated by $R$, then $(a),(b)$ imply:*

*(c) $\mathrm{Ke}\, U = \{x \in C \mid u(x) = 0$ for all $u \in U\} = 0$.*

*If $R$ is a cogenerator in $\mathbf{M}_R$, then* (c) $\Rightarrow$ (b).

**Proof.** (a) $\Leftrightarrow$ (b) It can be derived from 7.11 that the finite topologies in $C^*$ and $\mathrm{End}^C(C)$ can be identified.

(a) $\Rightarrow$ (c) Let $C$ be cogenerated by $R$. Then, for any $0 \neq x \in C$, there exists $f \in C^*$ such that $f(x) \neq 0$. Then, for some $u \in U$, $u(x) = f(x) \neq 0$, that is, $x \notin \mathrm{Ke}\, U$, and hence $\mathrm{Ke}\, U = 0$.

$(c) \Rightarrow (b)$ Let $R$ be a cogenerator in $\mathbf{M}_R$. Let $f \in C^*$ and $x_1, \ldots, x_n \in C$. Suppose that

$$f\rightharpoonup(x_1, \ldots, x_n) \notin U\rightharpoonup(x_1, \ldots, x_n) \subset C^n.$$

Then there exists an $R$-linear map $g : C^n \to R$ such that

$$g(f\rightharpoonup(x_1, \ldots, x_n)) \neq 0 \text{ and } g(U\rightharpoonup(x_1, \ldots, x_n)) = 0.$$

For each $u \in U$ (by 8.6),

$$0 = \sum\nolimits_i g_i(u\rightharpoonup x_i) = \sum\nolimits_i u(x_i\leftharpoonup g_i) = u(\sum\nolimits_i x_i\leftharpoonup g_i),$$

where $g_i : C \to C^n \xrightarrow{g} R$, and this implies $\sum_i x_i\leftharpoonup g_i = 0$ and

$$0 \neq g(f\rightharpoonup(x_1, \ldots, x_n)) = \sum\nolimits_i g_i(f\rightharpoonup x_i) = \sum\nolimits_i f(x_i\leftharpoonup g_i) = f(\sum\nolimits_i x_i\leftharpoonup g_i) = 0,$$

contradicting the choice of $g$. $\square$

**9.7. Dense subalgebras of $C^*$.** *For a subalgebra $T \subset C^*$ the following are equivalent:*

*(a) ${}_RC$ is locally projective and $T$ is dense in $C^*$;*

*(b) $\mathbf{M}^C = \sigma[{}_TC]$.*

*If $T$ is an ideal in $C^*$, then (a),(b) are equivalent to:*

*(c) $C$ is an s-unital $T$-module and $C$ satisfies the $\alpha$-condition.*

**Proof.** (a) $\Leftrightarrow$ (b) There are embeddings $\mathbf{M}^C \subset \sigma[{}_{C^*}C] \subset \sigma[{}_TC]$. Now $\mathbf{M}^C = \sigma[{}_{C^*}C]$ is equivalent to the $\alpha$-condition while $\sigma[{}_TC] = \sigma[{}_{C^*}C]$ corresponds to the density property.

(a) $\Leftrightarrow$ (c) For an ideal $T$ the density property is equivalent to s-unitality of the $T$-module $C$. $\square$

Combining the properties of the trace functor observed in 4.8 with the characterisation of dense ideals in 4.3, we obtain:

**9.8. The rational functor exact.** *Let $T = \mathrm{Rat}^C(C^*)$. The following statements are equivalent:*

*(a) the functor $\mathrm{Rat}^C : {}_{C^*}\mathbf{M} \to \mathbf{M}^C$ is exact;*

*(b) $\mathbf{M}^C$ is closed under extensions in ${}_{C^*}\mathbf{M}$ and the class*

$$\{X \in {}_{C^*}\mathbf{M} \mid \mathrm{Rat}^C(X) = 0\}$$

*is closed under factor modules;*

*(c) for every $N \in \mathbf{M}^C$ (with $N \subset C$), $TN = N$;*

*(d) for every $N \in \mathbf{M}^C$, the canonical map $T \otimes_{C^*} N \to N$ is an isomorphism;*

*(e) $C$ is an s-unital $T$-module;*

*(f) $T^2 = T$ and $T$ is a generator in $\mathbf{M}^C$;*

*(g) $TC = C$ and $C^*/T$ is flat as a right $C^*$-module;*

*(h) $T$ is a left $C$-dense subring of $C^*$.*

**9.9. Corollary.** *Assume that $\mathrm{Rat}^C$ is exact and let $T = \mathrm{Rat}^C(C^*) \subset C^*$.*

*(1) $\mathbf{M}^C$ is closed under small epimorphisms in ${}_{C^*}\mathbf{M}$.*

*(2) If $P$ is finitely presented in $\mathbf{M}^C$, then $P$ is finitely presented in ${}_{C^*}\mathbf{M}$.*

*(3) If $P$ is projective in $\mathbf{M}^C$, then $P$ is projective in ${}_{C^*}\mathbf{M}$.*

*(4) For any $M \in \mathbf{M}^C$, the canonical map ${}_{C^*}\mathrm{Hom}(C^*, M) \to {}_{C^*}\mathrm{Hom}(T, M)$ is injective.*

**Proof.** (1)–(3) follow from Corollary 4.9.

(4) By density, for every $f \in {}_{C^*}\mathrm{Hom}(C^*, M)$, $f(\varepsilon) = tf(\varepsilon) = f(t)$ for some $t \in T$, and hence $f(T) = 0$ implies $f(C^*) = 0$. □

Notice that the exactness of $\mathrm{Rat}^C$, that is, the density of $\mathrm{Rat}^C(C^*)$ in $C^*$, also has some influence on left $C$-comodules.

**9.10. Corollary.** *Assume that $\mathrm{Rat}^C$ is exact and let $T = \mathrm{Rat}^C(C^*) \subset C^*$.*

*(1) For any $N \in {}^C\mathbf{M}$, the canonical map $\mathrm{Hom}_{C^*}(C^*, N) \to \mathrm{Hom}_{C^*}(T, N)$ is injective.*

*(2) ${}^C\mathrm{Rat}(C^*) \subset T$ and equality holds if and only if $T \in {}^C\mathbf{M}$.*

**Proof.** (1) By the preceding remark, $C$ is also s-unital as a right $T$-module and hence the proof of Corollary 9.9(4) applies.

(2) By the density of $T \subset C^*$, $X \leftharpoondown T = X$, for each $X \in {}^C\mathbf{M}$ (see 4.3). This implies

$$\mathrm{Hom}_{C^*}(X, C^*) = \mathrm{Hom}_{C^*}(X \leftharpoondown T, C^*) = \mathrm{Hom}_{C^*}(X, C^* * T) = \mathrm{Hom}_{C^*}(X, T);$$

hence ${}^C\mathrm{Rat}(C^*) \subset T$ and ${}^C\mathrm{Rat}(C^*) = T$ provided $T \in {}^C\mathbf{M}$. □

The assertion in 4.10 yields here:

**9.11. Corollary.** *Suppose that* $\mathbf{M}^C$ *has a generator that is (locally) projective in* ${}_{C^*}\mathbf{M}$. *Then* $\mathrm{Rat}^C : {}_{C^*}\mathbf{M} \to \mathbf{M}^C$ *is an exact functor.*

Except when ${}_RC$ is finitely generated (i.e., $\mathrm{Rat}^C(C^*) = C^*$) the trace ideal does not contain a unit element. However, if $C$ is a direct sum of finitely generated left (and right) $C^*$-submodules, the trace ideal has particularly nice properties.

**9.12. Trace ideal and decompositions.** *Let* $T := \mathrm{Rat}^C(C^*)$ *and* $T' := {}^C\mathrm{Rat}(C^*)$.

(1) *If* $C$ *is a direct sum of finitely generated right* $C^*$*-modules, then* $T$ *is* $C$*-dense in* $C^*$ *and there is an embedding*

$$\gamma : T' \to \bigoplus_\Lambda T' * e_\lambda \subset T,$$

*for a family of orthogonal idempotents* $\{e_\lambda\}_\Lambda$ *in* $T$.

(2) *If* $C$ *is a direct sum of finitely generated right* $C^*$*-modules and of finitely generated left* $C^*$*-modules, then* $T = T'$ *and* $T$ *is a projective generator both in* $\mathbf{M}^C$ *and* ${}^C\mathbf{M}$.

**Proof.** (1) Under the given conditions there exist orthogonal idempotents $\{e_\lambda\}_\Lambda$ in $C^*$ with $C = \bigoplus_\Lambda e_\lambda \rightharpoonup C$, where all $e_\lambda \rightharpoonup C$ are finitely generated right $C^*$-modules. By the Finiteness Theorem 8.11, the $e_\lambda \rightharpoonup C$ are finitely generated as $R$-modules, and they are $R$-projective as direct summands of $C$. Now it follows from 9.5(1) that $e_\lambda \in T$. Clearly $C$ is an s-unital left $T$-module and hence the density property follows (see 4.3).

Consider the assignment $\gamma : T' \to \bigoplus_\Lambda T' * e_\lambda$, $t \mapsto \sum_\Lambda t * e_\lambda$. For any $t \in T'$, $t * C^*$ is finitely $R$-generated and so $t * e_\lambda = 0$ for almost all $\lambda \in \Lambda$. Hence $\gamma$ is a well-defined map. Assume $\gamma(t) = 0$. Then, for any $c \in C$, $0 = t * e_\lambda(c) = t(e_\lambda \rightharpoonup c)$, for all $\lambda \in \Lambda$, implying $t = 0$.

(2) By symmetry, (1) implies $T = T'$ and so $T = \bigoplus_\Lambda T * e_\lambda$ and $T = \bigoplus_\Omega f_\omega * T$, where the $\{f_\omega\}_\Omega$ are orthogonal idempotents in $C^*$, and the $C \leftharpoonup f_\omega$ are finitely $R$-generated (hence $f_\omega \in T'$). Clearly each $T * e_\lambda$ is a projective left $T$-module and $f_\omega * T$ a projective right $T$-module. Now the density property implies that $T$ is a projective generator both in $\mathbf{M}^C$ and in ${}^C\mathbf{M}$ (see 9.8). □

Notice that, in 9.12, $e_\lambda \in T'$ need not imply that $C \leftharpoonup e_\lambda$ is finitely $R$-generated, unless we know that $R$ is Noetherian (see 9.5).

**9.13. Decompositions over Noetherian rings.** *Let* $R$ *be Noetherian,* $T = \mathrm{Rat}^C(C^*)$ *and* $T' = {}^C\mathrm{Rat}(C^*)$. *Then the following are equivalent:*

(a) $C_{C^*}$ *and* ${}_{C^*}C$ *are direct sums of finitely generated* $C^*$*-modules;*

(b) $C_{C^*}$ *is a direct sum of finitely generated* $C^*$*-modules and* $T = T'$*;*

(c) *${}_{C^*}C$ is a direct sum of finitely generated $C^*$-modules and $T = T'$;*

(d) *$C = T \rightharpoonup C$ and $T = T'$ and is a ring with enough idempotents.*

*If these conditions hold, $T$ is a projective generator both in $\mathbf{M}^C$ and in ${}^C\mathbf{M}$.*

**Proof.** (a) $\Rightarrow$ (b) follows by 9.12.

(b) $\Rightarrow$ (d) Let $C = \bigoplus_\Lambda e_\lambda \rightharpoonup C$, with orthogonal idempotents $\{e_\lambda\}_\Lambda$ in $C^*$, where all $e_\lambda \rightharpoonup C$ are finitely $R$-generated. Then $e_\lambda \in T = T'$ and $T = \bigoplus_\Lambda T * e_\lambda$. For any $t \in T$, the module $t \rightharpoonup C$ is finitely $R$-generated (by 9.5) and so

$$t \rightharpoonup C \subset e_1 \rightharpoonup C \oplus \cdots \oplus e_k \rightharpoonup C, \text{ for some idempotents } e_i \in \{e_\lambda\}_\Lambda.$$

This implies $t = (e_1 + \cdots + e_k) * t \in \bigoplus_\Lambda e_\lambda * T$. So $\bigoplus_\Lambda T * e_\lambda = T = \bigoplus_\Lambda e_\lambda * T$, showing that $T$ is a ring with enough idempotents.

(d) $\Rightarrow$ (a) If $T = \bigoplus_\Lambda e_\lambda * T$, then

$$C = T \rightharpoonup C = \bigoplus\nolimits_\Lambda e_\lambda \rightharpoonup C,$$

and $e_\lambda \in T$ implies that $e_\lambda \rightharpoonup C$ is finitely $R$-generated. So, by 9.12, $T$ is dense in $C^*$, implying $C \leftharpoonup T = C$. Now symmetric arguments yield the decomposition of $C$ as a direct sum of finitely $R$-generated left $C^*$-modules.

(c) $\Leftrightarrow$ (a) The statement is symmetric to (d) $\Leftrightarrow$ (a).

If the conditions hold, the assertion follows by 9.12. $\square$

Fully invariant submodules of $C$ that are direct summands are precisely subcoalgebras that are direct summands, and they are of the form $e \rightharpoonup C$, where $e$ is a central idempotent in $C^*$. Hence 9.13 yields:

**9.14. Corollary.** *If $R$ is Noetherian, the following are equivalent:*

(a) *$C$ is a direct sum of finitely generated subcoalgebras;*

(b) *$C$ is a direct sum of finitely generated $(C^*, C^*)$-sub-bimodules;*

(c) *$T \rightharpoonup C = C$ and $T$ is a ring with enough central idempotents.*

# 10 Structure of comodules

Throughout this section we assume that $C$ is an $R$-coalgebra with ${}_RC$ locally projective (see 8.2).

Let $N$ be a right $C$-comodule. Then a $C$-comodule $Q$ is said to be $N$-*injective* provided $\mathrm{Hom}^C(-,Q)$ turns any monomorphism $K \to N$ in $\mathbf{M}^C$ into a surjective map. We recall characterisations from 3.3.

**10.1. Injectives in $\mathbf{M}^C$.** *(1) For $Q \in \mathbf{M}^C$ the following are equivalent:*

*(a) $Q$ is injective in $\mathbf{M}^C$;*

*(b) the functor $\mathrm{Hom}^C(-,Q) : \mathbf{M}^C \to \mathbf{M}_R$ is exact;*

*(c) $Q$ is $C$-injective (as left $C^*$-module);*

*(d) $Q$ is $N$-injective for any (finitely generated) subcomodule $N \subset C$;*

*(e) every exact sequence $0 \to Q \to N \to L \to 0$ in $\mathbf{M}^C$ splits.*

*(2) Every injective object in $\mathbf{M}^C$ is $C$-generated.*

*(3) Every object in $\mathbf{M}^C$ has an injective hull.*

A $C$-comodule $P$ is $N$-*projective* if $\mathrm{Hom}^C(P,-)$ turns any epimorphism $N \to L$ into a surjective map.

**10.2. Projectives in $\mathbf{M}^C$.** *(1) For $P \in \mathbf{M}^C$ the following are equivalent:*

*(a) $P$ is projective in $\mathbf{M}^C$;*

*(b) the functor $\mathrm{Hom}^C(P,-) : \mathbf{M}^C \to \mathbf{M}_R$ is exact;*

*(c) $P$ is $C^{(\Lambda)}$-projective, for any set $\Lambda$;*

*(d) every exact sequence $0 \to K \to N \to P \to 0$ in $\mathbf{M}^C$ splits.*

*(2) If $P$ is finitely generated and $C$-projective, then $P$ is projective in $\mathbf{M}^C$.*

Notice that projectives need not exist in $\mathbf{M}^C$. As observed in 7.18, projective objects in $\mathbf{M}^C$ (if they exist) are also projective in $\mathbf{M}_R$.

Over a Noetherian ring $R$, $C$ is left and right locally Noetherian as a $C^*$-module (by 8.15), and therefore we can apply 3.16 to obtain:

**10.3. $C$ as injective cogenerator in $\mathbf{M}^C$.** *If $R$ is Noetherian, then the following are equivalent:*

*(a) $C$ is an injective cogenerator in $\mathbf{M}^C$;*

*(b) $C$ is an injective cogenerator in ${}^C\mathbf{M}$;*

*(c) $C$ is a cogenerator both in $\mathbf{M}^C$ and ${}^C\mathbf{M}$.*

**10.4. $C$ as injective cogenerator in $\mathbf{M}_{C^*}$.** *If $R$ is Artinian, then the following are equivalent:*

*(a) $C$ is an injective cogenerator in $\mathbf{M}_{C^*}$;*

*(b) ${}_{C^*}C$ is Artinian and an injective cogenerator in $\mathbf{M}^C$;*

*(c) $C$ is an injective cogenerator in $\mathbf{M}^C$ and $C^*$ is right Noetherian.*

*If these conditions hold, then $C^*$ is a semiperfect ring and every right $C^*$-module that is finitely generated as an $R$-module belongs to ${}^C\mathbf{M}$.*

**Proof.** Since $R$ is Artinian, $C$ has locally finite length as a $C^*$-module.

(a) $\Rightarrow$ (b) Assume $C$ to be an injective cogenerator in $\mathbf{M}_{C^*}$. Then, by 10.3, $C$ is an injective cogenerator in $\mathbf{M}^C$. Now 3.17 implies that ${}_{C^*}C$ is Artinian.

(b) $\Rightarrow$ (a) and (b) $\Leftrightarrow$ (c) follow again from 3.17.

