COALGEBRAIC STRUCTURES IN MODULE THEORY

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ABSTRACT. Although coalgebras and coalgebraic structures are well-known for a long time it is only in recent years that they are getting new attention from people working in algebra and module theory. The purpose of this survey is to explain the basic notions of the coalgebraic world and to show their ubiquity in classical algebra. For this we recall the basic categorical notions and then apply them to linear algebra and module theory. It turns out that a number of results proven there were already contained in categorical papers from decades ago.

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INTRODUCTION

For an associative ring R we denote the category of left (resp. right) modules by ${}_R\mathbb{M}$ (resp. $\mathbb{M}_R$). Given rings R, S, for any (R,S)-bimodule ${}_RP_S$, the definition of the tensor product implies that there is a bijection, natural in $X \in {}_S\mathbb{M}$ and $N \in {}_R\mathbb{M}$,

$$\mathrm{Hom}_R(P \otimes_S X, N) \simeq \mathrm{Hom}_S(X, \mathrm{Hom}_R(P, N)).$$

In categorical terminolgy one says that

$$P \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad \mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M},$$

form an *adjoint pair of functors*.

This notion can be defined for any functors between two categories. Such a pair (F, G) of adjoint functors $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ leads (by composition) to endofunctors, more precisely *monads* and *comonads* on $\mathbb{A}$ and $\mathbb{B}$, respectively. In Section 1 we outline this general approach by defining *modules for monads* and *comodules for comonads*. We also study the situation when a monad is adjoint to a comonad and the resulting relations between the corresponding module and comodule categories. Properties of the composition of monads and comonads are controlled by distributive laws (entwinings) and this is described in Section 2.

In the following sections the general notions are brought back to module categories and this step provides natural comonads and comodule structures in these categories.

Adjoint functors between module categories are described by a tensor and a Hom functor and the properties derived from the categorical setting are explained in Section 3. Algebras and coalgebras can be understood as monads and comonads and the results coming out are formulated in Section 4. Hereby we observe in 4.10 that Abrams' characterisation of Frobenius algebras from 1999 can easily be seen as a byproduct of the Eilenberg-Moore paper from 1965. A short proof is given to show that, for a finitely generated module over a commutative ring R, the endomorphism ring is a Frobenius R-algebra (see 4.13).

Finally, in Section 5, possible definitions of an algebra structure on the tensor product of two algebras are explained by the entwinings of functors treated in Section 2. These methods also apply to the tensor product of coalgebras and bialgebras. In fact, the entwining structures introduced by Brzeziński and Majid in 1998 are obtained as an application of categorical results by van Osdol from 1971. The section concludes with a remark on the role of the Yang-Baxter equation for entwinings.

For notions from category theory we refer to Mac Lane [10] and for facts from module theory the reader may consult [16] or any other introductory texts on these fields.

1. Categorical setting

For convenience we recall the basic definitions. The observations on the interplay between adjoint pairs of functors, monads and comonads go back to the Eilenberg-Moore paper [9] from 1965 but for a long time were hardly exploited by people working on rings and modules. Parallel to the new appreciation of these techniques in classical algebra they also turn out to be of considerable interest in universal algebra and theoretical computer science (programming). In order to facilitate the understanding for people working in linear algebra, hints from this area are given along the way.

1.1. **Categories.** A category $\mathbb{A}$ consists of a class of *objects*, *morphism sets* $\mathrm{Mor}_{\mathbb{A}}(A,B)$ for any objects $A, B \in \mathbb{A}$, and an associative *composition*

$$\mathrm{Mor}_{\mathbb{A}}(A,B) \times \mathrm{Mor}_{\mathbb{A}}(B,C) \to \mathrm{Mor}_{\mathbb{A}}(A,C).$$

Moreover, any object $A \in \mathbb{A}$ is required to have an *identity morphism* I_A.

Besides of the category of sets with the set maps as morphisms, the reader is certainly familiar with the category ${}_k\mathbb{V}$ of vector spaces over any field k with the k-linear maps as morphisms.

Similar to homomorphisms between groups, categories can be related by

1.2. **Functors.** A (covariant) functor $F : \mathbb{A} \to \mathbb{B}$ between two categories assigns
an object $A \in \mathbb{A}$ to an object $F(A) \in \mathbb{B}$,
a morphism $f : A \to A'$ to a morphism $F(f) : F(A) \to F(A')$ of $\mathbb{B}$,

and has to respect the composition of morphisms as well as the identity morphisms on objects, that is, $F(I_A) = I_{F(A)}$ for any $A \in \mathbb{A}$. The identity functor on the category $\mathbb{A}$ is denoted by $I_{\mathbb{A}}$ or just I.

By definition, any functor F induces a map

$$F_{A,A'} : \mathrm{Mor}_{\mathbb{A}}(A, A') \to \mathrm{Mor}_{\mathbb{B}}(F(A), F(A')),$$

and F is called *faithful* (resp. *full*) if $F_{A,A'}$ is injective (resp. surjective) for any $A, A' \in \mathbb{A}$.

Any vector space $V \in {}_k\mathbb{V}$ defines a functor

$$V \otimes_k - : {}_k\mathbb{V} \to {}_k\mathbb{V}, \qquad \begin{array}{rcl} X & \mapsto & V \otimes_k X \\ X \xrightarrow{f} X' & \mapsto & V \otimes_k X \xrightarrow{I \otimes f} V \otimes_k X', \end{array}$$

and this is always faithful; it is full provided V has finite dimension.

Two functors between the same categories can be connected by

1.3. **Natural transformations.** Given two functors $F, G : \mathbb{A} \to \mathbb{B}$, a *natural transformation* $\psi : F \to G$ is defined as a family of morphisms $\psi_A : F(A) \to G(A)$, $A \in \mathbb{A}$, inducing, for any morphism $h : A \to A'$ in $\mathbb{A}$, commutativity of the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\psi_A"] \arrow[d, "F(h)"'] & G(A) \arrow[d, "G(h)"] \\ F(A') \arrow[r, "\psi_{A'}"] & G(A'). \end{tikzcd}$$

Let V and W be two k-vector spaces. Then for any k-linear map $\varphi : V \to W$, $\varphi \otimes - : V \otimes_k - \to W \otimes_k -$ is a natural transformation.

Using these basic notions we can already define

1.4. **Adjoint functors.** Given a functor $L : \mathbb{A} \to \mathbb{B}$, a functor $R : \mathbb{B} \to \mathbb{A}$ is said to be *right adjoint* to L if there are bijections, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\varphi_{A,B} : \mathrm{Mor}_{\mathbb{B}}(L(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, R(B)).$$

In this case (L, R) is said to be an *adjoint pair* of functors.