Assume the conditions hold. $C^*$ is f-semiperfect, being the endomorphism ring of a self-injective module (see 3.15). So $C^*/\mathrm{Jac}(C^*)$ is von Neumann regular and right Noetherian, and hence right (and left) semisimple. This implies that $C^*$ is semiperfect.

Let $L \in \mathbf{M}_{C^*}$ be finitely generated as an $R$-module. Then $L$ is finitely cogenerated as a $C^*$-module, and hence it is finitely cogenerated by $C$. This implies $L \in {}^C\mathbf{M}$. $\square$

The decomposition of left semisimple coalgebras as a direct sum of (indecomposable) subcoalgebras (see 8.13) can be extended to more general situations. Recall that a relation on any family of (co)modules $\{M_\lambda\}_\Lambda$ is defined by setting (cf. [3, 44.11])

$$M_\lambda \sim M_\mu \quad \text{if there exist nonzero morphisms } M_\lambda \to M_\mu \text{ or } M_\mu \to M_\lambda,$$

and the smallest equivalence relation determined by $\sim$ is given by

$$\begin{array}{ll} M_\lambda \approx M_\mu & \text{if there exist } \lambda_1, \ldots, \lambda_k \in \Lambda, \\ & \text{such that } M_\lambda = M_{\lambda_1} \sim \cdots \sim M_{\lambda_k} = M_\mu\,. \end{array}$$

**10.5. $\sigma$-decomposition of coalgebras.** *Let $R$ be a Noetherian ring.*

*(1) There exist a $\sigma$-decomposition $C = \bigoplus_\Lambda C_\lambda$ and a family of orthogonal central idempotents $\{e_\lambda\}_\Lambda$ in $C^*$ with $C_\lambda = C \leftharpoonup e_\lambda$, for each $\lambda \in \Lambda$.*

*(2) Each $C_\lambda$ is a subcoalgebra of $C$, $C_\lambda^* \simeq C^* * e_\lambda$, $\sigma[{}_{C^*}C_\lambda] = \sigma[{}_{C_\lambda^*}C_\lambda]$, and*

$$\mathbf{M}^C = \bigoplus\nolimits_\Lambda \sigma[{}_{C^*}C_\lambda] = \bigoplus\nolimits_\Lambda \mathbf{M}^{C_\lambda}.$$

*(3) $\mathbf{M}^C$ is indecomposable if and only if, for any two injective uniform $L, N \in \mathbf{M}^C$, $L \approx N$ holds.*

*(4) Assume that $R$ is Artinian. Then $\mathbf{M}^C$ is indecomposable if and only if, for any two simple $E_1, E_2 \in \sigma_{C^*}[C]$, $\widehat{E}_1 \approx \widehat{E}_2$ holds.*

**Proof.** (1),(2) By the Finiteness Theorem 8.11, $C$ is a locally Noetherian $C^*$-module. Now the decomposition of $\mathbf{M}^C$ $(=\sigma[_{C^*}C])$ follows from module theory (see 44.14. in [3]).

Clearly the resulting $\sigma$-decomposition of $C$ is a fully invariant decomposition, and hence it can be described by central idempotents in the endomorphism ring ($= C^*$; see [3, 44.1]). Fully invariant submodules $C_\lambda \subset C$ are in particular $R$-direct summands in $C$ and hence are subcoalgebras (by 8.6). It is straightforward to verify that $\mathrm{Hom}_R(C_\lambda, R) = C^*_\lambda \simeq C^* * e_\lambda$ is an algebra isomorphism. This implies $\sigma[_{C^*}C_\lambda] = \sigma[_{C^*_\lambda}C_\lambda] = \mathbf{M}^{C_\lambda}$.

(3) is a special case of [3, 44.14].

(4) follows from [3, 44.14](3). Notice that $\widehat{E}_1 \approx \widehat{E}_2$ can be described by extensions of simple modules (see [3, 44.11]). (The assertion means that the Ext quiver of simple modules in $\mathbf{M}^C$ is connected.) □

Transferring [3, 44.7] we obtain:

**10.6. Corollary.** *Let $C$ be a coalgebra with $\sigma$-decomposition $C = \bigoplus_\Lambda C_\lambda$. Then the left rational functor $\mathrm{Rat}^C$ is exact if and only if the left rational functors $\mathrm{Rat}^{C_\lambda}$ are exact, for each $C_\lambda$.*

Even for coalgebras $C$ over fields there need not be any projective comodules in $\mathbf{M}^C$. We discuss the existence of (enough) projectives in $\mathbf{M}^C$ and the projectivity of $C$ in $\mathbf{M}^C$ or in $_{C^*}\mathbf{M}$.

**Definition.** A coalgebra $C$ is called *right semiperfect* if every simple right comodule has a projective cover in $\mathbf{M}^C$. If $_RC$ is locally projective, this is obviously equivalent to the condition that every simple module in $\sigma[_{C^*}C]$ has a projective cover in $\sigma[_{C^*}C]$ (by 8.3), that is, $\mathbf{M}^C = \sigma[_{C^*}C]$ is a semiperfect category.

Notice that a right semiperfect coalgebra $C$ need not be a semiperfect left $C^*$-module as defined in 3.10. The following characterisations can be shown.

**10.7. Right semiperfect coalgebras.** *The following are equivalent:*

*(a) $C$ is a right semiperfect coalgebra;*

*(b) $\mathbf{M}^C$ has a generating set of local projective modules;*

*(c) every finitely generated module in $\mathbf{M}^C$ has a projective cover.*

*If $R$ is a perfect ring, then (a)-(c) are equivalent to:*

*(d) $\mathbf{M}^C$ has a generating set of finitely generated $C$-projective comodules.*

**Proof.** See 41.14., 41.16 and 41.22 in [3]. □

As an obvious application of 10.5 we obtain:

**10.8. $\sigma$-decomposition of semiperfect coalgebras.** *Let $R$ be Noetherian and $C$ with $\sigma$-decomposition $C = \bigoplus_\Lambda C_\lambda$. Then $C$ is a right semiperfect coalgebra if and only if the $C_\lambda$ are right semiperfect coalgebras, for all $\lambda \in \Lambda$.*

We finally turn to the question of when $C$ itself is projective in $\mathbf{M}^C$ or ${}_{C^*}\mathbf{M}$. Since $C$ is a balanced $(C^*, C^*)$-bimodule, we can use standard module theory to obtain some properties of $C$ as a locally projective $C^*$-module.

**10.9. $C$ locally projective as $C^*$-module.**

*(1) If $C$ is locally projective as a left $C^*$-module, then $C$ is a generator in ${}^C\mathbf{M}$.*

*(2) If $C$ is locally projective as a left and right $C^*$-module, then both* $\mathrm{Rat}^C$ *and* ${}^C\mathrm{Rat}$ *are exact.*

**Proof.** (1) If ${}_{C^*}C$ is locally projective, then, by [3, 42.10], $C_{C^*}$ is a generator in $\sigma[C_{C^*}] = {}^C\mathbf{M}$.

(2) Assume that both ${}_{C^*}C$ and $C_{C^*}$ are locally projective. Then, by (1), ${}_{C^*}C$ is a locally projective generator in $\sigma[{}_{C^*}C]$, and, by 9.11, $\mathrm{Rat}^C$ is an exact functor. Similar arguments show that ${}^C\mathrm{Rat}$ is exact. $\square$

**10.10. $C$ projective in $\mathbf{M}^C$.** *Assume that $C$ is projective in $\mathbf{M}^C$.*

*(1) If $C^*$ is an f-semiperfect ring, or $C$ is $C$-injective, then $C$ is a direct sum of finitely generated left $C^*$-modules.*

*(2) If $C^*$ is a right self-injective ring, then $C$ is a generator in ${}^C\mathbf{M}$.*

*(3) If $C^*$ is a semiperfect ring, then ${}_RC$ is finitely generated.*

**Proof.** (1) This is a decomposition property of projective modules with f-semiperfect endomorphism rings (see [3, 41.19]). If $C$ is self-injective, then $C^*$ is f-semiperfect.

(2) As a self-injective ring, $C^*$ is f-semiperfect. By 9.12, (1) implies that $C$ is s-unital over the right trace ideal $T'$, and so $T'$ is a generator (by 9.8). Moreover, right injectivity of $C^*$ implies that $T' = \mathrm{Tr}({}^C\mathbf{M}, C^*) = \mathrm{Tr}(C_{C^*}, C^*)$, and so $C$ generates $T'$ (see [3, 42.7] for the definition of a trace).

(3) This follows from (1) and [3, 41.19]. $\square$

# 11 Coalgebras over QF rings

Recall that a QF ring $R$ is an Artinian injective cogenerator in $\mathbf{M}_R$. We consider $R$-coalgebras $C$ with ${}_RC$ locally projective. If $R$ is a QF ring, then this is equivalent to $C$ being projective as an $R$-module.

**11.1. Coalgebras over QF rings.** *If $R$ is a QF ring, then:*

*(1) $C$ is a (big) injective cogenerator in $\mathbf{M}^C$.*

*(2) Every comodule in $\mathbf{M}^C$ is a subcomodule of some direct sum $C^{(\Lambda)}$.*

*(3) $C^*$ is an f-semiperfect ring.*

*(4) $K := \mathrm{Soc}_{C^*}C \trianglelefteq C$ and $\mathrm{Jac}(C^*) = \mathrm{Hom}_R(C/K, R)$.*

*(5) $C^*$ is right self-injective if and only if $C$ is flat as left $C^*$-module.*

**Proof.** (1),(2) By 7.17, $C$ is injective in $\mathbf{M}^C$. Over a QF ring $R$, every $R$-module $M$ is contained in a free $R$-module $R^{(\Lambda)}$. This yields, for any right $C$-comodule, an injection $\varrho^M : M \to M \otimes_R C \subset R^{(\Lambda)} \otimes_R C \simeq C^{(\Lambda)}$.

(3) The endomorphism ring of any self-injective module is f-semiperfect (see 3.15).

(4) By 8.15, ${}_{C^*}C$ is locally of finite length and hence has an essential socle. By the Hom-tensor relations (see 7.9),

$$\mathrm{Jac}(C^*) = \mathrm{Hom}^C(C/K, C) \simeq \mathrm{Hom}_R(C/K, R).$$

(5) For any $N \in \mathbf{M}_{C^*}$, there is an isomorphism $\mathrm{Hom}_R(N \otimes_{C^*} C, R) \simeq \mathrm{Hom}_{C^*}(N, \mathrm{Hom}_R(C, R)) = \mathrm{Hom}_{C^*}(N, C^*)$ (cf. [3, 40.18]). So, if $C^*$ is right self-injective, the functor $\mathrm{Hom}_R(- \otimes_{C^*} C, R) : \mathbf{M}_{C^*} \to \mathbf{M}_R$ is exact. Since $R$ is a cogenerator in $\mathbf{M}_R$, this implies that $- \otimes_{C^*} C$ is exact, that is, ${}_{C^*}C$ is flat. Similar arguments yield the converse conclusion. □

By 4.17, For any injective cogenerator, fully invariant decompositions (coalgebra decompositions) are $\sigma$-decompositions (see [3, 44.8]). Consequently, 10.5 yields:

**11.2. $\sigma$-decomposition of $C$.** *If $R$ is a QF ring, then:*

*(1) $C$ has fully invariant decompositions with $\sigma$-indecomposable summands.*

*(2) Each fully invariant decomposition (= decomposition into coalgebras) is a $\sigma$-decomposition.*

*(3) $C$ is $\sigma$-indecomposable if and only if $C$ has no nontrivial fully invariant decomposition, that is, $C^*$ has no nontrivial central idempotents.*

*(4) If $C$ is cocommutative, then $C = \bigoplus_\Lambda \widehat{E}_\lambda$ is a fully invariant decomposition, where $\{E_\lambda\}_\Lambda$ is a minimal representing set of simple comodules in $\mathbf{M}^C$, and $\widehat{E}_\lambda$ denotes the injective hull of $E_\lambda$.*

**Proof.** By 11.1, $C$ is an injective cogenerator in $\sigma[{}_{C^*}C]$, and so (1), (2) and (3) follow from 4.17 and 10.5. In (4), $C^*$ is a commutative algebra by assumption, and so the assertion follows from [3, 43.7]. $\square$

Over QF rings there is a bijective correspondence between closed subcategories of $\mathbf{M}^C$ and $(C^*, C^*)$-sub-bimodules in $C$. However, the latter need not be pure $R$-submodules of $C$, and hence they may not be subcoalgebras. Recall that injectivity of $C$ in $\mathbf{M}^C$ implies $\mathrm{Tr}(\sigma[N], C) = \mathrm{Tr}(N, C)$, for any $N \in \mathbf{M}^C$.

**11.3. Correspondence relations.** *Let $R$ be a QF ring and $N \in \mathbf{M}^C$. Then:*

(1) $\sigma[N] = \sigma[\mathrm{Tr}(N, C)]$.

(2) *The map $\sigma[N] \mapsto \mathrm{Tr}(N, C)$ yields a bijective correspondence between closed subcategories of $\mathbf{M}^C$ and $(C^*, C^*)$-sub-bimodules of $C$.*

(3) *$\sigma[N]$ is closed under essential extensions (injective hulls) in $\mathbf{M}^C$ if and only if $\mathrm{Tr}(N, C)$ is a $C^*$-direct summand of ${}_{C^*}C$. In this case $\mathrm{Tr}(N, C)$ is a subcoalgebra of $C$.*

(4) *$N$ is a semisimple comodule if and only if $\mathrm{Tr}(N, C) \subset \mathrm{Soc}({}_{C^*}C)$.*

(5) *If $R$ is a semisimple ring, then $\mathrm{Tr}(N, C)$ is a subcoalgebra of $C$.*

**Proof.** Since $R$ is a QF ring, ${}_{C^*}C$ has locally finite length and is an injective cogenerator in $\mathbf{M}^C$. Hence (1)–(4) follow from 4.13. Furthermore, if $R$ is semisimple, the $(C^*, C^*)$-sub-bimodule $\mathrm{Tr}(N, C)$ is an $R$-direct summand in $C$ and so is a subcoalgebra by 8.6. This proves assertion (5). $\square$

Since over a QF ring $R$ any $R$-coalgebra $C$ is an injective cogenerator in $\mathbf{M}^C$ and ${}^C\mathbf{M}$ (by 11.1), the results from 10.4 simplify to the following.

**11.4. $C$ injective in $\mathbf{M}_{C^*}$.** *If $R$ is QF, the following are equivalent:*

(a) *$C$ is injective in $\mathbf{M}_{C^*}$;*

(b) *$C$ is an injective cogenerator in $\mathbf{M}_{C^*}$;*

(c) *${}_{C^*}C$ is Artinian;*

(d) *$C^*$ is a right Noetherian ring.*

**Proof.** In view of the preceding remark the equivalence of (b), (c) and (d) follows from 10.4. The implication (b) $\Rightarrow$ (a) is trivial, and (a) $\Rightarrow$ (c) needs an argument from module theory (see [3, 8.3]). $\square$

For finitely generated comodules, injectivity and projectivity in $\mathbf{M}^C$ may extend to injectivity, resp. projectivity, in ${}_{C^*}\mathbf{M}$.

**11.5. Finitely presented modules over QF rings.** *Let $R$ be a QF ring and $M \in \mathbf{M}^C$.*

*(1) If $M$ is projective in $\mathbf{M}^C$, then $M^*$ is $C$-injective as a right $C^*$-module and $\mathrm{Rat}^C(M^*)$ is injective in ${}^C\mathbf{M}$.*

*(2) If $M$ is finitely generated as an $R$-module, then:*

*(i) if $M$ is injective in $\mathbf{M}^C$, then $M^*$ is projective in $\mathbf{M}_{C^*}$.*

*(ii) $M$ is injective in $\mathbf{M}^C$ if and only if $M$ is injective in ${}_{C^*}\mathbf{M}$.*

*(iii) $M$ is projective in $\mathbf{M}^C$ if and only if $M$ is projective in ${}_{C^*}\mathbf{M}$.*

**Proof.** (1) Consider any diagram with exact row in ${}^C\mathbf{M}$,

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "f"] & N \\ & M^*, & \end{tikzcd}$$

where $N$ is finitely generated as an $R$-module. Applying $(-)^* = \mathrm{Hom}_R(-, R)$ we obtain – with the canonical map $\Phi_M : M \to M^{**}$ – the diagram

$$\begin{tikzcd} M \arrow[r] & M^{**} \arrow[d, "f^*"] & \\ N^* \arrow[r] & K^* \arrow[r] & 0, \end{tikzcd}$$

where the lower row is in $\mathbf{M}^C$ and hence can be extended commutatively by some right comodule morphism $g : M \to N^*$. Again applying $(-)^*$ - and recalling that the composition $M^* \xrightarrow{\Phi_{M^*}} M^{***} \xrightarrow{(\Phi_M)^*} M^*$ yields the identity (by [3, 40.23]) - we see that $g^*$ extends $f$ to $N$. This proves that $M^*$ is $N$-injective for all modules $N \in {}^C\mathbf{M}$ that are finitely presented as $R$-modules.

In particular, by the Finiteness Theorem 8.11, every finitely generated $C^*$-submodule of $C$ is finitely generated - hence finitely presented – as an $R$-module. So $M^*$ is $N$-injective for all these modules, and hence it is $C$-injective as a right $C^*$-module (see 10.1). Notice that $M^*$ need not be in ${}^C\mathbf{M}$ (not rational). It is straightforward to show that $\mathrm{Rat}^C(M^*)$ is an injective object in ${}^C\mathbf{M}$.

(2)(i) We know that $M \subset R^k$, for some $k \in \mathbb{N}$, and so there is a monomorphism in $\mathbf{M}^C$, $M \xrightarrow{\varrho^M} M \otimes_R C \longrightarrow R^k \otimes_R C \simeq C^k$, that splits in $\mathbf{M}^C$ and hence in ${}_{C^*}\mathbf{M}$ (by 8.1). So the dual sequence $(C^*)^k \to M^* \to 0$ splits in $\mathbf{M}_{C^*}$, and hence $M^*$ is projective in $\mathbf{M}_{C^*}$.

(ii) Let $M$ be injective in $\mathbf{M}^C$. Then $M^*$ is projective in $\mathbf{M}_{C^*}$ (by (i)). Consider any monomorphism in $M \to X$ in ${}_{C^*}\mathbf{M}$. Then $X^* \to M^* \to 0$ is exact and splits in $\mathbf{M}_{C^*}$, and hence, in the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\simeq"] & X \arrow[d] \\ 0 \arrow[r] & M^{**} \arrow[r] & X^{**}, \end{tikzcd}$$

the bottom row splits in ${}_{C^*}\mathbf{M}$ and as a consequence so does the upper row, proving that $M$ is injective in ${}_{C^*}\mathbf{M}$.

(iii) Let $M$ be projective in $\mathbf{M}^C$. Since $M^*$ is in ${}^C\mathbf{M}$ (by 7.10), we know from (1) that it is injective in ${}^C\mathbf{M}$. Now we conclude, by the right-hand version of (i), that $M \simeq M^{**}$ is projective in ${}_{C^*}\mathbf{M}$. $\square$

As shown in 11.5, for coalgebras over QF rings, finitely generated projective modules in $\mathbf{M}^C$ are in fact projective in ${}_{C^*}\mathbf{M}$. This is the key to the fact that in this case right semiperfect coalgebras are characterised by the exactness of the left trace functor (so also by all the equivalent properties of the trace functor given in 9.8).

**11.6. Right semiperfect coalgebras over QF rings.** *Let $R$ be QF and $T = \mathrm{Rat}^C(C^*)$. Then the following are equivalent:*

*(a) $C$ is a right semiperfect coalgebra;*

*(b) $\mathbf{M}^C$ has a generating set of finitely generated modules that are projective in ${}_{C^*}\mathbf{M}$;*

*(c) injective hulls of simple* left *$C$-comodules are finitely generated as $R$-modules;*

*(d) the functor* $\mathrm{Rat}^C : {}_{C^*}\mathbf{M} \to \mathbf{M}^C$ *is exact;*

*(e) $T$ is left $C$-dense in $C^*$;*

*(f)* $\mathrm{Ke}\, T = \{x \in C \mid T(x) = 0\} = 0$.

**Proof.** (a) $\Leftrightarrow$ (b) If $C$ is right semiperfect, there exists a generating set of finitely generated projective modules in $\mathbf{M}^C$ (see 10.7). By 11.5, all these are projective in ${}_{C^*}\mathbf{M}$. The converse conclusion is immediate.

(a) $\Rightarrow$ (c) Let $U$ be a simple left $C$-comodule with injective hull $U \to \widehat{U}$ in ${}^C\mathbf{M}$. Applying $\mathrm{Hom}_R(-, R)$ we obtain a small epimorphism in ${}_{C^*}\mathbf{M}$,

$$\widehat{U}^* \to U^* \to 0,$$

where $U^*$ is a simple left $C^*$-module. Moreover, since $R$ is QF, we know that $\widehat{U}$ is a direct summand of $C_{C^*}$, and so $\widehat{U}^*$ is a direct summand of $C^*$, and hence is projective in ${}_{C^*}\mathbf{M}$. By assumption there exists a projective cover $P \to U^*$ in $\mathbf{M}^C$. Since $P$ is finitely generated as an$R$-module and projective in $\mathbf{M}^C$, it is also projective in ${}_{C^*}\mathbf{M}$ (by 11.5), and hence $\widehat{U}^* \simeq P$. So $\widehat{U}^*$ is finitely generated as an $R$-module and so is $\widehat{U}$.

(c) $\Rightarrow$ (a) Let $V \subset C$ be a simple left $C^*$-submodule. Then $V^*$ is a simple right $C^*$-module in ${}^C\mathbf{M}$. Let $V^* \to K$ be its injective hull in ${}^C\mathbf{M}$. By assumption, $K$ is a finitely generated $R$-module, and so $K^*$ is a projective $C^*$-module (by 11.5) and $K^* \to V^{**} \simeq V$ is a projective cover in $\mathbf{M}^C$.

(b) $\Rightarrow$ (d) The assumption implies that $\mathbf{M}^C$ has a generator that is projective in ${}_{C^*}\mathbf{M}$, and the assertion follows from 9.11.

(d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) These equivalences follow from 9.8 and 9.6.

(d) $\Rightarrow$ (c) Let $V \subset C$ be a simple left $C^*$-submodule. Then $U = V^*$ is a simple left $C$-comodule and there is a projective cover $\widehat{U}^* \to V$ in ${}_{C^*}\mathbf{M}$ (see proof (a) $\Rightarrow$ (c)). By 9.9, (d) implies that $\mathbf{M}^C$ is closed under small epimorphisms and hence $\widehat{U}^* \in \mathbf{M}^C$. $\square$

The conditions on left $C^*$-modules (right $C$-comodules) posed in the preceding theorem imply remarkable properties of the left $C$-comodules.

**11.7. Left side of right semiperfect coalgebras.** *Let $C$ be right semiperfect, $R$ a QF ring and $T = \mathrm{Rat}^C(C^*)$. Then:*

*(1) the injective hull of any $X \in {}^C\mathbf{M}$ is finitely $R$-generated, provided $X$ is finitely $R$-generated.*

*(2) For every $X \in {}^C\mathbf{M}$ that is finitely $R$-generated, $\mathrm{Hom}_{C^*}(T, X) \simeq X$.*

*(3) For every $M \in {}^C\mathbf{M}$, the trace of $\mathbf{M}^C$ in $M^*$ is nonzero.*

*(4) Any module in ${}^C\mathbf{M}$ has a maximal submodule and has a small radical.*

**Proof.** (1) Let $X \in {}^C\mathbf{M}$ be finitely generated as an $R$-module. Then $X$ has finite uniform dimension, and so its injective hull in ${}^C\mathbf{M}$ is a finite direct sum of injective hulls of simple modules, which are finitely generated by 11.6(c).

(2) By (1), the $C$-injective hull $\widehat{X}$ of $X$ is finitely $R$-generated and hence is $C^*$-injective (see 11.5). So any $f \in \mathrm{Hom}_{C^*}(T, X)$ can be uniquely extended to some $h : C^* \to \widehat{X}$ and $h(\varepsilon) \in \widehat{X}$, which is s-unital over $T$ (see 9.10). Hence

$$h(\varepsilon) \in h(\varepsilon) \cdot C^* = h(\varepsilon) \cdot T = h(T) = f(T) \subset X,$$

showing that $h \in \mathrm{Hom}_{C^*}(C^*, X) \simeq X$.

(3) For every simple submodule $S \subset M$ with injective hull $\widehat{S}$ in ${}^C\mathbf{M}$, there are commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d, "i"'] & M \arrow[dl, "j"] \\ & \widehat{S} & \end{tikzcd} , \qquad \begin{tikzcd} M^* \arrow[r] & S^* \\ \widehat{S}^* \arrow[u, "j^*"] \arrow[ur, "i^*"'] & \end{tikzcd} ,$$

where $i$ is injective and $j$ is nonzero. By 7.10, $\widehat{S}^*$ belongs to $\mathbf{M}^C$ and so does its nonzero image under $j^*$.

(4) Let $M \in {}^C\mathbf{M}$. By (3), there exists a simple submodule $Q \subset M^*$ with $Q \in \mathbf{M}^C$. Then $\operatorname{Ke} Q = \{m \in M \mid Q(m) = 0\}$ is a maximal $C^*$-submodule of $M$. This shows that all modules in ${}^C\mathbf{M}$ have maximal submodules, and hence every proper submodule of $M$ is contained in a maximal $C^*$-submodule. This implies that $\operatorname{Rad}(M)$ is small in $M$. □

**11.8. Finiteness properties.** *Let $R$ be a QF ring.*

*(1) If $C$ is right semiperfect and there are only finitely many nonisomorphic simple right $C$-comodules, then ${}_RC$ is finitely generated.*

*(2) If $C$ is right semiperfect and any two nonzero subalgebras have non-zero intersection (i.e., $C$ is* irreducible*), then ${}_RC$ is finitely generated.*

*(3) ${}_RC$ is finitely generated if and only if $\mathbf{M}^C$ has a finitely generated projective generator.*

*(4) $C^*$ is an algebra of finite representation type if and only if there are only finitely many nonisomorphic finitely generated indecomposable modules in $\mathbf{M}^C$.*

**Proof.** (1) Since $C_{C^*}$ is self-injective, the socle of $C_{C^*}$ is a finitely generated $R$-module by [3, 41.23]. Hence $\operatorname{Soc}(C_{C^*})$ has finite uniform dimension, and since $\operatorname{Soc}(C_{C^*}) \trianglelefteq C$, $C$ is a finite direct sum of injective hulls of simple modules in ${}^C\mathbf{M}$ that are finitely generated $R$-modules by 11.7.

(2) Under the given condition there exists only one simple right $C$-comodule (up to isomorphisms), and the assertion follows from (1).

(3) If ${}_RC$ is finitely generated, then $\mathbf{M}^C = {}_{C^*}\mathbf{M}$. Conversely, assume there exists a finitely generated projective generator $P$ in $\mathbf{M}^C$. Then $P$ is semiperfect and there are only finitely many simples in $\mathbf{M}^C$. Now (1) applies.

(4) One implication is obvious. Assume there are only finitely many nonisomorphic finitely generated indecomposables in $\mathbf{M}^C$. Since $C$ is subgenerated by its finitely generated submodules, this implies that $\mathbf{M}^C$ has a finitely generated subgenerator. Now [10, 54.2] implies that there is a progenerator in $\mathbf{M}^C$, and hence ${}_RC$ is finitely generated by (3). □

Unlike in the case of associative algebras, right semiperfectness is a strictly one-sided property for coalgebras – it need not imply left semiperfectness. The next proposition describes coalgebras that are both right and left semiperfect.

**11.9. Left and right semiperfect coalgebras.** *Let $R$ be a QF ring, $T = \operatorname{Rat}^C(C^*)$ and $T' = {}^C\operatorname{Rat}(C^*)$. The following are equivalent:*

*(a) $C$ is a left and right semiperfect coalgebra;*

*(b) all left $C$-comodules and all right $C$-comodules have projective covers;*

*(c) $T = T'$ and is dense in $C^*$;*

*(d) ${}_{C^*}C$ and $C_{C^*}$ are direct sums of finitely generated $C^*$-modules.*

*Under these conditions, $T$ is a ring with enough idempotents, and it is a generator in $\mathbf{M}^C$.*

**Proof.** (b) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (b) By [3, 41.16], all finitely generated projective modules in $\mathbf{M}^C$ are semiperfect in $\mathbf{M}^C$. According to 3.10 and 3.11, a direct sum of projective semiperfect modules in $\mathbf{M}^C$ is semiperfect provided it has a small radical. Since this is the case by 11.7, we conclude that every module in $\mathbf{M}^C$ has a projective cover. Similar arguments apply to the category ${}^C\mathbf{M}$.

(a) $\Leftrightarrow$ (c) This is obvious by the characterisation of exactness of the rational functor in 9.8 and 11.6.

(c) $\Leftrightarrow$ (d) follows from 9.12.

The final assertions follow from 9.12 and 9.8. □

For cocommutative coalgebras we can combine 11.9 with 11.2(4).

**11.10. Cocommutative semiperfect coalgebras.** *Let $R$ be QF and $C$ cocommutative. The following are equivalent:*

*(a) $C$ is semiperfect;*

*(b) $C$ is a direct sum of finitely generated $C^*$-modules;*

*(c) $C$ is a direct sum of finitely $R$-generated subcoalgebras;*

*(d) every uniform subcomodule ($C^*$-submodule) of $C$ is finitely $R$-generated.*

The trace functors combined with the dual functor $(-)^*$ define covariant functors ${}^C\mathrm{Rat} \circ (-)^* : \mathbf{M}^C \to {}^C\mathbf{M}$ and $\mathrm{Rat}^C \circ (-)^* : {}^C\mathbf{M} \to \mathbf{M}^C$. Over QF rings, these functors clearly are exact if and only if ${}^C\mathrm{Rat}$, respectively $\mathrm{Rat}^C$, is exact, that is, $C$ is left or right semiperfect. In this case they yield dualities between subcategories of $\mathbf{M}^C$ and ${}^C\mathbf{M}$.

Over a QF ring, projective comodules in $\mathbf{M}^C$ that are finitely generated as left $C^*$-modules are also projective in ${}_{C^*}\mathbf{M}$ (see 11.5). Moreover, any direct sum of copies of $C$ is $C$-injective as a left and right $C^*$-module.

**11.11. Projective coalgebras over QF rings.** *If $R$ is QF, the following are equivalent:*

*(a) $C$ is a submodule of a free left $C^*$-module;*

*(b) $C$ (or every right $C$-comodule) is cogenerated by $C^*$ as a left $C^*$-module;*

*(c) there exists a family of left nondegenerate $C$-balanced bilinear forms $C \times C \to R$;*

*(d) in $\mathbf{M}^C$ every (indecomposable) injective object is projective;*

*(e) $C$ is projective in $\mathbf{M}^C$;*

*(f) $C$ is projective in ${}_{C^*}\mathbf{M}$.*

*If these conditions are satisfied, then $C$ is a left semiperfect coalgebra and $C$ is a generator in ${}^C\mathbf{M}$.*

**Proof.** (a) $\Leftrightarrow$ (b) By 8.15, $C$ is a direct sum of injective hulls of simple modules in $\mathbf{M}^C$. If $C$ is cogenerated by $C^*$, then each of these modules is contained in a copy of $C^*$, and hence $C$ is contained in a free $C^*$-module. Recall from 11.1 that $C$ is a cogenerator in $\mathbf{M}^C$ and hence $C^*$ cogenerates any $N \in \mathbf{M}^C$ provided it cogenerates $C$.

(b) $\Leftrightarrow$ (c) This is shown in [3, 6.6(2)].

(c) $\Rightarrow$ (f) Let $U$ be a simple left $C^*$-submodule of $C$ with injective hull $\widehat{U} \subset C$ in $\mathbf{M}^C$. Then $\widehat{U}$ is a finitely generated $R$-module by [3, 6.6(3)]. Now we conclude from 11.5 that $\widehat{U}$ is injective in ${}_{C^*}\mathbf{M}$. Being cogenerated by $C^*$, we observe in fact that $\widehat{U}$ is a direct summand of $C^*$, and hence it is projective in ${}_{C^*}\mathbf{M}$. This implies that $C$ is projective in ${}_{C^*}\mathbf{M}$.

(f) $\Rightarrow$ (a) and (f) $\Rightarrow$ (e) are obvious, and so is (d) $\Leftrightarrow$ (e) (by 11.1).

(e) $\Rightarrow$ (f) $C$ is a direct sum of injective hulls $\widehat{U} \subset C$ of simple submodules $U \subset C$. By (e), $\widehat{U}$ is projective in $\mathbf{M}^C$. Since it has a local endomorphism ring, we know from 3.9 that it is finitely generated as a $C^*$-module and hence finitely generated as an $R$-module (by 8.11). Now we conclude from 11.5 that $\widehat{U}$ is projective in ${}_{C^*}\mathbf{M}$ and so is $C$.

Finally, assume these conditions hold. By the proof of 11.11, the injective hulls of simple modules in $\mathbf{M}^C$ are finitely generated $R$-modules. By 10.7, this characterises left semiperfect coalgebras, implying that the right trace ideal $T' := {}^C\mathrm{Rat}(C^*)$ is a generator in ${}^C\mathbf{M}$. Now, by 11.5(1), $T'$ is injective in ${}^C\mathbf{M}$, and hence it is generated by $C$ and therefore $C$ is a generator in ${}^C\mathbf{M}$. $\square$

**11.12. Corollary.** *Let $R$ be QF and $C$ projective in $\mathbf{M}^C$. Then the following are equivalent:*

*(a) ${}_{C^*}C$ contains only finitely many nonisomorphic simple submodules;*

*(b) $\mathrm{Soc}({}_{C^*}C)$ is finitely generated as an $R$-module;*

*(c) $C^*$ is a semiperfect ring;*

*(d) ${}_RC$ is finitely generated.*

**Proof.** (a) $\Rightarrow$ (b) By the Finiteness Theorem, the homogeneous components of the socle of $C$ are finitely generated a $R$-modules.

(b) $\Rightarrow$ (c) We know that $C^*$ is f-semiperfect. Clearly $\mathrm{Soc}({}_{C^*}C) \trianglelefteq C$, and hence $C$ has a finite uniform dimension as a left $C^*$-module. This implies that $C^*$ is semiperfect.

(c) $\Rightarrow$ (a) For any semiperfect ring there are only finitely many simple left (or right) modules (up to isomorphisms).

(c) $\Rightarrow$ (d) is shown in 10.10(3).

(d) $\Rightarrow$ (b) follows from the fact that $R$ is Noetherian. $\square$

From 11.1 we know that, over a QF ring $R$, $C$ is always an injective cogenerator in $\mathbf{M}^C$. Which additional properties make $C$ a projective generator?