The adjointness of functors can also be expressed by

1.5. **Unit and counit of an adjunction.** Given an adjunction as in 1.4, one may consider

$$\eta_A := \varphi_{A,L(A)}(I_{L(A)}) : A \to LR(A), \quad \varepsilon_B := \varphi^{-1}_{R(B),B}(I_{R(B)}) : LR(A) \to A,$$

yielding natural transformations, called the *unit* and the *counit*,

$$\eta : I_{\mathbb{A}} \to RL, \quad \varepsilon : LR \to I_{\mathbb{B}},$$

which satisfy the *triangular identities*

$$\begin{tikzcd} L \arrow[r, "L\eta"] \arrow[dr, "="'] & LRL \arrow[d, "\varepsilon L"] \\ & L, \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\eta R"] \arrow[dr, "="'] & RLR \arrow[d, "R\varepsilon"] \\ & R. \end{tikzcd}$$

Conversely, with these transformations the map φ can be described by

$$\begin{array}{llll} \varphi: & L(A) \xrightarrow{f} B & \longmapsto & A \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B), \\ \varphi^{-1}: & A \xrightarrow{h} R(B) & \longmapsto & L(A) \xrightarrow{L(h)} LR(B) \xrightarrow{\varepsilon_B} B. \end{array}$$

Functors which have left or right adjoints also respect limits and colimits, respectively:

1.6. **Properties of adjoint functors.** *If (L, R) be an adjoint pair of functors, then*

(i) *L preserves epimorphisms and coproducts,*

(ii) *R preserves monomorphisms and products.*

The functor $V \otimes_k -$ has a right adjoint $\mathrm{Hom}_k(V, -)$ provided by the canonical bijection, where $X, Y \in {}_k\mathbb{V}$,

$$\mathrm{Hom}_k(V \otimes_k X, Y) \to \mathrm{Hom}_k(X, \mathrm{Hom}_k(V, Y),$$

and counit and unit are given as k-linear maps

$$\begin{array}{lll} \varepsilon_X: & V \otimes_k \mathrm{Hom}_k(V, X) \to X, & v \otimes f \mapsto f(v), \\ \eta_Y: & Y \to \mathrm{Hom}_k(V, V \otimes Y), & y \mapsto [v \mapsto v \otimes y]. \end{array}$$

It follows from 1.6 that $\mathrm{Hom}_k(V, -)$ can only be a *right adjoint* provided V has finite dimension, that is, $\mathrm{Hom}_k(V, X) \simeq \mathrm{Hom}_k(V, k) \otimes_k X$.

There are some properties of units and counits which are easily seen to correspond to special properties of the functors R and L.

1.7. **Properties of unit and counit.** *Let (L, R) be an adjoint pair of functors with unit η and counit ε.*

(1) (i) *ε_B is an epimorphism $\Leftrightarrow$ R is a faithful functor.*

(ii) *ε_B is a coretraction $\Leftrightarrow$ R is a full functor.*

(iii) *ε_B is an isomorphism $\Leftrightarrow$ R is full and faithful.*

(2) (i) *η_A is a monomorphism $\Leftrightarrow$ L is a faithful functor.*

(ii) *η_A is a retraction $\Leftrightarrow$ L is a full functor.*

(iii) *η_A is an isomorphism $\Leftrightarrow$ L is full and faithful.*

(3) *ε and η are isomorphisms $\Leftrightarrow$ L is an equivalence (with inverse R).*

Some properties of unit and counit have a weaker impact on the related functors.

1.8. **More on unit and counit.** *Let (L, R) be an adjoint pair as in* 1.7*.*

(1) *If ε_B is a monomorphism for all $B \in \mathbb{B}$, then*

(i) *$\eta R : R \to RLR$ is an isomorphism;*

(ii) *$LR : \mathbb{B} \to \mathbb{B}$ preserves monomorphisms.*

(2) *If η_A is an epimorphism for all $A \in \mathbb{A}$, then*

(i) *$\varepsilon L : LRL \to L$ is an isomorphism;*

(ii) *$RL : \mathbb{A} \to \mathbb{A}$ preserves epimorphisms.*

Composing adjoint functors yields special endofunctors.

1.9. **Related endofunctors.** Let (L, R) be an adjoint pair (as in 1.7).

(1) $RL : \mathbb{A} \to \mathbb{A}$ is an endofunctor with natural transformations (product and unit)
$$R\varepsilon L : RLRL \to RL, \quad \eta : I \to RL.$$

(2) $LR : \mathbb{B} \to \mathbb{B}$ is an endofunctor with natural transformations (coproduct and counit)
$$\delta := L\eta R : LR \to LRLR, \quad \varepsilon : LR \to I.$$

The properties shared by these structures lead to the notions of

1.10. **Monads and comonads.**

(1) A *monad* on a category $\mathbb{A}$ is a triple (F, m, η) where $F : \mathbb{A} \to \mathbb{A}$ is an endofunctor, $m : FF \to F$ and $\eta : I_{\mathbb{A}} \to F$ are natural transformations, called the *product* and *unit*, inducing commutativity of the diagrams
$$\begin{tikzcd} FFF \arrow[r, "mF"] \arrow[d, "Fm"'] & FF \arrow[d, "m"] \\ FF \arrow[r, "m"'] & F, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "="'] & FF \arrow[d, "m"] & F \arrow[l, "\eta F"'] \arrow[dl, "="] \\ & F & \end{tikzcd} .$$

(2) A *comonad* on a category $\mathbb{A}$ is a triple (G, δ, ε) where $G : \mathbb{A} \to \mathbb{A}$ is an endofunctor, $\delta : G \to GG$ and $\varepsilon : G \to I_{\mathbb{A}}$ are natural transformations, called the *coproduct* and *counit*, inducing commutativity of the diagrams
$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & GG \arrow[d, "G\delta"] \\ GG \arrow[r, "\delta G"'] & GGG, \end{tikzcd} \qquad \begin{tikzcd} & G \arrow[dl, "="'] \arrow[d, "\delta"] \arrow[dr, "="] & \\ G & GG \arrow[l, "\varepsilon G"] \arrow[r, "G\varepsilon"'] & G. \end{tikzcd}$$

For a k-vector space, the (endo)functor $V \otimes_k -$ has a monad structure provided there are a k-linear map $V \otimes_k V \to V$ satisfying the associativity condition and a unit morphism $e : R \to V$, that is, V ia a k-algebra with unit $e(1)$. A comonad structure on $V \otimes_k -$ is obtained by reversing the arrows.

1.11. **Modules for monads.** Let (F, m, η) be a monad on $\mathbb{A}$. An *F-module* (A, ϱ_A) is an object $A \in \mathbb{A}$ with a morphism $\varrho_A : F(A) \to A$ inducing commutativity of the diagrams
$$\begin{tikzcd} FF(A) \arrow[r, "m_A"] \arrow[d, "F(\varrho_A)"'] & F(A) \arrow[d, "\varrho_A"] \\ F(A) \arrow[r, "\varrho_A"'] & A, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "="'] & F(A) \arrow[d, "\varrho_A"] \\ & A. \end{tikzcd}$$

A *morphism* between two F-modules (A, ϱ_A) and $(A', \varrho_{A'})$ is a morphisms $f : A \to A'$ in $\mathbb{A}$ with commutative diagram
$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\varrho_A"'] & F(A') \arrow[d, "\varrho_{A'}"] \\ A \arrow[r, "f"'] & A'. \end{tikzcd}$$

With these morphisms, the F-modules form a category which we denote by $\mathbb{A}_F$ (Eilenberg-Moore category). We write $\mathrm{Mor}_F(A,A') := \mathrm{Mor}_{\mathbb{A}_F}(A,A')$ for short.