**11.13. $C$ as a projective generator in $\mathbf{M}^C$.** *Let $R$ be QF and $T = \mathrm{Rat}^C(C^*)$. The following are equivalent:*

*(a) $C$ is projective as left and right $C$-comodule;*

*(b) $C$ is a projective generator in $\mathbf{M}^C$;*

*(c) $C$ is a projective generator in ${}^C\mathbf{M}$;*

*(d) $C = TC$ and $T$ has enough idempotents and is an injective cogenerator in $\mathbf{M}^C$.*

**Proof.** (a) $\Rightarrow$ (b) This is obtained from 11.11 and 10.9.

(b) $\Rightarrow$ (a) By 11.11, $C$ is projective as a left $C^*$-module and hence $C^*$ is $C$-injective as a right $C^*$-module (by 11.5). To show that $C$ is projective as a right $C^*$-module we show that $C^*$ cogenerates $C$ as a right $C^*$-module. For this it is enough to prove that each simple submodule $U \subset C_{C^*}$ is embedded in $C^*$. By 8.11, $U$ is a finitely generated $R$-module. Clearly $U^*$ is a simple module in $\mathbf{M}^C$, and hence there is a $C^*$-epimorphism $C \to U^*$. From this we obtain an embedding $U \simeq U^{**} \subset C^*$, which proves our assertion.

(a) $\Leftrightarrow$ (c) is clear by symmetry.

(a) $\Rightarrow$ (d) From the above discussion we know that $C$ is a left and right semiperfect coalgebra. Hence $T$ is a ring with enough idempotents and $\mathbf{M}^C = \sigma[{}_{C^*}T]$ by 11.9. Since $C$ is projective, $C \subset T^{(\Lambda)}$, and hence $T$ is a cogenerator in $\mathbf{M}^C$. $T$ is injective in $\mathbf{M}^C$ by 11.5.

(d) $\Rightarrow$ (b) Since $T$ is projective in $\mathbf{M}^C$, injective hulls of simple modules in $\mathbf{M}^C$ are projective, and so $C$ is projective in $\mathbf{M}^C$. $T$ is injective, and hence it is generated by $C$. By our assumptions $T$ is a generator in $\mathbf{M}^C$ and so is $C$. $\square$

In case $C$ is finitely $R$-generated, $\mathbf{M}^C = {}_{C^*}\mathbf{M}$ and we obtain:

**11.14. $C$ as a projective generator in ${}_{C^*}\mathbf{M}$.** *If $R$ is QF, the following are equivalent:*

*(a) $C$ is a projective generator in ${}_{C^*}\mathbf{M}$;*

*(b) $C$ is a generator in ${}_{C^*}\mathbf{M}$;*

*(c) $C$ is a generator in $\mathbf{M}^C$ and ${}_RC$ is finitely generated;*

*(d) $C^*$ is a QF algebra and ${}_RC$ is finitely generated.*

**Proof.** (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c) As a generator in ${}_{C^*}\mathbf{M}$, $C$ is finitely generated as a module over its endomorphism ring $C^*$, and hence ${}_RC$ is finitely generated.

(c) $\Rightarrow$ (d) Clearly $C^*$ is left (and right) Artinian. By assumption, $C$ is an injective generator in ${}_{C^*}\mathbf{M}$. This implies that $C^*$ is self-injective and hence QF.

(d) $\Rightarrow$ (a) As a QF ring, $C^*$ is an injective cogenerator in $\mathbf{M}^C = {}_{C^*}\mathbf{M}$. From this it is easy to see that $C$ is a projective generator in ${}_{C^*}\mathbf{M}$. $\square$

# 12 Bialgebras

In this section we are concerned with the compatibility of algebra and coalgebra structures on a given $R$-module. In particular, we define *bialgebras* and study their most elementary properties.

**12.1. Bialgebras.** An $R$-module $B$ that is an algebra $(B, \mu, \iota)$ and a coalgebra $(B, \Delta, \varepsilon)$ is called a *bialgebra* if $\Delta$ and $\varepsilon$ are algebra morphisms or, equivalently, $\mu$ and $\iota$ are coalgebra morphisms. For $\Delta$ to be an algebra morphism one needs commutativity of the diagrams

$$\begin{tikzcd}
B \otimes_R B \arrow[rr, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] & & B \arrow[dd, "\Delta"] \\
(B \otimes_R B) \otimes_R (B \otimes_R B) \arrow[d, "I_B\otimes\mathsf{tw}\otimes I_B"'] & & \\
(B \otimes_R B) \otimes_R (B \otimes_R B) \arrow[rr, "\mu\otimes\mu"] & & B \otimes_R B,
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\iota"] \arrow[d, "\simeq"'] & B \arrow[d, "\Delta"] \\
R \otimes_R R \arrow[r, "\iota\otimes\iota"] & B \otimes_R B,
\end{tikzcd}$$

where $\mathsf{tw}$ denotes the twist map. Similarly, $\varepsilon$ is an algebra morphism if and only if the following two diagrams

$$\begin{tikzcd}
B \otimes_R B \arrow[r, "\mu"] \arrow[d, "\varepsilon\otimes\varepsilon"'] & B \arrow[d, "\varepsilon"] \\
R \otimes_R R \arrow[r, "\simeq"] & R,
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[dr, "\varepsilon"] & \\
R \arrow[ur, "\iota"] \arrow[rr, "="] & & R
\end{tikzcd}$$

are commutative. The same set of diagrams makes $\mu$ and $\iota$ coalgebra morphisms. For the units $1_B \in B, 1_R \in R$ and for all $a, b \in B$, the above diagrams explicitly mean that

$$\begin{aligned}
\Delta(1_B) &= 1_B \otimes 1_B, \qquad & \varepsilon(1_B) &= 1_R, \\
\Delta(ab) &= \Delta(a)\Delta(b), \qquad & \varepsilon(ab) &= \varepsilon(a)\varepsilon(b).
\end{aligned}$$

Note that this implies that, in any $R$-bialgebra $B$, $R$ is a direct summand of $B$ as an $R$-module and hence $B$ is a generator in $\mathbf{M}_R$.

**12.2. Bialgebra morphisms.** An $R$-linear map $f : B \to B'$ of bialgebras is called a *bialgebra morphism* if $f$ is both an algebra and a coalgebra morphism.

An $R$-submodule $I \subset B$ is a *sub-bialgebra* if it is a subalgebra as well as a subcoalgebra. $I$ is a *bi-ideal* if it is both an ideal and a coideal.

*Let $f : B \to B'$ be a bialgebra morphism. Then:*

(1) *If $f$ is surjective, then* $\operatorname{Ke} f$ *is a bi-ideal in $B$.*

(2) $\operatorname{Im} f$ *is a subcoalgebra of $B'$.*

A remarkable feature of a bialgebra $B$ is that the tensor product of $B$-modules is again a $B$-module. First, recall that an $R$-module $N$ is a $B$-module if there is an algebra morphism $B \to \mathrm{End}_R(N)$.

**12.3. Tensor product of $B$-modules.** *Let $K, L$ be right modules over an $R$-bialgebra $B$.*

*(1) $K \otimes_R L$ has a right $B$-module structure by the map*

$$B \xrightarrow{\Delta} B \otimes_R B \to \mathrm{End}_R(K) \otimes_R \mathrm{End}_R(L) \to \mathrm{End}_R(K \otimes_R L)\,;$$

*we denote this module by $K \otimes_R^b L$. The right action of $B$ is given by*

$$\,!\, : K \otimes_R L \otimes_R B \longrightarrow K \otimes_R L, \quad k \otimes l \otimes b \mapsto (k \otimes l)\Delta b,$$

*where the product on the right side is taken componentwise, that is,*

$$(k \otimes l)\,!\,b := (k \otimes l)\,\Delta b = \sum kb_{\underline{1}} \otimes lb_{\underline{2}}\,.$$

*(2) For any morphisms $f : K \to K'$, $g : L \to L'$ in $\mathbf{M}_B$, the tensor product map $f \otimes g : K \otimes_R^b L \to K' \otimes_R^b L'$ is a morphism in $\mathbf{M}_B$.*

**Proof.** (1) follows easily from the definitions. Assertion (2) is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
K \otimes_R^b L \otimes_R B \arrow[r, "f\otimes g\otimes I_B"] \arrow[d, "!"'] & K' \otimes_R^b L' \otimes_R B \arrow[d, "!"] \\
K \otimes_R^b L \arrow[r, "f\otimes g"] & K' \otimes_R^b L',
\end{tikzcd}$$

which follows immediately from $B$-linearity of $f$ and $g$. □

Of course similar constructions apply to left $B$-modules $K, L$, in which case the left $B$-multiplication is given by

$$\,!\, : B \otimes_R K \otimes_R L \longrightarrow K \otimes_R L, \quad b \otimes k \otimes l \mapsto \Delta b(k \otimes l)\,.$$

Explicitly, the product comes out as $b\,!\,(k \otimes l) = \sum b_{\underline{1}}k \otimes b_{\underline{2}}l$.

Dually, the tensor product of comodules has a special comodule structure.

**12.4. Tensor product of $B$-comodules.** *Let $K, L$ be right comodules over an $R$-bialgebra $B$.*

*(1) $K \otimes_R L$ has a right $B$-comodule structure by the map (tensor over $R$)*

$$\varrho^{K\otimes L} : K \otimes L \xrightarrow{\mathsf{tw}_{23}\circ(\varrho^K\otimes\varrho^L)} K \otimes L \otimes B \otimes B \xrightarrow{I_K\otimes I_L\otimes\mu} K \otimes L \otimes B,$$

*where $\mathsf{tw}_{23} = I_K \otimes \mathsf{tw} \otimes I_B$. This comodule is denoted by $K \otimes_R^c L$. Thus, explicitly, for all $k \otimes l \in K \otimes_R^c L$,*

$$\varrho^{K\otimes_R L}(k \otimes l) = \sum k_{\underline{0}} \otimes l_{\underline{0}} \otimes k_{\underline{1}} l_{\underline{1}}.$$

(2) *For any morphisms $f : K \to K'$, $g : L \to L'$ in $\mathbf{M}^B$, the tensor product map $f \otimes g : K \otimes^c_R L \to K' \otimes^c_R L'$ is a morphism in $\mathbf{M}^B$.*

**Proof.** (1) This is proved by computing for all $k \in K$, $l \in L$,

$$\begin{aligned}(I_K \otimes I_L \otimes \Delta) \circ \varrho^{K \otimes_R L}(k \otimes l) &= \sum k_{\underline{0}} \otimes l_{\underline{0}} \otimes \Delta(k_{\underline{1}} l_{\underline{1}}) \\ &= \sum k_{\underline{0}} \otimes l_{\underline{0}} \otimes k_{\underline{11}} l_{\underline{11}} \otimes k_{\underline{12}} l_{\underline{12}} \\ &= \sum k_{\underline{00}} \otimes l_{\underline{00}} \otimes k_{\underline{01}} l_{\underline{01}} \otimes k_{\underline{11}} l_{\underline{11}} \\ &= (\varrho^{K \otimes_R L} \otimes I_B) \circ \varrho^{K \otimes_R L}(k \otimes l).\end{aligned}$$

To prove (2), take any $k \in K$, $l \in L$ and compute

$$\begin{aligned}\varrho^{K' \otimes_R L'} \circ (f \otimes g)(k \otimes l) &= \sum f(k)_{\underline{0}} \otimes g(l)_{\underline{0}} \otimes f(k)_{\underline{1}} g(l)_{\underline{1}} \\ &= \sum f(k_{\underline{0}}) \otimes g(l_{\underline{0}}) \otimes k_{\underline{1}} l_{\underline{1}} \\ &= (f \otimes g \otimes I_B) \circ \varrho^{K \otimes_R L}(k \otimes l).\end{aligned}$$

This shows that $f \otimes g$ is a comodule morphism, as required. $\square$

The coaction constructed in 12.4 is known as a *diagonal coaction* of a bialgebra $B$ on the tensor product of its comodules.

In contrast to coalgebras, for a bialgebra $B$, any $R$-module $K$ can be considered as $B$-comodule by $K \to K \otimes_R B$, $k \mapsto k \otimes 1_B$ (trivial coaction). In particular, the ring $R$ is a right $B$-comodule, and this draws attention to those maps $B \to R$ that are comodule morphisms.

**Definition.** An element $t \in B^*$ is called a *left integral on* $B$ if it is a left comodule morphism.

Recall that the rational part of $B^*$ is denoted by $\mathrm{Rat}^B(B^*) = T$ and $\varrho^T : T \to T \otimes_R B$ denotes the corresponding coaction.

**12.5. Left integrals on** $B$**.** *Let $B$ be an $R$-bialgebra and $t \in B^*$.*

(1) *The following are equivalent:*

(a) *$t$ is a left integral on $B$;*

(b) $(I_B \otimes t) \circ \Delta = \iota \circ t$.

*If $B$ is cogenerated by $R$ as an $R$-module, then (a) is equivalent to:*

(c) *For every $f \in B^*$, $f * t = f(1_B)t$.*

(2) *Assume that ${}_RB$ is locally projective.*

(i) *If $t \in T$, then $t$ is a left integral on $B$ if and only if $\varrho^T(t) = t \otimes 1_B$.*

(ii) *If $R$ is Noetherian or if $t(B) = R$, then any left integral $t$ on $B$ is rational, that is, $t \in T$.*

**Proof.** (1) (a) $\Leftrightarrow$ (b) The map $t$ is left colinear if and only if there is a commutative diagram

$$\begin{tikzcd} B \arrow[r, "t"] \arrow[d, "\Delta"'] & R \arrow[d, "\iota"] \\ B \otimes_R B \arrow[r, "I_B \otimes t"] & B \end{tikzcd} \qquad \begin{tikzcd} b \arrow[r, mapsto] \arrow[d, mapsto] & t(b) \arrow[d, mapsto] \\ \Delta(b) \arrow[r, mapsto] & (I_B \otimes t) \circ \Delta(b) = t(b)1_B . \end{tikzcd}$$

The commutativity of this diagram is expressed by condition (b).

(b) $\Leftrightarrow$ (c) For any $f \in B^*$ and $b \in B$,

$$\begin{aligned} f * t(b) &= (f \otimes t) \circ \Delta(b) &&= f((I_B \otimes t) \circ \Delta(b)), \\ f(1_B)t(b) &= f(t(b)1_B) &&= f(\iota \circ t(b)). \end{aligned}$$

From this (b) $\Rightarrow$ (c) is obvious. If $B$ is cogenerated by $R$, then (c) $\Rightarrow$ (b).

(2)(i) If $t \in T$, that is, $t$ is rational, then $f * t = (I_T \otimes f) \circ \varrho^T(t)$, for any $f \in B^*$, and (1)($c$) implies

$$(I_T \otimes f)(\varrho^T(t)) = (I_T \otimes f)(t \otimes 1_B).$$

By local projectivity ($\alpha$-condition; see 8.2) this means $\varrho^T(t) = t \otimes 1_B$. The converse conclusion is obvious.

(ii) By (1)(c), $B^* * t \subset R\,1_B$. If $R$ is Noetherian, this implies that $B^* * t$ is finitely presented as an $R$-module, and by 9.5(2) this implies that the element $t \in \mathrm{Rat}^B(B^*) = T$. If $t(B) = R$, then $t \rightharpoonup B = (I_B \otimes t)\Delta(B) = \iota \circ t(B) = R\,1_B$ is finitely presented as an $R$-module and $t \in T$ by 9.5(1). $\square$

Throughout $B$ will denotes an $R$-bialgebra with product $\mu$, coproduct $\Delta$, unit map $\iota$ and counit $\varepsilon$.

**12.6. $B$-Hopf modules.** An $R$-module $M$ is called a *right $B$-Hopf module* if $M$ is

(i) a right $B$-module with an action $\varrho_M : M \otimes_R B \to M$,

(ii) a right $B$-comodule with a coaction $\varrho^M : M \to M \otimes_R B$,

(iii) for all $m \in M, b \in B$, $\varrho^M(mb) = \varrho^M(m)\Delta(b)$.

Condition (iii) means that $\varrho^M : M \to M \otimes^b_R B$ is $B$-linear and is equivalent to the requirement that the multiplication $\varrho_M : M \otimes^c_R B \to M$ is $B$-colinear, or to the commutativity of either of the diagrams

$$\begin{tikzcd} M \otimes_R B \arrow[r, "\varrho^M \otimes I_B"] \arrow[d, "\varrho_M"'] & M \otimes^b_R B \otimes_R B \arrow[d, "!"] \\ M \arrow[r, "\varrho^M"] & M \otimes_R B\,, \end{tikzcd} \qquad \begin{tikzcd} M \otimes^c_R B \arrow[r, "\varrho^{M \otimes^c B}"] \arrow[d, "\varrho_M"'] & M \otimes^c_R B \otimes_R B \arrow[d, "\varrho_M \otimes I_B"] \\ M \arrow[r, "\varrho^M"] & M \otimes_R B\,. \end{tikzcd}$$

An $R$-linear map $f: M \to N$ between right $B$-Hopf modules is a *Hopf module morphism* if it is both a right $B$-module and a right $B$-comodule morphism. Denoting these maps by

$$\mathrm{Hom}_B^B(M,N) = \mathrm{Hom}_B(M,N) \cap \mathrm{Hom}^B(M,N),$$

there are characterising exact sequences in $\mathbf{M}_R$,

$$0 \to \mathrm{Hom}_B^B(M,N) \to \mathrm{Hom}_B(M,N) \xrightarrow{\gamma} \mathrm{Hom}_B(M, N \otimes_R^b B),$$

where $\gamma(f) = \varrho^N \circ f - (f \otimes I_B) \circ \varrho^M$ or, equivalently,

$$0 \to \mathrm{Hom}_B^B(M,N) \to \mathrm{Hom}^B(M,N) \xrightarrow{\delta} \mathrm{Hom}^B(M \otimes_R^c B, N),$$

where $\delta(g) = \varrho_N \circ (g \otimes I_B) - g \circ \varrho_M$.