1.12**. Free and forgetful functors.** Given any monad (F,m,η) on $\mathbb{A}$, there are the *free functor* and the *forgetful functor*,

$$\begin{aligned} &\phi_F : \mathbb{A} \to \mathbb{A}_F, \quad A \mapsto (F(A), m_A : FF(A) \to F(A)),\\ &U_F : \mathbb{A}_F \to \mathbb{A}, \quad (A, \varrho_A) \mapsto A, \end{aligned}$$

and (ϕ_F, U_F) is an adjoint pair by the bijection, for $A \in \mathbb{A}$ and $B \in \mathbb{A}_F$,

$$\mathrm{Mor}_F(\phi_F(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, U_F(B)), \quad f \mapsto f \cdot \eta_A.$$

Reversing the arrows in the constructions considered above we get

1.13**. Comodules for comonads.** Let (G,δ,ε) be a comonad on $\mathbb{A}$. A *G-comodule* (X,ϱ^X) is an object $X \in \mathbb{A}$ with a morphism $\varrho^X : X \to G(X)$ and commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\varrho^X"] \arrow[d, "\varrho^X"'] & G(X) \arrow[d, "\delta_X"] \\ G(X) \arrow[r, "G(\varrho^X)"'] & GG(X), \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varrho^X"] \arrow[dr, "="'] & G(X) \arrow[d, "\varepsilon_X"] \\ & X. \end{tikzcd}$$

Morphisms between comodules (X,ϱ^X) and $(X',\varrho^{X'})$ are defined as morphisms $h : X \to X'$ in $\mathbb{A}$ with a commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\varrho^X"'] & X' \arrow[d, "\varrho^{X'}"] \\ G(X) \arrow[r, "G(h)"'] & G(X'). \end{tikzcd}$$

These notions yield the (Eilenberg-Moore) category of G-comodules which we denote by $\mathbb{A}^G$. We write $\mathrm{Mor}^G(X,X') := \mathrm{Mor}_{\mathbb{A}^G}(X,X')$ for short.

To prevent misunderstandings we mention that the F-modules and G-comodules defined in 1.11 and 1.13 are also called *F-algebras* and *G-coalgebras* in category theory (e.g. [10]).

1.14**. Cofree and forgetful functors.** Given a comonad (G,δ,ε) on $\mathbb{A}$, there are the *cofree functor* and the *forgetful functor*,

$$\begin{aligned} &\phi^G : \mathbb{A} \to \mathbb{A}^G, \quad X \mapsto (G(X), \delta_X : G(X) \to GG(X)),\\ &U^G : \mathbb{A}^G \to \mathbb{A}, \quad (X, \varrho^X) \mapsto X, \end{aligned}$$

and (U^G, ϕ^G) is an adjoint pair by the bijection, for $M \in \mathbb{A}^G$ and $X \in \mathbb{A}$,

$$\mathrm{Mor}^G(M, \phi^G(X)) \to \mathrm{Mor}_{\mathbb{A}}(U^G(M), X), \quad h \mapsto \varepsilon_X \cdot h.$$

A nice correspondence between monads and comonads comes out for endofunctors which are adjoint to each other.

1.15**. Adjoint endofunctors.** Let (F,G) be an adjoint pair of endofunctors on a category $\mathbb{A}$ with bijection

$$\varphi_{X,Y} : \mathrm{Mor}_{\mathbb{A}}(F(X), Y) \to \mathrm{Mor}_{\mathbb{A}}(X, G(Y)),$$

and $\eta : I_{\mathbb{A}} \to GF$, $\varepsilon : FG \to I_{\mathbb{A}}$ as unit and counit.

Assume (F, m, e) to be a monad. Then, for $X, Y \in \mathbb{A}$, we have the diagrams

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{A}}(F(X), Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "\mathrm{Mor}(m_X, Y)"'] & \mathrm{Mor}_{\mathbb{A}}(X, G(Y)) \arrow[d, dotted, "\mathrm{Mor}(X, ?)"] \\
\mathrm{Mor}_{\mathbb{A}}(FF(X), Y) \arrow[r, "\simeq"] & \mathrm{Mor}_{\mathbb{A}}(X, GG(Y)),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{A}}(F(X), Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "\mathrm{Mor}(e_X, Y)"'] & \mathrm{Mor}_{\mathbb{A}}(X, G(Y)) \arrow[dl, dotted, "\mathrm{Mor}(X, ??)"] \\
\mathrm{Mor}_{\mathbb{A}}(X, Y) & ,
\end{tikzcd}$$

in which the dotted morphisms exist by composition of the other morphisms. By the *Yoneda Lemma* it follows that they are induced by morphisms

$$\underline{\delta}_Y : G(Y) \to GG(Y) \quad \text{and} \quad \underline{\varepsilon} : G(Y) \to Y,$$

and these are explicitly given by the natural transformations

$$\underline{\delta} : G \xrightarrow{\eta G} GFG \xrightarrow{G\eta FG} GGFFG \xrightarrow{GGmG} GGFG \xrightarrow{GG\varepsilon} GG,$$
$$\underline{\varepsilon} : G \xrightarrow{eG} FG \xrightarrow{\varepsilon} I_{\mathbb{A}}\,,$$

yielding a comonad $(G, \underline{\delta}, \underline{\varepsilon})$.

By symmetry of the constructions, we obtain the first part in the next theorem.

1.16. **Theorem.** *Let (F, G) be an adjoint pair of endofunctors on $\mathbb{A}$ with unit $\eta : I \to GF$ and counit $\varepsilon : FG \to I$. Then F has a monad structure if and only if G allows for a comonad structure.*

In this case, the category of F-modules is isomorphic to the category of G-comodules by the functors

$$\mathbb{A}_F \to \mathbb{A}^G, \quad F(A) \xrightarrow{h} A \;\longmapsto\; A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A)\,,$$
$$\mathbb{A}^G \to \mathbb{A}_F, \quad A \xrightarrow{\rho} G(A) \;\longmapsto\; F(A) \xrightarrow{F(\rho)} FG(A) \xrightarrow{\varepsilon_A} A\,.$$

More about these structures may be found, for example, in [4, 17].

For any k-vector space V, $(V \otimes_k -, \mathrm{Hom}_k(V, -))$ forms an adjoint pair of endofunctors. Thus a monad structure on $V \otimes_k -$ (algebra) implies a comonad structure on $\mathrm{Hom}_k(V, -)$. Then, if V has finite dimension, $\mathrm{Hom}_k(V, k)$ is a k-coalgebra.

Recall that a functor is said to be a *Frobenius functor* provided it has a right adjoint which is also a left adjoint. Then a monad (F, m, η) is called a *Frobenius monad* if the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ is Frobenius. This corresponds to the condition that F is isomorphic to its right adjoint and, by 1.16, this means that F allows for a comonad structure and leads to the following characterisation of Frobenius monads (compare [13]).

1.17. **Proposition.** *For a monad (F, m, η) on $\mathbb{A}$, the following are equivalent:*

(a) *F is a Frobenius monad;*

(b) *F has a comonad structure (F, δ, ε) inducing commutativity of the diagrams*

$$\begin{tikzcd} FF \arrow[r, "F\delta"] \arrow[d, "m"'] & FFF \arrow[d, "mF"] \\ F \arrow[r, "\delta"] & FF, \end{tikzcd} \qquad \begin{tikzcd} FF \arrow[r, "\delta F"] \arrow[d, "m"'] & FFF \arrow[d, "Fm"] \\ F \arrow[r, "\delta"] & FF; \end{tikzcd}$$

(c) *F has a comonad structure (F, δ, ε) and (F, F) is an adjoint pair with counit and unit*

$$FF \xrightarrow{m} F \xrightarrow{\varepsilon} I, \quad I \xrightarrow{\eta} F \xrightarrow{\delta} FF.$$

Proof. (a)⇒(b) It follows from 1.16 that the category $\mathbb{A}_F$ is isomorphic to $\mathbb{A}^F$. This in turn implies that F-module morphisms are F-comodule morphisms, and vice versa. In particular, δ is an F-module morphism and this just means commutativity of the left hand diagram in (b), and m is an F-comodule morphisms meaning commutativity of the right hand diagram of (b).

The remaining implications are obtained by standard verifications. □

2. Lifting of functors

Given two monads (comonads) F and F' on a category $\mathbb{A}$, the composition $F'F$ is again an endofunctor on $\mathbb{A}$. Now one may ask when $F'F$ allows for a monad (comonad) structure. Also, what happens if we compose a monad with a comonad? This kind of questions can be related with lifting properties of endofunctors to the category of the corresponding modules or comodules, respectively, which can be described by *distributive laws* as considered in [3] (also called *entwinings*).

For the category of vector spaces these liftings come in to define suitable structures on the tensor product of algebras or coalgebras. For details we refer to Section 5.

2.1. Lifting of endofunctors. Let (F, m, η) be a monad, (G, δ, ε) a comonad, and T an endofunctor of the category $\mathbb{A}$. For functors

$$\overline{T} : \mathbb{A}_F \to \mathbb{A}_F \quad \text{and} \quad \widehat{T} : \mathbb{A}^G \to \mathbb{A}^G,$$

we have the diagrams

$$\begin{tikzcd} \mathbb{A}_F \arrow[r, "\overline{T}"] \arrow[d, "U_F"'] & \mathbb{A}_F \arrow[d, "U_F"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{A}, \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^G \arrow[r, "\widehat{T}"] \arrow[d, "U^G"'] & \mathbb{A}^G \arrow[d, "U^G"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{A}, \end{tikzcd}$$

and we say that $\overline{T}$ or $\widehat{T}$ are liftings of T provided the corresponding diagrams are commutative.

2.2. Lifting of monads to monads. *With the notation from* 2.1*,*

(1) *there exists a lifting $\overline{T} : \mathbb{A}_{\mathbb{F}} \to \mathbb{A}_{\mathbb{F}}$ of T if and only if there is a natural transformation $\lambda : FT \to TF$ with commutative diagrams*

$$\begin{tikzcd} FFT \arrow[rr, "m_T"] \arrow[d, "F\lambda"'] & & FT \arrow[d, "\lambda"] \\ FTF \arrow[r, "\lambda_F"] & TFF \arrow[r, "Tm"] & TF, \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "T\eta"'] & FT \arrow[d, "\lambda"] \\ & TF; \end{tikzcd}$$

(2) *if (T, m', η') is a monad, then a lifting $\overline{T}$ has a monad structure if and only if λ induces commutativity of the diagrams*

$$\begin{tikzcd} FTT \arrow[rr, "Fm'"] \arrow[d, "\lambda_T"'] & & FT \arrow[d, "\lambda"] \\ TFT \arrow[r, "T\lambda"] & TTF \arrow[r, "m'F"] & TF, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\eta'"] \arrow[dr, "\eta' F"'] & FT \arrow[d, "\lambda"] \\ & TF; \end{tikzcd}$$

In this case, λ is called a monad entwining (or distributive law in [3]) *and TF has a monad structure with product and unit*

$$TFTF \xrightarrow{T\lambda F} TTFF \xrightarrow{m'\,m} TF\ , \qquad I \xrightarrow{\eta'\,\eta} TF.$$

Similar results for the lifting of comonads are obtained by reversing the arrows in 2.2 and under the resulting conditions the composition of two comonads again gives a comonad. In this case λ may be called a *comonad entwining* (e.g. [17, 4.9]).

The conditions for lifting monads to a comodule category employ the first diagrams of the preceding cases (e.g. [17, Section 5]).

2.3. **Lifting of monads for comonads.** *With the notation from* 2.1*,*

(1) *there exists a lifting $\widehat{T} : \mathbb{A}^{\mathbb{G}} \to \mathbb{A}^{\mathbb{G}}$ of T if and only if there is a natural transformation $\lambda : TG \to GT$ with commutative diagrams*

$$\begin{tikzcd} TG \arrow[r, "T\delta"] \arrow[d, "\lambda"'] & TGG \arrow[r, "\lambda"] & GTG \arrow[d, "G\lambda"] \\ GT \arrow[rr, "\delta T"] & & GGT, \end{tikzcd} \qquad \begin{tikzcd} TG \arrow[r, "T\varepsilon"] \arrow[d, "\lambda"'] & T \\ GT; \arrow[ur, "\varepsilon T"'] & \end{tikzcd}$$

(2) *if (T, m, η) is a monad, then the lifting $\widehat{T}$ has a monad structure if and only if λ induces commutativity of the diagrams*

$$\begin{tikzcd} TTG \arrow[rr, "mG"] \arrow[d, "T\lambda"'] & & TG \arrow[d, "\lambda"] \\ TGT \arrow[r, "\lambda T"] & GTT \arrow[r, "Gm"] & GT, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "G\eta"'] & TG \arrow[d, "\lambda"] \\ & GT. \end{tikzcd}$$

In this case, λ is called a mixed entwining.

An endofunctor allowing for a monad and a comonad structure may be a Frobenius monad if the compatibility conditions in Proposition 1.17 are satisfied, or one may impose other compatibility requirements leading to the definition of (see [11, Definition 4.1])

2.4. **Bimonads.** A *bimonad* on a category $\mathbb{A}$ is an endofunctor $B : \mathbb{A} \to \mathbb{A}$ which has a monad structure (B, m, e) and a comonad structure (B, δ, ε) such that

(i) $\varepsilon : B \to I$ is a module structure morphism for I;

(ii) $e : 1 \to B$ is comodule structure morphism for I;

(iii) there is a mixed entwining (see 2.3) $\lambda : BB \to BB$ yielding commutativity of the diagram

$$\begin{tikzcd} BB \arrow[r, "m"] \arrow[d, "B\delta"'] & B \arrow[r, "\delta"] & BB \\ BBB \arrow[rr, "\lambda B"'] & & BBB. \arrow[u, "Bm"'] \end{tikzcd}$$

If the endofunctor B has a bimonad structure and allows for a right adjoint G, then, by 1.16, G allows for a monad and a comonad structure and it is easy to check that the compatibility conditions for B are transferred to G, that is G is again a bimonad (e.g. [11, 4]).

2.5. **Hopf monads.** Given a bimonad $\mathbf{B} = (B, m, e, \delta, \varepsilon)$, one defines *Hopf modules* as objects A in $\mathbb{A}$ allowing for module and comodule structure maps, $\varrho_A : B(A) \to A$ and $\varrho^A : A \to B(A)$, with commutative diagram (see [11, 4.2])

$$\begin{tikzcd} B(A) \arrow[r, "\varrho_A"] \arrow[d, "B(\varrho^A)"'] & A \arrow[r, "\varrho^A"] & B(A) \\ BB(A) \arrow[rr, "\lambda_A"'] & & BB(A). \arrow[u, "B(\varrho_A)"'] \end{tikzcd}$$

Morphisms between Hopf modules A and A' are to be module as well as comodule morphisms and the set of all of them is denoted by $\mathrm{Mor}_B^B(A, A')$. They lead to the category $\mathbb{A}_B^B$ of Hopf modules. By the conditions in 2.4, for any $A \in \mathbb{A}$, $B(A)$ has the structure of a Hopf module and thus B induces the (comparison) functor

$$K_B : \mathbb{A} \to \mathbb{A}_B^B, \quad A \mapsto (B(A), m_A, \delta_A),$$

which is full and faithful by the isomorphism (see 1.12, 1.14)

$$\mathrm{Mor}_B^B(B(A), B(A')) \simeq \mathrm{Mor}_B(B(A), A') \simeq \mathrm{Mor}_{\mathbb{A}}(A, A').$$

The bimonad $\mathbf{B}$ is called a *Hopf monad* provided it has an *antipode* and with mild restrictions on the category $\mathbb{A}$, this is the case if and only if K_B is an equivalence of categories (see [11, Section 5]).