*Left $B$-Hopf modules* and the corresponding morphisms are defined similarly, and it is obvious that $B$ is both a right and a left $B$-Hopf module.

**12.7. Trivial $B$-Hopf modules.** *Let $K$ be any $R$-module.*

*(1) $K \otimes_R B$ is a right $B$-Hopf module with the canonical structures*

$$I_K \otimes \Delta : K \otimes_R B \to (K \otimes_R B) \otimes_R B, \quad I_K \otimes \mu : (K \otimes_R B) \otimes_R B \to K \otimes_R B.$$

*(2) For any $R$-linear map $f: K \to K'$, the map $f \otimes I_B : K \otimes_R B \to K' \otimes_R B$ is a $B$-Hopf module morphism.*

**Proof.** We know that $K \otimes_R B$ is both a right $B$-module, and a comodule and the compatibility conditions are obvious from the properties of a bialgebra. It is clear that $f \otimes I_B$ is $B$-linear as well as $B$-colinear. □

**12.8. $B$-modules and $B$-Hopf modules.** *Let $N$ be any right $B$-module.*

*(1) The right $B$-module $N \otimes_R^b B$ is a right $B$-Hopf module with the canonical comodule structure*

$$I_N \otimes \Delta : N \otimes_R^b B \to (N \otimes_R^b B) \otimes_R B, \quad n \otimes b \mapsto n \otimes \Delta b.$$

*(2) For any $B$-linear map $f: N \to N'$, the map $f \otimes I_B : N \otimes_R^b B \to N' \otimes_R^b B$ is a $B$-Hopf module morphism.*

*(3) The map*

$$\gamma_N : N \otimes_R B \to N \otimes_R^b B, \quad n \otimes b \mapsto (n \otimes 1_B)\Delta(b) = (n \otimes 1_B)!b$$

*is a $B$-Hopf module morphism.*

**Proof.** (1) To show that $I_N \otimes \Delta$ is $B$-linear one needs to check the commutativity of the following diagram:

$$\begin{tikzcd}
N \otimes_R^b B \otimes_R B \arrow[r, "I_N\otimes\Delta\otimes I_B"] \arrow[d, "!"'] & (N \otimes_R^b B) \otimes_R^b B \otimes_R B \arrow[d, "!"] \\
N \otimes_R^b B \arrow[r, "I_N\otimes\Delta"] & (N \otimes_R^b B) \otimes_R^b B\,.
\end{tikzcd}$$

Evaluating this diagram at any $a, b \in B$ and $n \in N$ yields

$$\begin{aligned}
(I_N \otimes \Delta)\,((n \otimes b)\Delta(a)) &= \sum n a_{\underline{1}} \otimes (b a_{\underline{2}})_{\underline{1}} \otimes (b a_{\underline{2}})_{\underline{2}} \\
&= \sum n a_{\underline{1}} \otimes b_{\underline{1}} a_{\underline{2}} \otimes b_{\underline{2}} a_{\underline{3}} \\
&= ((I_N \otimes \Delta)(n \otimes b))\,\Delta(a),
\end{aligned}$$

by the multplicativity of $\Delta$ and the definition of the diagonal $B$-action on $(N \otimes_R^b B) \otimes_R^b B$ (cf. 12.3).

(2) It was shown in 7.8 that $f \otimes I_B$ is a comodule morphism, and we know from 12.3 that it is a $B$-module morphism.

(3) Clearly $\gamma_N$ is $B$-colinear, and for any $c \in B$,

$$\gamma_N(n \otimes bc) = (n \otimes 1_B)\Delta(bc) = (n \otimes 1_B)(\Delta b)(\Delta c) = \gamma_N(n \otimes b)\Delta(c),$$

showing that $\gamma_N$ is right $B$-linear. □

**12.9. $B$-comodules and $B$-Hopf modules.** *Let $L$ be a right $B$-comodule.*

*(1) The right $B$-comodule $L \otimes_R^c B$ is a right $B$-Hopf module with the canonical module structure*

$$I_L \otimes \mu : L \otimes_R^c B \otimes_R B \to L \otimes_R^c B, \quad n \otimes b \otimes a \mapsto n \otimes ba.$$

*(2) For any $B$-colinear map $f : L \to L'$, the map $f \otimes I_B : L \otimes_R^c B \to L' \otimes_R^c B$ is a $B$-Hopf module morphism.*

*(3) There is a $B$-Hopf module morphism*

$$\gamma^L : L \otimes_R^c B \to L \otimes_R B, \quad l \otimes b \mapsto \varrho^L(l)(1_B \otimes b).$$

**Proof.** (1) To prove the colinearity of $I_L \otimes \mu$ one needs to show the commutativity of the diagram

$$\begin{tikzcd}
L \otimes_R^c B \otimes_R^c B \arrow[r, "I_L\otimes\mu"] \arrow[d, "\varrho^{L\otimes^c B\otimes^c B}"'] & L \otimes_R^c B \arrow[d, "\varrho^{L\otimes^c B}"] \\
L \otimes_R^c B \otimes_R^c B \otimes_R B \arrow[r, "I_L\otimes\mu\otimes I_B"] & L \otimes_R^c B \otimes_R B\,,
\end{tikzcd}$$

which follows from the multiplicativity of $\Delta$.

(2) Clearly $f \otimes I_B$ is $B$-linear, and, as shown in 12.4, it is also $B$-colinear.

(3) Obviously $\gamma^L$ is right $B$-linear, and colinearity follows from the commutativity of the diagram (which is easily checked)

$$\begin{tikzcd}
L \otimes^c_R B \arrow[r, "\gamma^L"] \arrow[d, "\varrho^{L\otimes B}"'] & L \otimes_R B \arrow[d, "I_L \otimes \Delta"] \\
L \otimes^c_R B \otimes_R B \arrow[r, "\gamma^L \otimes I"] & L \otimes_R B \otimes_R B
\end{tikzcd}
\qquad
\begin{tikzcd}
l \otimes b \arrow[r, mapsto] \arrow[d, mapsto] & \sum l_{\underline{0}} \otimes l_{\underline{1}} b \arrow[d, mapsto] \\
\sum l_{\underline{0}} \otimes b_{\underline{1}} \otimes l_{\underline{1}} b_{\underline{2}} \arrow[r, mapsto] & \sum l_{\underline{0}} \otimes l_{\underline{1}} b_{\underline{1}} \otimes l_{\underline{2}} b_{\underline{2}}.
\end{tikzcd}$$

This completes the proof. □

Right $B$-Hopf modules together with $B$-Hopf module morphisms form a category that is denoted by $\mathbf{M}^B_B$.

**12.10. The category $\mathbf{M}^B_B$.** *Let $B$ be an $R$-bialgebra.*

*(1) The right $B$-Hopf module $B \otimes^b_R B$ is a subgenerator in $\mathbf{M}^B_B$.*

*(2) The right $B$-Hopf module $B \otimes^c_R B$ is a subgenerator in $\mathbf{M}^B_B$.*

*(3) For any $M \in \mathbf{M}^B_B$, $N \in \mathbf{M}_B$,*

$$\mathrm{Hom}^B_B(M, N \otimes^b_R B) \to \mathrm{Hom}_B(M, N),\ f \mapsto (I_N \otimes \varepsilon) \circ f,$$

*is an $R$-module isomorphism with inverse map $h \mapsto (h \otimes I_B) \circ \varrho^M$.*

*(4) For any $M \in \mathbf{M}^B_B$, $N \in \mathbf{M}^B$,*

$$\mathrm{Hom}^B_B(N \otimes^c_R B, M) \to \mathrm{Hom}^B(N, M),\ f \mapsto f(- \otimes 1_B),$$

*is an $R$-module isomorphism with inverse map $h \mapsto \varrho_M \circ (h \otimes I_B)$.*

*(5) For any $K, L \in \mathbf{M}_R$,*

$$\mathrm{Hom}^B_B(K \otimes_R B, L \otimes_R B) \to \mathrm{Hom}_R(K, L),\ f \mapsto (I_L \otimes \varepsilon) \circ f(- \otimes 1_B),$$

*is an $R$-module isomorphism with inverse map $h \mapsto h \otimes I_B$.*

**Proof.** (1) Let $M \in \mathbf{M}^B_B$. For a $B$-module epimorphism $f : B^{(\Lambda)} \to M$,

$$f \otimes I_B :\ B^{(\Lambda)} \otimes^b_R B \to M \otimes^b_R B$$

is an epimorphism in $\mathbf{M}^B_B$ (by 12.8), and so $M \otimes^b_R B$ is generated by

$$B^{(\Lambda)} \otimes^b_R B \simeq \left(B \otimes^b_R B\right)^{(\Lambda)}.$$

Moreover, $\varrho^M : M \to M \otimes^b_R B$ is a ($B$-splitting) Hopf module monomorphism, and so $M$ is subgenerated by $B \otimes^b_R B$.

(2) For any $M \in \mathbf{M}_B^B$, there is a comodule epimorphism $B^{(\Lambda)} \to M \otimes_R B$, and from this we obtain a Hopf module epimorphism

$$(B \otimes_R^c B)^{(\Lambda)} \simeq B^{(\Lambda)} \otimes_R^c B \to (M \otimes_R B) \otimes_R^c B.$$

Moreover, there is a Hopf module monomorphism $\varrho^M \otimes I_B : M \otimes_R^c B \to (M \otimes_R B) \otimes_R^c B$ and a Hopf module epimorphism $M \otimes_R^c B \to M$, and hence $M$ is subgenerated by $B \otimes_R^c B$.

(3) There is a commutative diagram with exact rows ($\otimes$ means $\otimes_R^b$),

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_B^B(M, N\otimes B) \arrow[r] \arrow[d, dashed] & \mathrm{Hom}^B(M, N\otimes B) \arrow[r, "\beta_1"] \arrow[d, "(I_N\otimes\varepsilon)\circ-"] & \mathrm{Hom}^B(M\otimes B, N\otimes B) \arrow[d, "(I_N\otimes\varepsilon)\circ-"] \\
0 \arrow[r] & \mathrm{Hom}_B(M, N) \arrow[r] & \mathrm{Hom}_R(M, N) \arrow[r, "\beta_2"] & \mathrm{Hom}_R(M\otimes B, N),
\end{tikzcd}$$

where $\beta_1(f) = f \circ \varrho_M - \varrho_{N\otimes B} \circ (f \otimes I_B)$ and $\beta_2(g) = g \circ \varrho_M - \varrho_N \circ (g \otimes I_B)$. As shown in 7.9, the second and third vertical maps are isomorphisms and hence the first one is also an isomorphism.

(4) Consider the commutative diagram with exact rows ($\otimes$ for $\otimes_R^c$),

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_B^B(N\otimes B, M) \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_B(N\otimes B, M) \arrow[r, "\gamma_1"] \arrow[d, "-\circ(-\otimes 1_B)"] & \mathrm{Hom}_B(N\otimes B, M\otimes B) \arrow[d, "-\circ(-\otimes 1_B)"] \\
0 \arrow[r] & \mathrm{Hom}^B(N, M) \arrow[r] & \mathrm{Hom}_R(N, M) \arrow[r, "\gamma_2"] & \mathrm{Hom}_R(N, M\otimes B),
\end{tikzcd}$$

where $\gamma_1(f) = \varrho^M \circ f - (f \otimes I_B) \circ \varrho^{N\otimes B}$ and $\gamma_2(g) = \varrho^M \circ g - (g \otimes I_B) \circ \varrho^N$. The second and third vertical maps are isomorphisms and hence the first one is an isomorphism, too.

(5) View $K$ as a trivial $B$-comodule. Then $K \otimes_R^c B \simeq K \otimes_R B$, and, by (4) and 7.9, $\mathrm{Hom}_B^B(K \otimes_R B, L \otimes_R B) \simeq \mathrm{Hom}^B(K, L \otimes_R B) \simeq \mathrm{Hom}_R(K, L)$, as required. $\square$

**12.11. $\mathbf{M}_B^B$ for $B_R$ flat.** *Let $B$ be flat as an $R$-module and $M, N \in \mathbf{M}_B^B$. Then:*

*(1) $\mathbf{M}_B^B$ is a Grothendieck category.*

*(2) The functor $\mathrm{Hom}_B^B(M, -) : \mathbf{M}_B^B \to \mathbf{M}_R$ is left exact.*

*(3) The functor $\mathrm{Hom}_B^B(-, N) : \mathbf{M}_B^B \to \mathbf{M}_R$ is left exact.*

**Proof.** (1) For any morphism $f : M \to N$ in $\mathbf{M}_B^B$, $\mathrm{Ke}\, f$ is a $B$-submodule as well as a $B$-subcomodule (since $B_R$ flat) and hence $\mathrm{Ke}\, f \in \mathbf{M}_B^B$.

(2) Any exact sequence $0 \to X \to Y \to Z$ in $\mathbf{M}_B^B$ induces the commutative diagram with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_B^B(M,X) \arrow[r] \arrow[d] & \mathrm{Hom}_B^B(M,Y) \arrow[r] \arrow[d] & \mathrm{Hom}_B^B(M,Z) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_B(M,X) \arrow[r] \arrow[d] & \mathrm{Hom}_B(M,Y) \arrow[r] \arrow[d] & \mathrm{Hom}_B(M,Z) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_B(M, X \otimes_R^b B) \arrow[r] & \mathrm{Hom}_B(M, Y \otimes_R^b B) \arrow[r] & \mathrm{Hom}_B(M, Z \otimes_R^b B).
\end{tikzcd}$$

The columns are simply the defining sequences of $\mathrm{Hom}_B^B(M,-)$ in 12.6. The second and third rows are exact because of the left exactness of $\mathrm{Hom}_B(M,-)$ and $-\otimes_R B$. Now the diagram lemmata imply that the first row is exact.

(3) This is shown with a similar diagram that uses $-\otimes_R^c B$ instead of $-\otimes_R^b B$. $\square$

**12.12. Coinvariants of comodules.** For $M \in \mathbf{M}^B$, the *coinvariants* of $B$ in $M$ are defined as

$$M^{coB} := \left\{ m \in M \mid \varrho^M(m) = m \otimes 1_B \right\} = \mathrm{Ke}\,(\varrho^M - (- \otimes 1_B)).$$

This is clearly an $R$-submodule of $M$ and there is an isomorphism

$$\mathrm{Hom}^B(R, M) \to M^{coB}, \quad f \mapsto f(1),$$

where $R$ is considered as a $B$-comodule. In particular, this implies that $B^{coB} = R1_B$. Furthermore, for any $R$-module $K$,

$$\mathrm{Hom}^B(K, M) \simeq \mathrm{Hom}_R(K, M^{coB}),$$

where $K$ is considered as a trivial $B$-comodule.