Moreover, if the functor part B of a Hopf monad has a right adjoint G, then G again allows for the structure of a Hopf monad ([4], [6]).

3. Functors between module categories

In this section we apply our general results to adjoint pairs of functors between module categories. Here R and S denote any associative rings with units.

3.1. **Adjoint pair of functors.** Attached to any (R, S)-bimodule ${}_RP_S$, there is an adjoint pair of functors between ${}_S\mathbb{M}$ and ${}_R\mathbb{M}$,

$$P \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad \mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}.$$

The adjunction is given by the canonical bijection, for $N \in {}_R\mathbb{M}$, $X \in {}_S\mathbb{M}$,

$$\begin{array}{rcl} \varphi_{X,N} : \mathrm{Hom}_R(P \otimes_S X, N) & \to & \mathrm{Hom}_S(X, \mathrm{Hom}_R(P, N)), \\ f & \mapsto & [x \mapsto f(- \otimes x)]. \end{array}$$

Counit and unit of this adjunction come out as

$$\begin{aligned}\varepsilon_M : &\quad P \otimes_S \mathrm{Hom}_R(P, M) \to M, \quad p \otimes f \mapsto f(p),\\ \eta_X : &\quad X \to \mathrm{Hom}_R(P, P \otimes_S X), \quad x \mapsto [p \mapsto p \otimes x].\end{aligned}$$

It follows by the Eilenberg-Watts theorem (e.g. [6, 39.4]) that any covariant functor from ${}_R\mathbb{M}$ to ${}_S\mathbb{M}$ which preserves epimorphisms and direct sums is isomorphic to a tensor functor and hence any adjoint pair of functors between these categories is of the type considered above.

Since the ring S is a generator in ${}_S\mathbb{M}$, the functor $P \otimes_S -$ is determined by its value on S, that is by $P \otimes_S S \simeq P$. However, in general the functor $\mathrm{Hom}_R(P, -)$ need not be determined by its value on R, that is $P^* := \mathrm{Hom}_R(P, R)$.

We also have a functor $- \otimes_R P : \mathbb{M}_R \to \mathbb{M}_S$ and natural transformations

$$\beta : P^* \otimes_R - \to \mathrm{Hom}_R(P, -), \quad \alpha : - \otimes_R P \to \mathrm{Hom}_R(P^*, -),$$

given by the familiar morphisms for $M \in {}_R\mathbb{M}$ and $N \in \mathbb{M}_R$,

$$\begin{aligned}\beta_M : P^* \otimes_R M \to \mathrm{Hom}_R(P, M), &\quad f \otimes m \mapsto [p \mapsto f(p)m],\\ \alpha_N : N \otimes_R P \to \mathrm{Hom}_R(P^*, N), &\quad n \otimes p \mapsto [g \mapsto n \cdot g(p)].\end{aligned}$$

Recall that a *dual basis* for a projective R-module P consists of families $\{p_\lambda\}_\Lambda$ of elements in P and $\{p^*_\lambda\}_\Lambda$ of elements in P^*, such that $p = \sum_\Lambda p^*_\lambda(p)p_\lambda$ for any $p \in P$. P is finitely generated and projective if and only if it has a finite dual basis (i.e., Λ is a finite set).

3.2. **Proposition.** *For a left R-module P, the following are equivalent:*

(a) $\mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_\mathbb{Z}\mathbb{M}$ *has a right adjoint functor;*
(b) ${}_RP$ *is finitely generated and projective;*
(c) $\beta_P : P^* \otimes_R P \to \mathrm{Hom}_R(P, P)$ *is an isomorphism;*
(d) $\beta_P : P^* \otimes_R P \to \mathrm{Hom}_R(P, P)$ *is surjective;*
(e) $\beta : P^* \otimes_R - \to \mathrm{Hom}_R(P, -)$ *is a natural isomorphism.*

Proof. We just note that (d) implies the existence of a finite dual basis for P. The remaining conclusions are obvious. □

Recall that an R-module P is said to be *locally projective* provided for any diagram of left R-modules with exact bottom row

$$\begin{tikzcd} F \arrow[r, "i"] & P \arrow[d, "g"] & \\ L \arrow[r, "f"] & N \arrow[r] & 0, \end{tikzcd}$$

where F is finitely generated, there is a homomorphism $h : P \to L$ with $g \cdot i = f \cdot h \cdot i$ (e.g. [6, 42.9]). Clearly projective modules are locally projective modules, and the latter are projective provided they are finitely generated.

3.3. **Proposition.** *Let P be a left R-module.*

(1) *The following are equivalent:*
  (a) $\alpha : N \otimes_R P \to \mathrm{Hom}_R(P^*, N)$ *is injective for all* $N \in \mathbb{M}_R$*;*
  (b) P *is a locally projective R-module.*
(2) *The following are equivalent:*

(a) $\alpha : - \otimes_R P \to \mathrm{Hom}_R(P^*, -)$ *is a natural isomorphism;*

(b) $\alpha_{P^*} : P^* \otimes_R P \to \mathrm{Hom}_R(P^*, P^*)$ *is an isomorphism;*

(c) P *is finitely generated and projective.*

Proof. (1) is shown in [6, 42.10] (with more equivalent characterisations).
(2) is shown similarly to Proposition 3.2 □

Now we investigate the properties of the unit and counit.

3.4. **Proposition.** *We refer to the notations in* 3.1.

(1) *The following are equivalent:*

(a) ε_M *is an epimorphism (isomorphism) for all* $M \in {}_R\mathbb{M}$*;*

(b) $\mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}$ *is faithful;*

(c) $\mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}$ *is full and faithful;*

(d) P *is a generator in* ${}_R\mathbb{M}$.

(2) *The following are equivalent:*

(a) η_X *is an isomorphism for all* $X \in {}_S\mathbb{M}$*;*

(b) $P \otimes_S -$ *is full and faithful (faithfully flat).*

(3) *The following are equivalent:*

(a) η *and* ε *are (natural) isomorphisms;*

(b) $P \otimes_S -$ *is an equivalence (with inverse* $\mathrm{Hom}_R(P, -)$*);*

(c) ${}_RP$ *is a finitely generated, projective generator and* $S \simeq \mathrm{End}_R(P)$.

Proof. (1) The equivalence of (a) and (b) follows from 1.7. The equivalence of (b) and (c) is a particular property of full module (and Grothendieck) categories. The equivalence of (c) and (d) can be taken as definition of generators in categories. It is easy to see that in ${}_R\mathbb{M}$ this is equivalent to the fact that, for any R-module M, there is an epimorphism $P^{(\Lambda)} \to M$ (e.g. [16, 13.6]).
(2) and (3) follow essentially from 1.7. □

According to 1.9, the endofunctors

$$Hom_R(P, P \otimes_S -) : {}_S\mathbb{M} \to {}_S\mathbb{M}, \quad P \otimes_S \mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_R\mathbb{M},$$

allow for a monad and comonad structure, respectively, and thus induce the corresponding module and comodule categories.

Recall that a monad (comonad) is said to be *idempotent* provided the product (coproduct) is a natural isomorphism. It is clear that the above monad and comonad are idempotent in case ε and η are isomorphisms (i.e. $\mathrm{Hom}_R(P, -)$ induces an equivalence).

Elaborating the observations from 1.8 one obtains (see [8, Section 4]):

3.5. **Proposition.** *Let* P *be a left* R*-module and* $S = \mathrm{End}_R(P)$*. Then the following are equivalent:*

(a) ε_M *is a monomorphism for all* $M \in {}_R\mathbb{M}$ *and*
η_X *is an epimorphism for all* $X \in {}_S\mathbb{M}$*;*

(b) *the monad* $Hom_R(P, P \otimes_S -)$ *is idempotent;*

(c) *the comonad* $P \otimes_S \mathrm{Hom}_R(P, -)$ *is idempotent;*

(d) $(P \otimes_S -, \mathrm{Hom}_R(P,-))$ *induces an equivalence*

$$P \otimes_S - : {}_S\mathbb{M}_{Hom_R(P,P\otimes_S -)} \longrightarrow {}_R\mathbb{M}^{P\otimes_S \mathrm{Hom}_R(P,-)}.$$

These properties characterise P as a $*$-*module* (in the sense of Menini-Orsatti). A $*$-module P that is in addition a *subgenerator* in ${}_R\mathbb{M}$ (generates all injectives) is known as a *tilting module*. The equivalence noticed above implies the Brenner-Butler equivalence (see [8]).

4. Module and comodule categories

In the following we apply our general results to describe algebras and coalgebras and their (co-) module categories and this will throw new light on familiar notions from this part of algebra.

To avoid some technicalities, in this section R will always denote a commutative ring with unit and we sometimes write $\mathbb{M}$ for ${}_R\mathbb{M}$ (but see Remark 4.11). This also ensures that, for any R-modules M, N, we have the canonical twist map,

$$\mathsf{tw}_{M,N} : M \otimes_R N \to N \otimes_R M,\ m \otimes n \mapsto n \otimes m.$$

4.1**. Algebras and module categories.** An R-*algebra* is a triple (A, m, e) with an R-module A, an associative multiplication $m : A \otimes_R A \to A$, and a unit morphism $e : R \to A$ (where $e(1_R) = 1_A$). Then

$$A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M},\ X \mapsto A \otimes_R X,$$

is an endofunctor and the triple $(A \otimes_R -, m \otimes I, e \otimes I)$ is a monad on ${}_R\mathbb{M}$.

Now (left) A-modules are defined as R-modules M with an R-linear map

$$\rho_M : A \otimes_R M \to M,$$

satisfying the associativity and unitality conditions (see 1.11).

Morphisms $f : M \to N$ between A-modules are defined as R-linear maps subject to the compatibility conditions from 1.11 which here simply read as

$$f(am) = af(m), \text{ for } a \in A \text{ and } m \in M.$$

These notions define the category of (left) A-modules ${}_A\mathbb{M}$ $(=\mathbb{M}_{A\otimes_R -})$.

The free module functor is by definition

$$A \otimes_R - : {}_R\mathbb{M} \to {}_A\mathbb{M},\ X \mapsto (A \otimes_R X, m \otimes I_X),$$

and is left adjoint to the functor $U_A : {}_A\mathbb{M} \to {}_R\mathbb{M}$, forgetting the A-module structure, by the bijection, for $M \in {}_A\mathbb{M}$, $X \in {}_R\mathbb{M}$,

$$\mathrm{Hom}_A(A \otimes_R X, M) \to \mathrm{Hom}_R(X, M), \quad f \mapsto f \cdot (e \otimes I_X).$$

4.2**. Coalgebras and comodule categories.** An R-*coalgebra* is a triple (C, Δ, ε) with an R-module C, a coassociative coproduct $\Delta : C \to C \otimes_R C$, and a counit $\varepsilon : C \to R$. Then

$$C \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M},\ X \mapsto C \otimes_R X,$$

is an endofunctor and $(C \otimes_R -, \Delta \otimes I, \varepsilon \otimes I)$ forms a comonad on ${}_R\mathbb{M}$.

Now *(left) C-comodules* are defined as R-modules M with R-linear maps

$$\varrho^M : M \to C \otimes_R M,$$

satisfying the coassociativity and counitality conditions from 1.13.

Morphisms $g : M \to N$ of C-comodules are defined as R-linear maps subject to the compatibility condition formulated in 1.13. These notions define the category of (left) C-comodules denoted by ${}^C\mathbb{M}$ ($=\mathbb{M}^{C\otimes_R -}$).

The *free comodule functor* is defined as

$$C \otimes_R - : {}_R\mathbb{M} \to {}^C\mathbb{M}, \; X \mapsto (C \otimes_R X, \delta \otimes I_X),$$

and is right adjoint to the forgetful functor $U^C : {}^C\mathbb{M} \to {}_R\mathbb{M}$ by the bijection

$$\mathrm{Hom}^C(M, C \otimes_R X) \to \mathrm{Hom}_R(M, X), \quad f \mapsto \varepsilon_X \cdot f,$$

for $X \in {}_R\mathbb{M}$, $M \in {}^C\mathbb{M}$.

Notice that *coalgebras* in module categories are (usually) defined for a tensor functor whereas *comonads* may be defined for any endofunctor as will be the case in our next topic.

4.3. **Adjoint endofunctors.** By 3.1, for any R-module A, the two endofunctors $A \otimes_R -$ and $\mathrm{Hom}_R(A, -)$ form an adjoint pair by the canonical isomorphism, for any $X, Y \in {}_R\mathbb{M}$,

$$\varphi_{X,Y} : \mathrm{Hom}_R(A \otimes_R X, Y) \longrightarrow \mathrm{Hom}_R(X, \mathrm{Hom}_R(A, Y)).$$

As pointed out in 1.16, *the following are equivalent for* $A \in {}_R\mathbb{M}$ *:*

(a) $A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ *is a monad* *(A is an R-algebra);*

(b) $\mathrm{Hom}_R(A, -) : {}_R\mathbb{M} \to {}_R\mathbb{M}$ *is a comonad (with coproduct* $\mathrm{Hom}(m, -)$*).*

Moreover, there is an isomorphism between the corresponding module and comodule categories

$$T : {}_A\mathbb{M} \longrightarrow \mathbb{M}^{\mathrm{Hom}_R(A,-)}$$

given by the assignments

$$A \otimes_R N \xrightarrow{\rho_N} N \quad \longmapsto \quad N \xrightarrow{\nu_N} \mathrm{Hom}_R(A, A \otimes_R N) \xrightarrow{[A,\rho_N]} \mathrm{Hom}_R(A, N);$$

$$N \xrightarrow{\rho^N} \mathrm{Hom}_R(A, N) \quad \longmapsto \quad A \otimes_R N \xrightarrow{I\otimes\rho^N} A \otimes_R \mathrm{Hom}_R(A, N) \xrightarrow{\varepsilon_N} N.$$

This equivalence shows that the theory of A-modules over an R-algebra A could be build up completely as a theory of comodules over the comonad $\mathrm{Hom}_R(A, -)$. In general, the latter need not be representable as a tensor product unless A is finitely generated and projective (see Proposition 3.2). In the latter case 4.3 takes the following form.