The last isomorphism follows by the fact that $f \in \mathrm{Hom}^B(K, M)$ is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "-\otimes 1_B"'] & M \arrow[d, "\varrho^M"] \\
K \otimes_R B \arrow[r, "f \otimes I_B"] & M \otimes_R B
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, mapsto] \arrow[d, mapsto] & f(k) \arrow[d, mapsto] \\
k \otimes 1_B \arrow[r, mapsto] & f(k) \otimes 1_B = \varrho^M(f(k))\,.
\end{tikzcd}$$

**12.13. Coinvariants of Hopf modules.** *For any $M \in \mathbf{M}_B^B$, the map*

$$\nu_M : \operatorname{Hom}_B^B(B, M) \to M^{coB}, \ f \mapsto f(1_B),$$

*is an $R$-module isomorphism with the inverse $\omega_M : m \mapsto [b \mapsto mb]$. Furthermore, the diagram*

$$\begin{array}{ccc} \operatorname{Hom}_B^B(B,M) \otimes_R B & \longrightarrow & M \\ {\scriptstyle \nu_M \otimes I_B}\downarrow & & \| \\ M^{coB} \otimes_R B & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} f \otimes b & \longmapsto & f(b) \\ \downarrow & & \| \\ f(1_B) \otimes b & \longrightarrow & f(1_B)b \end{array}$$

*is commutative. In particular, $\operatorname{Hom}_B^B(B,B) \xrightarrow{\simeq} B^{coB} = R\,1_B$ is a ring isomorphism.*

**Proof.** The isomorphism $\nu_M$ is obtained from the following commutative diagram of $R$-module maps with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}_B^B(B,M) & \longrightarrow & \operatorname{Hom}_B(B,M) & \xrightarrow{\gamma_1} & \operatorname{Hom}_B(B, M \otimes_R B) \\ & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & M^{coB} & \longrightarrow & M & \xrightarrow{\gamma_2} & M \otimes_R B, \end{array}$$

where $\gamma_1(f) = \varrho^M \circ f - (f \otimes I_B) \circ \Delta$, that is, the top row is the defining sequence of $\operatorname{Hom}_B^B(B,M)$, and $\gamma_2(m) = \varrho^M(m) - m \otimes 1_B$. $\square$

**12.14. Coinvariants of trivial Hopf modules.**

*(1) For any $K \in \mathbf{M}_R$, $\operatorname{Hom}_B^B(B, K \otimes_R B) \simeq K$ as $R$-modules.*

*(2) For all $L \in \mathbf{M}_R$ and $M \in \mathbf{M}_B^B$, there are $R$-module isomorphisms*

$$\operatorname{Hom}_B^B(L \otimes_R B, M) \simeq \operatorname{Hom}_R(L, M^{coB}) \ \text{ and } \ \operatorname{End}_B^B(B \otimes_R B) \simeq \operatorname{End}_R(B).$$

*(3) There is an adjoint pair of functors*

$$- \otimes_R B : \mathbf{M}_R \to \mathbf{M}_B^B, \quad \operatorname{Hom}_B^B(B, -) : \mathbf{M}_B^B \to \mathbf{M}_R,$$

*and $\operatorname{Hom}_B^B(B, - \otimes_R B) \simeq I_{\mathbf{M}_R}$.*

**Proof.** (1) Consider $R$ as a $B$-comodule as in 12.10(4). Then the Hom-tensor relation 7.9(1) implies

$$\operatorname{Hom}_B^B(B, K \otimes_R B) \simeq \operatorname{Hom}_B^B(R \otimes_R^c B, K \otimes_R B) \simeq \operatorname{Hom}^B(R, K \otimes_R B) \simeq K.$$

(2) Combining 12.10(4) and 12.12, one obtains the chain of isomorphisms

$$\operatorname{Hom}_B^B(L \otimes_R B, M) \simeq \operatorname{Hom}^B(L, M) \simeq \operatorname{Hom}_R(L, M^{coB}).$$

(3) By 12.13, the adjointness is just an interpretation of the isomorphism in (2), and, by (1), the composition of the two functors is isomorphic to the identity functor on $\mathbf{M}_R$. $\square$

**12.15. Coinvariants and $B$-modules.** *For any $N \in \mathbf{M}_B$, the map*

$$\nu'_{N\otimes B} : \mathrm{Hom}_B^B(B, N \otimes_R^b B) \to N,\ \ f \mapsto (I_N \otimes \varepsilon) \circ f(1_B)\,,$$

*is an $R$-isomorphism with the inverse $n \mapsto [b \mapsto \sum n b_{\underline{1}} \otimes b_{\underline{2}}]$. Furthermore, the diagram*

$$\begin{tikzcd}
\mathrm{Hom}_B^B(B, N \otimes_R^b B) \otimes_R B \arrow[r] \arrow[d, "\nu'_{N\otimes B}\otimes I_B"'] & N \otimes_R^b B \arrow[d, equal] & g \otimes b \arrow[r, mapsto] \arrow[d] & g(b) \arrow[d, equal] \\
N \otimes_R B \arrow[r, "\gamma_N"] & N \otimes_R^b B & (I_N \otimes \varepsilon) g(1_B) \otimes b \arrow[r, mapsto] & g(1_B)\Delta b\,,
\end{tikzcd}$$

*where $\gamma_N$ is described in 12.8(3), is commutative. This yields in particular*

$$(B \otimes_R^b B)^{coB} \simeq \mathrm{Hom}_B^B(B, B \otimes_R^b B) \simeq B\,,$$

*and the commutative diagram*

$$\begin{tikzcd}
\mathrm{Hom}_B^B(B, B \otimes_R^b B) \otimes_R B \arrow[r] \arrow[d] & B \otimes_R^b B \arrow[d, equal] & h \otimes a \arrow[r, mapsto] \arrow[d] & h(a) \arrow[d, equal] \\
B \otimes_R B \arrow[r, "\gamma_B"] & B \otimes_R^b B & (I_B \otimes \varepsilon) h(1_B) \otimes a \arrow[r, mapsto] & h(1_B)\Delta a.
\end{tikzcd}$$

**Proof.** By 12.10, $\mathrm{Hom}_B^B(B, N \otimes_R^b B) \simeq \mathrm{Hom}_B(B, N) \simeq N$ and commutativity of the diagrams is shown by a straightforward computation. $\square$

**12.16. Invariants.** Let $A$ be an $R$-algebra $A$ and $\varphi : A \to R$ a ring morphism. Considering $R$ as a left $A$-module, one may ask for the $A$-morphisms from $R \to M$, where $M \in {}_A\mathbf{M}$. Define the *invariants* of $M$ by

$${}^A M = \{m \in M \mid am = \varphi(a)m \ \text{ for all } a \in A\}.$$

Then the map ${}_A\mathrm{Hom}(R, M) \to {}^A M$, $f \mapsto f(1)$, is an $R$-module isomorphism.

**12.17. Invariants for bialgebras.** For any bialgebra $B$, the counit $\varepsilon$ is a ring morphism and hence induces a left and right $B$-module structure on $R$. Therefore, for any left $B$-module $M$, the *invariants of* $M$ corresponding to $\varepsilon$ come out as

$${}^B M = \{m \in M \mid bm = \varepsilon(b)m \ \text{ for all } b \in B\}.$$

Furthermore, the map ${}_B\mathrm{Hom}(R, M) \to {}^B M$, $f \mapsto f(1)$, is an $R$-module isomorphism. The left invariants ${}^B B$ of $B$ are called *left integrals in* $B$,

$${}_B\mathrm{Hom}\,(R, B) \simeq {}^B B = \{c \in B \mid bc = \varepsilon(b)c \ \text{ for all } b \in B\}.$$

*Right invariants* and *right integrals in* $B$ are defined symmetrically .

On the other hand, for the dual algebra $B^*$, the map $\varphi : B^* \to R$, $f \mapsto f(1_B)$, is a ring morphism. Coinvariants of right $B$-comodules are closely related to invariants of left $B^*$-modules corresponding to $\varphi$.

**12.18. Invariants and coinvariants.** *Let $B$ be a bialgebra that is locally projective as an $R$-module (cf. 8.2).*

(1) *For any $M \in \mathbf{M}^B$, ${}^{B^*}M = M^{coB}$.*

(2) *For the trace ideal $T = \mathrm{Rat}^B(B^*)$, ${}^{B^*}T = T^{coB}$.*

(3) *If $B_R$ is finitely generated, then ${}^{B^*}B^* = (B^*)^{coB}$.*

**Proof.** (1) Let $m \in {}^{B^*}M$ and $f \in B^*$. From $f \rightharpoonup m = \sum m_{\underline{0}} f(m_{\underline{1}})$ we conclude

$$(I_M \otimes f)\varrho^M(m) = (I_M \otimes f)(m \otimes 1_B).$$

Now local projectivity of $B$ implies that $\varrho_M(m) = m \otimes 1_B$, that is, $m \in M^{coB}$, as required. Conversely, take any $m \in M^{coB}$ and compute

$$f \rightharpoonup m = (I_M \otimes f)\varrho^M(m) = mf(1_B) = m\varphi(f).$$

This shows that $m \in {}^{B^*}M$, and therefore ${}^{B^*}M = M^{coB}$.

(2) From the definition of the trace ideal we know that $T \in \mathbf{M}^B$; hence the assertion follows from (1).

(3) If $B_R$ is finitely generated and projective, then $T = B^*$ and the assertion follows from (2). □

Every Hopf module $M \in \mathbf{M}_B^B$ is a right $B$-comodule, and hence it is a left $B^*$-module (in the canonical way). This yields an action of $B^{op} \otimes_R B^*$ on $M$,

$$B^{op} \otimes_R B^* \otimes_R M \to M, \ (a \otimes f) \otimes m \mapsto (a \otimes f)\varrho^M(m) = \sum m_{\underline{0}} a f(m_{\underline{1}}).$$

This action is obviously an $R$-linear map, but it does not make $M$ a module with respect to the canonical algebra product in $B^{op} \otimes_R B^*$. On the other hand, there exists a different multiplication on $B^{op} \otimes_R B^*$ that makes $M$ a module over the new algebra. Denote this product by "?". For all $a \in B$, $f, g \in B^*$, and $m \in M$, a product ? has to satisfy the associative law

$$\begin{aligned}
[(a \otimes f)?(b \otimes g)](m) = (a \otimes f)((b \otimes g)m) &= \textstyle\sum (a \otimes f)(m_{\underline{0}} b g(m_{\underline{1}})) \\
&= \textstyle\sum m_{\underline{0}} b_{\underline{1}} a\, f(m_{\underline{1}} b_{\underline{2}}) g(m_{\underline{2}}) \\
&= \textstyle\sum m_{\underline{0}} b_{\underline{1}} a\, (b_{\underline{2}} \rightharpoondown f) * g(m_{\underline{1}}) \\
&= [\textstyle\sum b_{\underline{1}} a \otimes (b_{\underline{2}} \rightharpoondown f) * g](m).
\end{aligned}$$

From this we can see how the multiplication ? on $B^{op} \otimes_R B^*$ should be constructed in order to possess the desired properties.

**12.1. Smash product $B^{op}\#B^*$.** *Consider an algebra $B^{op}\#B^*$, which is isomorphic to the tensor product $B^{op} \otimes_R B^*$ as an $R$-module and has the product*

$$(a\#f)(b\#g) := ((\Delta b)(a\#f))\,(1_B\#g) = \sum b_{\underline{1}} a\#(b_{\underline{2}} \rightharpoondown f) * g,$$

*where $a\#f = a \otimes f$ is the notation. Then $B^{op}\#B^*$ is an associative $R$-algebra with unit $1_B\#\varepsilon$, and the maps*

$$\begin{array}{rcl} B^{op} & \to & B^{op}\#B^*, \quad a \mapsto a\#\varepsilon, \\ B^* & \to & B^{op}\#B^*, \quad f \mapsto 1_B\#f, \end{array}$$

*are injective ring morphisms, making every left $B\#B^*$-module a right $B$-module and a left $B^*$-module. The algebra $B^{op}\#B^*$ is called a* smash product.

*Every $M \in \mathbf{M}_B^B$ is a left $B^{op}\#B^*$-module, and therefore $\mathbf{M}_B^B$ is embedded in $\sigma_{B^{op}\#B^*}[B \otimes_R^b B] \subset {}_{B^{op}\#B^*}\mathbf{M}$. If $B_R$ is locally projective, then*

$$\mathbf{M}_B^B = \sigma_{B\#B^*}[B \otimes_R^b B] = \sigma_{B\#B^*}[B \otimes_R^c B].$$

*In particular, $\mathbf{M}_B^B = {}_{B^{op}\#B^*}\mathbf{M}$ provided that $B_R$ is finitely generated and projective.*

**Proof.** The first assertions are immediate consequences of the action considered above and the definition of the product $\#$. The local projectivity implies that the right $B$-comodule structures correspond to left $B^*$-module structures.

If $B_R$ is finitely generated and projective, then there is a right coaction (see 7.10) $B^* \to \mathrm{End}_R(B) \simeq B^* \otimes_R B,\ g \mapsto (I_{B^*} \otimes g) \circ \Delta$. The map

$$B^* \otimes_R^c B \to B^{op}\#B^*, \quad f \otimes b \mapsto b\#f,$$

is an isomorphism of left $B^{op}\#B^*$-modules. Indeed, note that, for any $b, x \in B$ and $f, g \in B^*$,

$$\sum (b \rightharpoondown f)(g_{\underline{0}}(x)g_{\underline{1}}) = (b \rightharpoondown f)(I_B \otimes g)\Delta(x) = (b \rightharpoondown f) * g(x).$$

Using these identities we compute

$$\begin{array}{rcl} (a\#f)(g \otimes b) & = & (a \otimes f)\varrho^{B^* \otimes B}(g \otimes b) = \sum g_{\underline{0}} \otimes b_{\underline{1}} a f(g_{\underline{1}} b_{\underline{2}}) \\ & \mapsto & \sum b_{\underline{1}} a\#(b_{\underline{2}} \rightharpoondown f)(g_{\underline{1}})g_{\underline{0}} = \sum b_{\underline{1}} a\#(b_{\underline{2}} \rightharpoondown f) * g \\ & = & (a\#f)(b\#g), \end{array}$$

that is, the map defined above is a morphism of left $B^{op}\#B^*$-modules. Clearly it is an isomorphism. Therefore, $B^{op}\#B^* \in \mathbf{M}_B^B$ and hence $\mathbf{M}_B^B = {}_{B^{op}\#B^*}\mathbf{M}$. $\square$

# 13 Antipodes and Hopf algebras

**13.1. The ring** $(\mathrm{End}_R(B), *)$**.** For any $R$-bialgebra $B$, $(\mathrm{End}_R(B), *)$ is an associative $R$-algebra with product, for $f, g \in \mathrm{End}_R(B)$,

$$f * g = \mu \circ (f \otimes g) \circ \Delta,$$

and unit $\iota \circ \varepsilon$, that is, $\iota \circ \varepsilon(b) = \varepsilon(b)1_B$, for any $b \in B$ (cf. 5.3). If $B$ is commutative and cocommutative, then $(\mathrm{End}_R(B), *)$ is a commutative algebra.

**Definitions.** An element $S \in \mathrm{End}_R(B)$ is called a *left (right) antipode* if it is left (right) inverse to $I_B$ with respect to the convolution product on $\mathrm{End}_R(B)$, that is, $S * I_B = \iota \circ \varepsilon$ (resp. $I_B * S = \iota \circ \varepsilon$). In case $S$ is a left and right antipode, it is called an *antipode*. The corresponding conditions are

$$\mu \circ (S \otimes I_B) \circ \Delta = \iota \circ \varepsilon, \quad \mu \circ (I_B \otimes S) \circ \Delta = \iota \circ \varepsilon.$$

Explicitly, for all $b \in B$, an antipode $S$ satisfies the following equalities:

$$\sum S\left(b_{\underline{1}}\right) b_{\underline{2}} = \varepsilon(b)1_B = \sum b_{\underline{1}} S\left(b_{\underline{2}}\right) .$$

Left and right antipodes need not be unique, whereas an antipode is unique whenever it exists. A bialgebra with an antipode is called a *Hopf algebra.*

Antipodes are related to the right Hopf module morphism

$$\gamma_B : B \otimes_R B \to B \otimes_R^b B,\; a \otimes b \mapsto (a \otimes 1_B)\Delta b = \sum a b_{\underline{1}} \otimes b_{\underline{2}}.$$

Notice that $\gamma_B$ is also a left $B$-module morphism in an obvious way.

**13.2. Existence of antipodes.** *Let $B$ be an $R$-bialgebra.*

*(1) $B$ has a right antipode if and only if $\gamma_B$ has a left inverse in ${}_B\mathbf{M}$.*

*(2) If $B$ has a left antipode, then $\gamma_B$ has a right inverse in ${}_B\mathbf{M}$.*

*(3) $\gamma_B$ is an isomorphism if and only if $B$ has an antipode.*

**Proof.** (1) If $\beta$ is a left inverse of $\gamma_B$, for all $b \in B$, $1_B \otimes b = \beta \circ \gamma_B(1_B \otimes b) = \beta(\Delta b)$ holds. This implies that $\iota \circ \varepsilon(b) = (I_B \otimes \varepsilon) \circ \beta(\Delta b)$. Then $S = (I_B \otimes \varepsilon) \circ \beta(1_B \otimes -) : B \to B$ is a right antipode since

$$\mu \circ (I_B \otimes S) \circ \Delta(b) = \sum b_{\underline{1}}((I_B \otimes \varepsilon)\beta(1_B \otimes b_{\underline{2}})) = (I_B \otimes \varepsilon) \circ \beta(\Delta b) = \iota \circ \varepsilon(b),$$

where we used that $\beta$ is left $B$-linear.

Now suppose that $S : B \to B$ is a right antipode. Then

$$\beta : B \otimes_R^b B \to B \otimes_R B, \quad a \otimes b \mapsto (a \otimes 1_B)(S \otimes I_B)(\Delta b) = \sum a S(b_{\underline{1}}) \otimes b_{\underline{2}},$$

is a left inverse of $\gamma_B$, since for any $b \in B$,

$$\begin{aligned}\beta \circ \gamma_B(1_B \otimes b) = \beta(\Delta b) &= (\mu \otimes I_B) \circ (I_B \otimes S \otimes I_B) \circ (I_B \otimes \Delta)(\Delta b) \\ &= (\mu \otimes I_B) \circ (I_B \otimes S \otimes I_B) \circ (\Delta \otimes I_B)(\Delta b) \\ &= \sum \mu \circ (S \otimes I_B)(\Delta b_{\underline{1}}) \otimes b_{\underline{2}} \\ &= \sum \varepsilon(b_{\underline{1}}) 1_B \otimes b_{\underline{2}} \;=\; 1_B \otimes b\,.\end{aligned}$$

(2) Let $S$ be a left antipode, that is, $\mu \circ (S \otimes I_B)(\Delta b) = \iota \circ \varepsilon(b)$, for $b \in B$. Then

$$\beta : B \otimes^b_R B \to B \otimes_R B, \quad 1_B \otimes b \mapsto (S \otimes I_B)(\Delta b) = \sum S(b_{\underline{1}}) \otimes b_{\underline{2}},$$

is a right inverse of $\gamma_B$, since

$$\begin{aligned}\gamma_B \circ \beta(1_B \otimes b) = \gamma_B((S \otimes I_B)(\Delta b)) &= \sum S(b_{\underline{1}}) b_{\underline{2}} \otimes b_{\underline{3}} \\ &= \sum \varepsilon(b_{\underline{1}}) 1_B \otimes b_{\underline{2}} \;=\; 1_B \otimes b\,.\end{aligned}$$

(3) Suppose that $\gamma_B$ is bijective. Take any $f \in \mathrm{End}_R(B)$ and observe that if $\mu \circ (I_B \otimes f)(\Delta b) = 0$, for all $b \in B$, then $f = 0$. Indeed, any element in $B \otimes_R B$ can be written as a sum of elements of the form $(a \otimes 1_B)(\Delta b)$ and

$$\mu \circ (I_B \otimes f)((a \otimes 1_B)(\Delta b)) = a(\mu((I_B \otimes f)(\Delta b))) = 0,$$

implying $\mu(I_B \otimes f)(B \otimes_R B) = Bf(B) = 0$, and so $f = 0$, as claimed.