4.4. $\mathrm{Hom}_R(A, -)$ **as tensor functor.** Let A be an R-algebra with finite dual R-module basis $\{a_\lambda\}_\Lambda$, $\{a^*_\lambda\}_\Lambda$. Then $\mathrm{Hom}_R(A, -) \simeq A^* \otimes_R -$ and as a consequence $(A \otimes_R -, A^* \otimes_R -)$ is an adjoint pair of endofunctors with

counit $\varepsilon : A \otimes_R A^* \to R, \quad a \otimes f \mapsto f(a),$

unit $\eta : R \to A^* \otimes_R A, \quad 1 \mapsto \sum_\Lambda a^*_\lambda \otimes a_\lambda.$

By the observations in 1.16 the following are equivalent:

(a) $A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ *is a monad (A is an R-algebra);*

(b) $A^* \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ *is a comonad (A^* is an R-coalgebra, with counit* $A^* \to R,\ f \mapsto f(1_A)$*).*

Moreover, the categories ${}_A\mathbb{M}$ and ${}^{A^*}\mathbb{M}$ are equivalent by the assignments

$$\begin{array}{rcl} A\otimes_R N \xrightarrow{\rho_N} N & \longmapsto & N \xrightarrow{\eta_N} A^*\otimes_R A\otimes_R N \xrightarrow{A^*\otimes\rho_N} A^*\otimes_R N; \\ N \xrightarrow{\rho^N} A^*\otimes_R N & \longmapsto & A\otimes_R N \xrightarrow{I\otimes\rho^N} A\otimes_R A^*\otimes_R N \xrightarrow{\varepsilon_N} N. \end{array}$$

We now look at adjoints of the comonads induced by coalgebras. By our general knowledge about adjoint endofunctors we have (see 1.16):

4.5. **Proposition.** *For any R-module C, the following are equivalent:*

(a) $C\otimes_R - : {}_R\mathbb{M}\to {}_R\mathbb{M}$ *has a comonad structure (C is an R-coalgebra);*

(b) $\mathrm{Hom}_R(C,-) : {}_R\mathbb{M}\to {}_R\mathbb{M}$ *has a monad structure.*

The corresponding module and comodule categories have distinct properties.

4.6. **Proposition.** *Let (C,Δ,ε) be an R-coalgebra, put $[C,-] := \mathrm{Hom}_R(C,-)$.*

(1) *Consider the category ${}^C\mathbb{M}$ of left C-comodules.*
   - (i) *${}^C\mathbb{M}$ has colimits, coproducts and cokernels;*
   - (ii) *${}^C\mathbb{M}$ is abelian provided C_R is flat;*
   - (iii) *in ${}^C\mathbb{M}$ monomorphisms need not be injective maps.*

(2) *Consider the category $\mathbb{M}_{[C,-]}$ of $\mathrm{Hom}_R(C,-)$-modules.*
   - (i) *$\mathbb{M}_{[C,-]}$ has limits, products and kernels;*
   - (ii) *$\mathbb{M}_{[C,-]}$ is abelian provided C_R is projective;*
   - (iii) *in $\mathbb{M}_{[C,-]}$ epimorphisms need not be surjective maps.*

From this it is clear that the two categories need not be isomorphic. There is however a functor between them which allows for a left adjoint.

4.7. **Correspondence of categories.** *With the notations above, there is a functor*

$$\mathrm{Hom}^C(C,-) : {}^C\mathbb{M}\to \mathbb{M}_{[C,-]}, \quad M\mapsto \mathrm{Hom}^C(C,M).$$

This has a left adjoint $C\otimes_{[C,-]} -$ (contratensor product) and, for $X\in {}_R\mathbb{M}$,

$$\begin{array}{rcl} C\otimes_R X & \mapsto & \mathrm{Hom}^C(C, C\otimes_R X)\simeq \mathrm{Hom}_R(C,X), \\ \mathrm{Hom}_R(C,X) & \mapsto & C\otimes_{[C,-]} \mathrm{Hom}_R(C,X)\simeq C\otimes_R X. \end{array}$$

The *Kleisli category* of a monad F (comonad G) on any category $\mathbb{A}$ (see [10]) may be characterised as full subcategory $\widetilde{\mathbb{A}}_F\subset \mathbb{A}_F$ (resp. $\widetilde{\mathbb{A}}^G\subset\mathbb{A}^G$) whose objects are of the form $F(A)$ (resp. $G(A)$) for some $A\in\mathbb{A}$.

With these notions, 4.7 says that the Kleisli category of the monad $\mathrm{Hom}_R(C,-)$ is equivalent to the Kleisli category of the comonad $C\otimes_R -$ on $\mathbb{M}_R$ (see [4]).

Given an R-algebra A, the dual module A^* allows for a coalgebra structure provided A is finitely generated and projective as an R-module. The situation for coalgebras is different.

4.8. **Algebra structure on C^*.** Let (C,Δ,ε) be an R-coalgebra. Then the dual module $C^* = \mathrm{Hom}_R(C,R)$ has an R-algebra structure with convolution product, for $f,g\in C^*$, $f*g = (g\otimes f)\circ\Delta$, and unit ε. (The reader should be aware that this may also be defined as $(f\otimes g)\circ\Delta$ leading to the opposite ring structure on C^*).

4.9**. Monads and comonads for coalgebras.** Let (C, Δ, ε) be an R-coalgebra. By the algebra structure of C^*, $C^* \otimes_R -$ is a monad on $\mathbb{M}_R$ and so is $[C, -] := \mathrm{Hom}_R(C, -)$. By 3.1, these are related by a natural transformation, for $M \in {}_R\mathbb{M}$,

$$\beta_M : C^* \otimes_R M \to \mathrm{Hom}_R(C, M), \quad f \otimes m \mapsto [c \mapsto f(c)m].$$

It turns out that this is a monad morphism thus yielding a functor

$$\begin{array}{rcl} \mathbb{M}_{[C,-]} & \longrightarrow & {}_{C^*}\mathbb{M}, \\ \mathrm{Hom}_R(C, M) \xrightarrow{h} M & \longmapsto & C^* \otimes_R M \xrightarrow{\beta_M} \mathrm{Hom}_R(C, M) \xrightarrow{h} M, \end{array}$$

which is an equivalence if and only if ${}_RC$ is finitely generated and projective (see Proposition 3.2).

On the other hand, the R-algebra C^* induces a comonad $\mathrm{Hom}_R(C^*, -)$ on ${}_R\mathbb{M}$ that is related with $C \otimes_R -$ by a natural transformation (see 3.1)

$$\alpha_M : C \otimes_R M \to \mathrm{Hom}_R(C^*, M), \quad c \otimes m \mapsto [f \mapsto f(c)m].$$

It is straightforward to see that this is a comonad morphism and with the functor T from 4.3 we get a functor

$$\begin{array}{rcl} {}^C\mathbb{M} & \longrightarrow & \mathbb{M}^{(C^*,-)} \xrightarrow{T^{-1}} {}_{C^*}\mathbb{M}, \\ M \xrightarrow{\varrho^M} C \otimes_R M & \longmapsto & M \xrightarrow{\varrho^M} C \otimes_R M \xrightarrow{\alpha_M} \mathrm{Hom}_R(C^*, M) \\ & \longmapsto & C^* \otimes_R M \xrightarrow{I \otimes \varrho^M} C^* \otimes_R C \otimes_R M \xrightarrow{ev \otimes I} M, \end{array}$$

where $ev : C^* \otimes_R C \to R$ denotes the evaluation map.

This functor is full and faithful if and only if α_M is injective for any $M \in {}_R\mathbb{M}$. In this context, the injectivity condition for α is known as *α-condition* and just means that C is locally projective as an R-module (see 3.3). Since C is a subgenerator in ${}^C\mathbb{M}$ (every C-comodule is a subcomodule of a C-generated comodule), the α-condition implies that

$${}^C\mathbb{M} \simeq \sigma[{}_{C^*}C] \subseteq {}_{C^*}\mathbb{M},$$

where the right hand side denotes the full subcategory whose objects are subgenerated by the C^*-module C and these are precisely the *rational C^*-modules* (e.g. [6], [16]). Of course, equality holds on the right side if and only if ${}_RC$ is finitely generated and projective.

Clearly, if R is a field, C satisfies the α-condition and - as pointed out by Abrams and Weibel in [2] - C^* can be seen as a pro-object in the category of finite-dimensional algebras. In this case the isomorphism given coincides with [2, Theorem 4.3].