By (1), there exists a right antipode $S$, and for this we compute

$$\begin{aligned}&\mu \circ (I_B \otimes \mu \circ (S \otimes I_B) \circ \Delta)(\Delta b) \\ &\quad = \mu \circ (I_B \otimes \mu) \circ (I_B \otimes S \otimes I_B) \circ (I_B \otimes \Delta)(\Delta b) \\ &\quad = \mu \circ (\mu \otimes I_B) \circ (I_B \otimes S \otimes I_B) \circ (\Delta \otimes I_B)(\Delta b) \\ &\quad = \sum \varepsilon(b_{\underline{1}}) b_{\underline{2}} \;=\; b \;=\; \mu \circ (I_B \otimes \iota \circ \varepsilon)(\Delta b).\end{aligned}$$

By the preceding observation this implies $\mu \circ (S \otimes I_B) \circ \Delta = \iota \circ \varepsilon$, thus showing that $S$ is also a left antipode. $\square$

**13.3. Properties of antipodes.** *Let $H$ be a Hopf algebra with antipode $S$. Then:*

*(1) $S$ is an algebra anti-morphism, that is, for all $a, b \in H$, $S(ab) = S(b)S(a)$, and $S \circ \iota = \iota$.*

*(2) $S$ is a coalgebra anti-morphism, that is, $\mathsf{tw} \circ (S \otimes S) \circ \Delta = \Delta \circ S$ and $\varepsilon \circ S = \varepsilon$.*

*(3) If $S$ is invertible as a map, then, for any $b \in H$,*

$$\sum S^{-1}(b_{\underline{2}}) b_{\underline{1}} = \varepsilon(b) 1_H = \sum b_{\underline{2}} S^{-1}(b_{\underline{1}})\,.$$

**Proof.** (1) Consider the convolution algebra $\widetilde{H} := (\mathrm{Hom}_R(H \otimes_R H, H), \tilde{*})$ corresponding to the canonical coalgebra structure $\Delta_{H\otimes_R H}$ on $H \otimes_R H$ with the counit $\tilde{\varepsilon} = \varepsilon \otimes \varepsilon$. In particular, the unit in $\widetilde{H}$ comes out as

$$\tilde{\iota}: \; H \otimes_R H \xrightarrow{\varepsilon\otimes\varepsilon} R \xrightarrow{\iota} H\,.$$

In addition to the product $\mu : \; H \otimes_R H \to H$, consider the $R$-linear maps

$$\nu : H \otimes_R H \to H, \; a \otimes b \mapsto S(b)S(a), \quad \rho : H \otimes_R H \to H, \; a \otimes b \mapsto S(ab)\,.$$

To prove that $S$ is an anti-multiplicative map, it is sufficient to show that $\rho \tilde{*} \mu = \mu \tilde{*} \nu = \tilde{\iota} \circ \tilde{\varepsilon}$ (the identity in $\widetilde{H}$). By the uniqueness of inverse elements we are then able to conclude that $\nu = \rho$. Consider the $R$-linear maps

$$H \otimes_R H \xrightarrow{\Delta_{H\otimes H}} H \otimes_R H \otimes_R H \otimes_R H \underset{\mu\otimes\nu}{\overset{\rho\otimes\mu}{\rightrightarrows}} H \otimes_R H \xrightarrow{\mu} H\,.$$

Take any $a, b \in H$ and compute

$$\begin{aligned} a \otimes b \mapsto & \textstyle\sum a_{\underline{1}} \otimes b_{\underline{1}} \otimes a_{\underline{2}} \otimes b_{\underline{2}} \\ & \overset{\rho\otimes\mu}{\longmapsto} \textstyle\sum S(a_{\underline{1}}b_{\underline{1}})a_{\underline{2}}b_{\underline{2}} = S * I_H(ab) = \varepsilon(ab)1_H\,, \\ & \overset{\mu\otimes\nu}{\longmapsto} \textstyle\sum a_{\underline{1}}b_{\underline{1}}S\,(b_{\underline{2}})\,S\,(a_{\underline{2}}) \\ & \quad = \textstyle\sum a_{\underline{1}}S\,(a_{\underline{2}})\,\varepsilon(b) = \varepsilon(a)\varepsilon(b)1_H\,. \end{aligned}$$

Thus $\nu = \rho$, and $S$ is an anti-multiplicative map, that is, $S(ab) = S(a)S(b)$. Furthermore, $1_H = \iota \circ \varepsilon(1_H) = (I_H * S)(1_H) = S(1_H)$, so that $S$ is a unital map and hence an algebra anti-morphism.

(2) This is a dual statement to (1), and we use a similar technique as for the proof of (1). In this case consider the convolution algebra corresponding to $H$ as a coalgebra and $H \otimes_R H$ as an algebra, $(\mathrm{Hom}_R(H, H \otimes_R H), \underline{*})$. Let $\nu := \mathsf{tw} \circ (S \otimes S) \circ \Delta$ and $\rho := \Delta \circ S$. Direct computation verifies that $\rho \,\underline{*}\, \Delta = \iota_H \circ \varepsilon_{H\otimes H} = \Delta \,\underline{*}\, \nu$. From this we conclude that $\rho = \nu$, so that $S$ is an anti-comultiplicative map. Furthermore, for all $a \in H$, we know that $\varepsilon(\iota \circ \varepsilon(a)) = \varepsilon(a)$, and $\iota \circ \varepsilon(a) = \sum S(a_{\underline{1}})a_{\underline{2}}$. This implies

$$\varepsilon(a) = \varepsilon(\iota \circ \varepsilon(a)) = \sum \varepsilon\,(S(a_{\underline{1}}))\,\varepsilon(a_{\underline{2}}) = \varepsilon \circ S(a)\,,$$

hence $S$ is a coalgebra anti-morphism, as stated.

(3) Apply $S^{-1}$ to the defining properties of $S$. □

We now prove that Hopf algebras are precisely those $R$-bialgebras for which the category $\mathbf{M}_B^B$ is equivalent to $\mathbf{M}_R$. It is interesting to notice that this can be seen from a single isomorphism.

**13.4. Fundamental Theorem of Hopf algebras.** *For any $R$-bialgebra $B$ the following are equivalent:*

(a) *$B$ is a Hopf algebra (that is, $B$ has an antipode);*

(b) *$\gamma_B : B \otimes_R B \to B \otimes_R^b B,\ a \otimes b \mapsto (a \otimes 1_B)\Delta b$, is an isomorphism in $\mathbf{M}_B^B$;*

(c) *$\gamma^B : B \otimes_R^c B \to B \otimes_R B,\ a \otimes b \mapsto \Delta a(1_B \otimes b)$, is an isomorphism in $\mathbf{M}_B^B$;*

(d) *for every $M \in \mathbf{M}_B^B$, $M^{coB} \otimes_R B \to M,\ m \otimes b \mapsto mb$, is an isomorphism in $\mathbf{M}_B^B$;*

(e) *for every $M \in \mathbf{M}_B^B$, there is an isomorphism (in $\mathbf{M}_B^B$)*

$$\varphi_M :\ \mathrm{Hom}_B^B(B, M) \otimes_R B \to M, \quad f \otimes b \mapsto f(b);$$

(f) *$\varphi_{B\otimes B} : \mathrm{Hom}_B^B(B, B \otimes_R^b B) \otimes_R B \to B \otimes_R^b B$ is an isomorphism in $\mathbf{M}_B^B$;*

(g) *$\mathrm{Hom}_B^B(B, -) : \mathbf{M}_B^B \to \mathbf{M}_R$ is an equivalence (with inverse $- \otimes_R B$).*

*If $B$ is flat as an $R$-module, then (a)-(g) are equivalent to:*

(h) *$B$ is a (projective) generator in $\mathbf{M}_B^B$;*

(i) *$B$ is a subgenerator in $\mathbf{M}_B^B$, and $\varphi_M$ is injective for every $M \in \mathbf{M}_B^B$.*

*If $B_R$ is locally projective, then $(a) - (i)$ are equivalent to:*

(j) *$B$ is a subgenerator in $\mathbf{M}_B^B$ and the image of $B^{op}\# B^* \to \mathrm{End}_R(B)$ is dense (for the finite topology).*

*For any Hopf module $M$ over a Hopf algebra $B$, the coinvariants $M^{coB}$ are an $R$-direct summand of $M$.*

**Proof.** (a) $\Leftrightarrow$ (b) was shown in 13.2, and by symmetry (see 12.9) the same proof implies (a)$\Leftrightarrow$(c). (b) $\Leftrightarrow$ (f) is clear by 12.15.

(d) $\Leftrightarrow$ (e) This follows from the commutative diagram in 12.13.

(a) $\Rightarrow$ (d) For any $M \in \mathbf{M}_B^B$, consider $\phi : M \to M^{coB}, m \mapsto \sum m_{\underline{0}} S(m_{\underline{1}})$. The following equalities show that the image of $\phi$ is in $M^{coB}$:

$$\begin{aligned}\varrho^M(\phi(m)) = \varrho^M(\textstyle\sum m_{\underline{0}} S(m_{\underline{1}})) &= \textstyle\sum m_{\underline{0}} S(m_{\underline{3}}) \otimes m_{\underline{1}} S(m_{\underline{2}}) \\ &= \textstyle\sum m_{\underline{0}} S(m_{\underline{1}}) \otimes 1_B \ = \ \phi(m) \otimes 1_B.\end{aligned}$$

Now we show that the map

$$(\phi \otimes I_B) \circ \varrho^M : M \to M^{coB} \otimes_R B$$

is the inverse of the multiplication map $\varrho_M : M^{coB} \otimes_R B \to M$. For $m \in M$,

$$\varrho_M \circ (\phi \otimes I_B)(\varrho^M(m)) = \textstyle\sum \phi(m_{\underline{0}}) m_{\underline{1}} = \sum m_{\underline{0}} S(m_{\underline{1}}) m_{\underline{2}} = \sum m_{\underline{0}} \varepsilon(m_{\underline{1}}) = m.$$

On the other hand, for $n \otimes b \in M^{coB} \otimes_R B$,

$$\begin{aligned}(\phi \otimes I_B) \circ \varrho^M(nb) &= (\phi \otimes I_B)(\sum nb_{\underline{1}} \otimes b_{\underline{2}}) \;=\; \sum \phi(nb_{\underline{1}}) \otimes b_{\underline{2}} \\ &= \sum nb_{\underline{1}}\, S(b_{\underline{2}}) \otimes b_{\underline{3}} \;=\; \sum n\varepsilon(b_{\underline{1}}) \otimes b_{\underline{2}} \;=\; n \otimes b.\end{aligned}$$

(e) ⇒ (f) is trivial (take $M = B \otimes^b_R B$).

(e) ⇔ (g) From 12.14 we know $\mathrm{Hom}^B_B(B, - \otimes_R B) \simeq I_{{}_R\mathbf{M}}$. Condition (f) induces $\mathrm{Hom}^B_B(B, -) \otimes_R B \simeq I_{\mathbf{M}^B_B}$, and the two isomorphisms characterise an equivalence between $\mathbf{M}_R$ and $\mathbf{M}^B_B$.

(g) ⇒ (h) Obviously (g) always implies that $B$ is a generator in $\mathbf{M}^B_B$ and that $B$ is projective in $\mathbf{M}^B_B$ (that is, $\mathrm{Hom}^B_B(B, -) : \mathbf{M}^B_B \to \mathbf{M}_R$ preserves epimorphisms).

Now suppose that ${}_RB$ is flat. Then $\mathbf{M}^B_B$ has kernels and $\mathrm{Hom}^B_B(B, -)$ is a left exact functor.

(h) ⇒ (i) Suppose that $B$ is a generator in $\mathbf{M}^B_B$. Of course any generator is in particular a subgenerator. For any $M \in \mathbf{M}^B_B$, the set $\Lambda = \mathrm{Hom}_B(B, M)$ yields a canonical epimorphism

$$p : B^{(\Lambda)} \to M, \quad b_f \mapsto f(b).$$

Choosing $\Lambda' = \mathrm{Hom}_B(B, \mathrm{Ke}\, p)$ we form – with a similar map $p'$ – the exact sequence in $\mathbf{M}^B_B$,

$$B^{(\Lambda')} \xrightarrow{p'} B^{(\Lambda)} \xrightarrow{p} M \longrightarrow 0\,.$$

Now apply $\mathrm{Hom}^B_B(B, -)$ to obtain the exact sequence

$$\mathrm{Hom}^B_B(B, B^{(\Lambda')}) \longrightarrow \mathrm{Hom}^B_B(B, B^{(\Lambda)}) \longrightarrow \mathrm{Hom}^B_B(B, M) \longrightarrow 0\,.$$

By the choice of $\Lambda$ and $\Lambda'$, this sequence is exact. Now tensor with $- \otimes_R B$ to obtain the commutative diagram with exact rows ($\otimes$ for $\otimes_R$),

$$\begin{tikzcd}
\mathrm{Hom}^B_B(B, B^{(\Lambda')})\otimes B \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Hom}^B_B(B, B^{(\Lambda)})\otimes B \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Hom}^B_B(B,M)\otimes B \arrow[r] \arrow[d, "\varphi_M"] & 0 \\
B^{(\Lambda')} \arrow[r] & B^{(\Lambda)} \arrow[r] & M \arrow[r] & 0\,.
\end{tikzcd}$$

The first two vertical maps are bijective since $\mathrm{Hom}^B_B(B, -)$ commutes with direct sums. By the diagram properties this implies the bijectivity of $\varphi_M$.

(i) ⇒ (f) Assume that $B$ is a subgenerator in $\mathbf{M}^B_B$ and that $\varphi_M$ is injective for all $M \in \mathbf{M}^B_B$. Then clearly $\varphi_N$ is bijective for all $B$-generated objects $N$ in $\mathbf{M}^B_B$ and $M$ is a subobject of such an $N$. Choose an exact sequence

$0 \to M \to N \to L$ in $\mathbf{M}_B^B$ where $N$ and $L$ are $B$-generated. Then clearly $\varphi_N$ and $\varphi_L$ are bijective and there is a commutative diagram with exact rows,

$$\begin{CD}
0 @>>> \mathrm{Hom}_B^B(B,M)\otimes_R B @>>> \mathrm{Hom}_B^B(B,N)\otimes_R B @>>> \mathrm{Hom}_B^B(B,L)\otimes_R B \\
@. @VV{\varphi_M}V @VV{\varphi_N}V @VV{\varphi_L}V \\
0 @>>> M @>>> N @>>> L\,.
\end{CD}$$

From this we conclude that $\varphi_M$ is also bijective.

(h) $\Leftrightarrow$ (j) If $B_R$ is locally projective, then, by 4.7(g), $B$ is a generator in $\sigma[_{\mathrm{End}_R(B)}B]$. Moreover, $\mathbf{M}_B^B = \sigma[_{B^{op}\#B^*}B \otimes_R^b B]$. Now assume (h). Then $\mathbf{M}_B^B = \sigma[_{B^{op}\#B^*}B]$ and the density property follows by [3, 43.12]. On the other hand, given the density property and the subgenerating property of $B$, one has $\sigma[_{\mathrm{End}_R(B)}B] = \sigma[_{B^{op}\#B^*}B]$ and $B$ is a generator in $\mathbf{M}_B^B$.

The $R$-linear map $\phi : M \to M^{coB}$ considered in the proof (a)$\Rightarrow$(d) splits the inclusion $M^{coB} \to M$, thus proving the final statement. $\square$

Notice that parts of the characterisations in 13.4 apply to Hopf algebras that are not necessarily flat as $R$-modules (see 13.7 for such examples).

**13.5. Finitely generated Hopf algebras.** *For an $R$-bialgebra $B$ with $B_R$ finitely generated and projective, the following are equivalent:*

*(a) $B$ is a Hopf algebra;*

*(b) $\gamma_B : B \otimes_R B \to B \otimes_R^b B$ is surjective;*

*(c) $B$ has a left antipode;*

*(d) $B^{op}\#B^* \simeq \mathrm{End}_R(B)$;*

*(e) $B$ is a generator in $_{B^{op}\#B^*}\mathbf{M}$.*

**Proof.** (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c) follow from 13.2 and the fact that, for finitely generated projective $R$-modules, any surjective endomorphism is bijective.

(a) $\Rightarrow$ (d) As a generator in $\mathbf{M}_B^B = {}_{B^{op}\#B^*}\mathbf{M}$, $B$ is a faithful $B^{op}\#B^*$-module and the density property of $B^{op}\#B^*$ (see 13.4) implies $B^{op}\#B^* \simeq \mathrm{End}_R(B)$.

(e) $\Leftrightarrow$ (d) Since $B$ is a subgenerator in $\sigma[_{\mathrm{End}_R(B)}B]$, the assertion follows from 13.4(j).

(e) $\Rightarrow$ (a) Under the given conditions $\mathbf{M}_B^B = {}_{B^{op}\#B^*}\mathbf{M}$ (see 12.1) and the assertion again follows from the Fundamental Theorem 13.4. $\square$

Clearly, if $B$ is a finitely generated projective $R$-module, then $\mathbf{M}^B = {}_{B^*}\mathbf{M}$ has (enough) projectives and 12.5 implies the following corollary.

**13.6. Semigroup bialgebra.** Let $G$ be a semigroup with identity $e$. The *semigroup algebra* $R[G]$ is the $R$-module $R^{(G)}$ together with the maps (defined on the basis $G$ and linearly extended)

$$\mu : R[G] \times R[G] \longrightarrow R[G],\ (g,h) \mapsto gh \ \text{ and } \iota : R \to R[G],\ r \mapsto re\,.$$

Since $R[G]$ is a free $R$-module, there are also linear maps (see 5.6)

$$\Delta : R[G] \longrightarrow R[G] \otimes_R R[G],\ g \mapsto g \otimes g, \ \text{ and } \varepsilon : R[G] \longrightarrow R,\ 1 \mapsto 1, g \mapsto 0.$$

It is easily seen from the definitions that $\Delta$ and $\varepsilon$ are algebra morphisms.

If $G$ is a group, then $S : R[G] \to R[G],\ g \mapsto g^{-1}$, is an antipode, that is, in this case $R[G]$ is a Hopf algebra.

**13.7. Polynomial Hopf algebra.** As noticed in 5.8, for any commutative ring $R$, the polynomial algebra $R[X]$ is a coalgebra by

$$\begin{array}{ll} \Delta_2 : R[X] \otimes_R R[X] \to R[X], & 1 \mapsto 1,\ X^i \mapsto (X \otimes 1 + 1 \otimes X)^i, \\ \varepsilon_2 : R[X] \to R, & 1 \mapsto 1,\ X^i \mapsto 0, \quad i = 1, 2, \ldots . \end{array}$$

Together with the polynomial multiplication this yields a (commutative and cocommutative) bialgebra that is a Hopf algebra with antipode

$$S : R[X] \to R[X], \quad 1 \mapsto 1,\ X \to -X.$$

For any $a \in R$, denote by $J$ the ideal in $R[X]$ generated by $aX$. Since

$$\Delta_2(aX) = 1 \otimes aX + aX \otimes 1,\ \varepsilon_2(aX) = 0 \ \text{ and } S(aX) = -aX\,,$$

it is easily seen that $J$ is a Hopf ideal. Therefore $H = R[X]/J$ is a Hopf algebra over $R$. Notice that $H$ need no longer be projective or flat as an $R$-module. In particular, if $R$ is an integral domain and $0 \neq a \in R$, then $\mathrm{Hom}_R(R/aR, R) = 0$ and $H^* = \mathrm{Hom}(H, R) \simeq R$, and $H$-subcomodules of $H$ do not correspond to $H^*$-submodules.

# Chapter 3

# Exercises

**13.8. Exercises for Modules and Tensor Products.**

(1) Let $M, M', M''$ be right and $N, N', N''$ left $R$-modules. Prove:

(i) $I_M \otimes I_N = I_{M\otimes_R N}$.

(ii) For any morphism $f : M \to M'$, $f \otimes 0 = 0$.

(iii) For morphisms $f : M \to M'$, $f' : M' \to M''$ and $g : N \to N'$, $g' : N' \to N''$,

$$(f' \otimes g') \circ (f \otimes g) = f' \circ f \otimes g' \circ g.$$

(iv) If $f$ and $g$ are isomorphisms, then $f \otimes g$ is an isomorphism, and

$$(f \otimes g)^{-1} = f^{-1} \otimes g^{-1}.$$

(v) For $f_1,\, f_2 : M \to M'$ and $g_1,\, g_2 :\ N \to N'$,

$$(f_1 + f_2) \otimes g = f_1 \otimes g + f_2 \otimes g \text{ and } f \otimes (g_1 + g_2) = f \otimes g_1 + f \otimes g_2.$$

(2) Let $K \subset R_R$ and $L \subset {}_RR$ be right and left ideals of $R$. Prove

$$R/K \otimes_R R/L \simeq R/(K + L) \text{ as } \mathbb{Z}\text{-modules.}$$

Conclude that $\mathbb{Z}/m\mathbb{Z}\otimes_{\mathbb{Z}}/n\mathbb{Z} \simeq \mathbb{Z}/g(m,n)\mathbb{Z}$, where $g(m,n)$ denotes the greatest common divisor of $m, n \in \mathbb{Z}$.

(3) Let ${}_RM_S$ be a bimodule and ${}_SN$ an S-module. Prove: If ${}_RM$ and ${}_SN$ are flat modules, then ${}_RM \otimes_S N$ is a flat R-module.

(4) Let $\mu_{\mathbb{Q}} : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ and $\mu_{\mathbb{C}} : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ be the multiplication maps. Prove:

(i) $\mu_{\mathbb{Q}}$ and $\mu_{\mathbb{C}}$ are ring homomorphisms;

(ii) $\mu_{\mathbb{Q}}$ is an isomorphism, $\mu_{\mathbb{C}}$ is not monic.

(5) Prove:

(i) $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module.

(ii) For abelian torsion groups $M$ (every element has finite order), $M \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

(iii) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

(iv) For finite $\mathbb{Z}$-modules $K$, $L$, $\quad K \otimes_{\mathbb{Z}} L \simeq \mathrm{Hom}_{\mathbb{Z}}(K, L)$.

## 13.9. Exercises for Modules and Algebras.

(1) Let $I$ be an ideal of the ring $R$. Put $\mathrm{An}_M(I) = \{m \in M \mid Im = 0\}$ for a left $R$-module $M$. Prove:

(i) $\mathrm{An}_M(I)$ is an $R/I$-module.

(ii) The map $\psi : \mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{An}_M(I)$, $f \longmapsto f(1 + I)$, is an isomorphism of $R/I$-modules.

(iii) The assignment $M \longmapsto \mathrm{An}_M(I)$ defines a functor

$$\mathrm{Hom}_R(R/I, -) : R\text{–Mod} \longrightarrow R/I\text{–Mod}.$$

(iv) The functor $\mathrm{Hom}_R(R/I, -)$ respects essential monomorphisms and injective modules.

(2) Let $M$ be a left $R$-module and $M^* := \mathrm{Hom}_R(M, R)$. Prove:

(i) $M^*$ is a right $R$-module.

(ii) If $M$ is finitely generated and projective, then $M^*$ is also finitely generated and projective.

(iii) If $M$ is a generator in $R$-Mod, then $M^*$ is a generator in Mod-$R$.

(3) Prove that for a right ideal $J \subset R$ the following are equivalent:

(a) $R/J$ is flat as right $R$-module;

(b) the exact sequence $0 \to J \to R \to R/J \to 0$ is pure in Mod-$R$;

(c) for every left ideal $I \subset R$, we have $JI = J \cap I$.

(4) Let $T$ be any associative ring (without unit). A left $T$-module $N$ is called *s-unital* if $u \in Tu$ for every $u \in N$. $T$ itself is called *left s-unital* if it is s-unital as a left $T$-module.

Prove that for a left $T$-module $N$, the following are equivalent:

(a) $N$ is an s-unital $T$-module;

(b) for any $n_1, \ldots, n_k \in N$, there exists $t \in T$ with $n_i = tn_i$ for all $i \leq k$;

(c) for any set $\Lambda$, $N^{(\Lambda)}$ is an s-unital $T$-module.

**Hint** for $(a) \Rightarrow (b)$: Assume the assertion holds for $k - 1$ elements. Choose $t_k \in T$ such that $t_k n_k = n_k$ and put $m_i = n_i - t_k n_i$, for all $i \leq k$. Choose $t' \in T$ satisfying $m_i = t' m_i$, for all $i \leq k - 1$. Then consider $t := t' + t_k - t' t_k \in T$.

(5) Prove that for an ideal $T$ in an algebra $A$, the following are equivalent:

(a) $T$ is left s-unital;

(b) for every left ideal $I$ of $A$, $TI = T \cap I$;

(c) $A/T$ is a flat right $A$-module.

**13.10. Exercises for Coalgebras.**

(1) Let $g : A \to A'$ be an $R$-algebra morphism. Prove that, for any $R$-coalgebra $C$,
$$\mathrm{Hom}(C, g) : \mathrm{Hom}_R(C, A) \to \mathrm{Hom}_R(C, A')$$
is an $R$-algebra morphism.

(2) Let $f : C \to C'$ be an $R$-coalgebra morphism. Prove: If $f$ is bijective then $f^{-1}$ is also a coalgebra morphism.

(3) Let $C$ be a free $R$-module with basis $\{g_i, d_i \mid i \in \mathbb{N}\}$ and define
$$\begin{array}{rl} \Delta : C \to C \otimes_R C, & g_i \mapsto g_i \otimes g_i, \\ & d_i \mapsto g_i \otimes d_i + d_i \otimes g_{i+1}; \\ \varepsilon : C \to R & g_i \mapsto 1, \\ & d_i \mapsto 0\,. \end{array}$$
Prove that $(C, \Delta, \varepsilon)$ is a coalgebra.

(4) Let $C$ be a free $R$-module with basis $\{s, c\}$ and define
$$\begin{array}{rl} \Delta : C \to C \otimes_R C, & s \mapsto s \otimes c + c \otimes s, \\ & c \mapsto c \otimes c - s \otimes s; \\ \varepsilon : C \to R & s \mapsto 0, \\ & c \mapsto 1\,. \end{array}$$
Prove that $(C, \Delta, \varepsilon)$ is a coalgebra.

(5) Let $C$ be an $R$-module and $C^* = \mathrm{Hom}_R(C, R)$. For an $R$-submodule $D \subset C$ define
$$D^\perp := \{f \in C^* \mid f(D) = 0\} = \mathrm{Hom}_R(C/D, R) \subset C^*\,,$$
and for any subset $J \subset C^*$, put
$$J^\perp := \bigcap \{\mathrm{Ke}\, f \mid f \in J\} \subset C\,.$$
Prove that $D \subset D^{\perp\perp}$, and $D = D^{\perp\perp}$, provided that $C/D$ is cogenerated by $R$.

(6) Now let $C$ be an $R$-coalgebra and $D \subset C$ be an $R$-submodule. Prove:
   (i) If $D$ is a left $C^*$-submodule of $C$, then $D^\perp$ is a right $C^*$-submodule.
   (ii) If $D$ is a $(C^*, C^*)$-sub-bimodule of $C$, then $D^\perp$ is an ideal in $C^*$.
   (iii) If $D$ is a coideal in $C$, then $D^\perp$ is a subalgebra of $C^*$.

   (**Hint:** Recall that for $f, g \in C^*$ and $c \in C$, $f * g(c) = f(g \rightharpoonup c)$.)

(7) With the notation above prove:
   (i) If $J \subset C^*$ is a right (left) ideal, then $J^\perp$ is a left (right) $C^*$-submodule of $C$.
   (ii) If $J \subset C^*$ is an ideal, then $J^\perp$ is a $(C^*, C^*)$-sub-bimodule of $C$.

(8) Let $P, N$ be $R$-modules where $P$ is projective. Prove:

(i) For any index set $\Lambda$, the canonical map $R^\Lambda \otimes_R P \to P^\Lambda$ is injective.

(ii) For any family $f_\lambda \in \mathrm{Hom}_R(N, R)$, $\lambda \in \Lambda$,

$$\bigcap\nolimits_\Lambda (\mathrm{Ke}\, f_\lambda \otimes_R P) = (\bigcap\nolimits_\Lambda \mathrm{Ke}\, f_\lambda) \otimes_R P.$$

(9) Let $C = (C, \Delta, \varepsilon)$ be an $R$-coalgebra. Prove that the category of right $C$-comodules is equivalent to the category of left comodules over the opposite coalgebra $C^{cop} = (C, \mathsf{tw} \circ \Delta, \varepsilon)$.

(**Hint:** If $\varrho^M : M \to M \otimes_R C$ defines a right $C$-comodule then $\mathsf{tw} \circ \varrho^M : M \to C \otimes_R M$ yields a left $C^{cop}$-comodule.)

(10) Let $R$ be a von Neumann regular ring and $f : M_R \to M'_R$ and $g : {}_RN \to {}_RN'$ two $R$-module homomorphisms. Prove that

$$\mathrm{Ke}\, f \otimes g = \mathrm{Ke}\, f \otimes_R N + M \otimes_R \mathrm{Ke}\, g,$$

(11) Let $R$ be von Neumann regular, $C \in \mathbf{M}_R$, $C^* = \mathrm{Hom}_R(C, R)$, $D \subset C$, and $J \subset C^*$. Recall the definitions

$$D^\perp := \{f \in C^* \mid f(D) = 0\} \subset C^*, \quad J^\perp := \bigcap \{\mathrm{Ke}\, f \mid f \in J\} \subset C.$$

Prove:

(i) $(J_1 \otimes_R J_2)^\perp = J_1^\perp \otimes_R C + C \otimes_R J_2^\perp$.

(ii) If $C$ is a coalgebra and $J \subset C^*$ is a subalgebra, then $J^\perp$ is a coideal.

(iii) If $C$ is a coalgebra and $R$ is semisimple, then $D \subset C$ is a coideal if and only if $D^\perp$ is a subalgebra.

(12) Let $(A, \mu, \iota)$ be an $R$-algebra with ${}_RA$ is finitely generated and projective with dual basis $a_1, \ldots, a_n \in A$ and $\pi_1, \ldots, \pi_n \in A^*$, where $\mathrm{Hom}_R(-, R) = (-)^*$. Recall the isomorphism

$$\psi : A^* \otimes_R A^* \to (A \otimes_R A)^*, \quad f \otimes g \mapsto [a \otimes b \mapsto f(a)g(b)].$$

Prove:

(i) Describe $\psi^{-1}$ in terms of the dual basis.

(ii) $A^*$ is a coalgebra with coproduct

$$A^* \xrightarrow{\mu^*} (A \otimes_R A)^* \xrightarrow{\psi^{-1}} A^* \otimes_R A^*$$

and counit

$$\varepsilon := \iota^* : A^* \to R, \quad f \mapsto f(1_A).$$

(iii) The dual of the coalgebra $A^*$ is isomorphic (as an algebra) to $A$.

(13) For $C \in \mathbf{M}_R$, $C^* = \mathrm{Hom}_R(C, R)$, and subsets $D \subset C$, $S \subset C^*$ recall the definitions

$$D^\perp := \{f \in C^* \mid f(D) = 0\} \subset C^*, \quad S^\perp := \bigcap \{\mathrm{Ke}\, f \mid f \in S\} \subset C\,.$$

Prove:

(i) $S^\perp = < S >^\perp$ and $D^\perp = < D >^\perp$, where $< X >$ is the $R$-linear closure of $X$.

(ii) $S^\perp = ((S^\perp)^\perp)^\perp$ and $D^\perp = ((D^\perp)^\perp)^\perp$.

(14) Let $K$ be a field and $V, W$ vector spaces over $K$. Prove:
For any $u \in V \otimes_K W$ there exist $n \in \mathbb{N}$, linearly independent $x_1, \ldots, x_n \in V$, and linearly independent $y_1, \ldots, y_n \in W$, such that

$$u = \sum_{i=1}^{n} x_i \otimes y_i.$$

(15) Let $K$ be a field and $C$ a $K$-coalgebra. An non-zero element $g \in C$ is called *grouplike* if $\Delta(g) = g \otimes g$. Prove:

(i) For any grouplike element $g \in C$, $\varepsilon(g) = 1$.

(ii) The set of grouplike elements is linearly independent.

(16) Let $A$ be a finite dimensional algebra over the field $K$ and $A^*$ the dual coalgeba of $A$. Prove that the grouplike elements of $A^*$ correspond to the algebra morphisms $A \to K$.

(17) Let $A$ be a ring and $M$ a faithful left $A$-module. Prove that for an ideal $T \subset A$ the following are equivalent:

(a) $T$ is $M$-dense in $A$, i.e., for any $a \in A$ and $m_1, \ldots, m_n \in M$ there exists $t \in T$ such that

$$tm_i = am_i, \text{ for all } i = 1, \ldots, n.$$

(b) $M$ is s-unital as $T$-module.

**13.11. Exercises for Hopf Algebras.**

(1) Let $H$ be a Hopf algebra with antipode $S$ that is finitely generated and projective as an $R$-module. Show that $H^*$ with the canonical structure maps is again a Hopf algebra.

(2) Prove that for a Hopf algebra $H$ with antipode $S$, the following are equivalent:

(a) for any $h \in H$, $\sum S(h_{\underline{2}})h_{\underline{1}} = \varepsilon(h)1_H$;

(b) for any $h \in H$, $\sum h_{\underline{2}}S(h_{\underline{1}}) = \varepsilon(h)1_H$;

(c) $S \circ S = I_H$.

(3) Let $H$, $K$ be Hopf algebras with antipodes $S_H$, $S_K$, respectively. Prove that, for any bialgebra morphism $f : H \to K$, $S_K \circ f = f \circ S_H$.

(4) Let $H$ be a Hopf $R$-algebra that is finitely generated and projective as an $R$-module. Prove:

(i) The antipode $S$ of $H$ is bijective.

(ii) The right coinvariants $(H^*)^{coH}$ of $H^*$ form a finitely generated projective $R$-module of rank 1.

(iii) If $(H^*)^{coH} \simeq R$, then $H \simeq H^*$ as left $H$-modules (that is, $H$ is a Frobenius algebra).

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