The constructions of this section can also be formulated in a general categorical context and are investigated in [12].

Recall that a *Frobenius algebra* over a field k is defined as a finite dimensional k-algebra A for which $A \simeq A^* = \mathrm{Hom}_k(A, k)$ as (left) A-modules. This notion can be readily extended to algebras over rings and from Proposition 1.17 we immediately obtain:

4.10**. Proposition.** *For an R-algebra (A, m, e), the following are equivalent:*

(a) $A \otimes_R -$ *is a Frobenius monad on* ${}_R\mathbb{M}$*;*

(b) *A is finitely generated and projective as R-module and allows for a coalgebra structure (A, δ, ε) inducing commutativity of the diagrams*

$$\begin{tikzcd} A \otimes_R A \arrow[r, "I_A \otimes \delta"] \arrow[d, "m"'] & A \otimes_R A \otimes_R A \arrow[d, "m \otimes I_A"] \\ A \arrow[r, "\delta"] & A \otimes_R A, \end{tikzcd} \qquad \begin{tikzcd} A \otimes_R A \arrow[r, "\delta \otimes I_A"] \arrow[d, "m"'] & A \otimes_R A \otimes_R A \arrow[d, "I_A \otimes m"] \\ A \arrow[r, "\delta"] & A \otimes_R A; \end{tikzcd}$$

(c) *A has a comonad structure (A, δ, ε) and $(A \otimes_R -, A \otimes_R -)$ is an adjoint pair of functors with counit and unit*

$$A \otimes_R A \xrightarrow{m} A \xrightarrow{\varepsilon} R, \quad R \xrightarrow{e} A \xrightarrow{\delta} A \otimes_R A.$$

For algebras over fields, the equivalence of (a) and (b) was shown in 1999 by Abrams (see [1, Theorem 2.1]). He understood the two diagrams in (b) as conditions which make δ a left and right A-module morphism. This interpretation is not possible in general, since for a monad F on an arbitrary category $\mathbb{A}$, one can not distinguish between left and right F-modules. From the categorical setting, the left hand diagram should be seen as condition to make δ an A-module morphism whereas the right hand diagram makes m an A-comodule morphism.

We note that the proof of the general situation only uses elementary notions from category theory and is completely contained (although not formulated in this way) in Eilenberg-Moore [9]. Thus Abrams theorem from 1999 was implicitly already around in more generality in 1965.

4.11. **Remark.** For most of the results of this section commutativity of the ring R is not essential. In case R is not commutative, some obvious adaptions have to be made; in particular, R-algebras are to be replaced by *R-rings* (monads on ${}_R\mathbb{M}$) and R-coalgebras by *R-corings* (comonads on ${}_R\mathbb{M}$) (see [6]).

With the knowledge gained in this section let us come back to the monad and comonad on ${}_S\mathbb{M}$ and ${}_R\mathbb{M}$, respectively, considered in Proposition 3.5. In the next proposition R need not necessarily be commutative.

4.12. $\mathrm{Hom}_R(P, -)$ **as tensor functor.** Let ${}_RP_S$ be an (R, S)-bimodule with finite dual R-module basis $\{p^*_\lambda\}_\Lambda$, $\{p_\lambda\}_\Lambda$. Then $P^* \otimes_R - \simeq \mathrm{Hom}_R(P, -)$ and thus, by 3.1, $(P \otimes_R -, P^* \otimes_R -)$ is an adjoint pair of functors with

counit $\varepsilon : P \otimes_R P^* \to R, \quad p \otimes f \mapsto f(p),$

unit $\eta : R \to P^* \otimes_R P, \quad 1 \mapsto \sum_\Lambda p^*_\lambda \otimes p_\lambda.$

(1) $P^* \otimes_R P \otimes_S -$ is a monad on ${}_S\mathbb{M}$ (S-ring) and there is an isomorphism

$$P^* \otimes_R P \xrightarrow{\simeq} End_R(P), \quad f \otimes p \mapsto [x \mapsto f(x)p].$$

This induces a product $\underline{m}$ on $P^* \otimes_R P$, for $f, g \in P^*$, $p, q \in P$,

$$\underline{m} : (g \otimes q) \otimes (f \otimes p) \mapsto g \otimes f(q)p.$$

(2) $P \otimes_S P^* \otimes_R -$ is a comonad on ${}_R\mathbb{M}$ (R-coring) with coproduct

$$\underline{\delta} : P \otimes_S P^* \to P \otimes_S P^* \otimes_R P \otimes_S P^*, \quad p \otimes f \mapsto \sum_\Lambda p \otimes p^*_\lambda \otimes p_\lambda \otimes f.$$

For a commutative ring R, any R-module P can be considered as (R, R)-bimodule and the canonical isomorphism $\mathsf{tw}_{P,P^*} : P \otimes_R P^* \to P^* \otimes_R P$ allows to transfer the comonad structure defined on $P \otimes_R P^*$ to $P^* \otimes_R P \simeq \mathrm{End}_R(P)$ (keeping the same symbol).

4.13. $\mathrm{End}_R(P)$ **as Frobenius algebra.** *Let P be a finitely generated and projective module over a commutative ring R. Then $\mathrm{End}_R(P)$ is a Frobenius R-algebra.*

Proof. We refer to the notation from 4.12. The left hand diagram in 4.10 comes out as (writing $\otimes$ for $\otimes_R$)

$$\begin{array}{ccc} P^*\otimes P\otimes P^*\otimes P & \xrightarrow{I\otimes\underline{\delta}} & P^*\otimes P\otimes P^*\otimes P\otimes P^*\otimes P \\ \downarrow{\scriptstyle \underline{m}} & & \downarrow{\scriptstyle \underline{m}\otimes I} \\ P^*\otimes P & \xrightarrow{\underline{\delta}} & P^*\otimes P\otimes P^*\otimes P \end{array}$$

and this is commutative, since for $g,f\in P^*$ and $p,q\in P$,

$$\begin{array}{rcl} (\underline{m}\otimes I)\cdot(I\otimes\underline{\delta})[g\otimes q\otimes f\otimes p] &=& \sum_\Lambda g\otimes f(q)p_\lambda\otimes p^*_\lambda\otimes p, \\ (\underline{\delta}\cdot\underline{m})[g\otimes q\otimes f\otimes p] &=& \sum_\Lambda g\otimes p_\lambda\otimes p^*_\lambda\otimes f(q)p, \end{array}$$

and since $f(q)\in R$, the two expressions are equal.

Similarly one can see that the right hand diagram in 4.10 is commutative. □

Putting $P=R^n$ in 4.13, for any $n\in\mathbb{N}$, $\mathrm{End}_R(P)$ is just the $n\times n$-matrix ring over R and we retrieve the (known) fact that finite matrix rings over commutative rings are Frobenius algebras. In this case the counit $\varepsilon: P^*\otimes_R P\to R$ is just the trace of the matrix.

A further application of 4.12 for not necessarily commutative rings yields

4.14. **Sweedler's coring** [14]. Let $(A,m,1_A)$ be any ring and $h:R\to A$ a ring homomorphism. Then the (A,R)-bimodule A is finitely generated and projective as left A-module and $A\otimes_R A$ becomes an A-coring with coproduct and counit

$$\begin{array}{rlll} \delta: A\otimes_R A & \to A\otimes_R A\otimes_R A, & a\otimes b\mapsto\ a\otimes 1_A\otimes b\,, \\ \varepsilon=m: A\otimes_R A & \to A, & a\otimes b\mapsto ab\,, \end{array}$$

where we have identified $A\otimes_A A\simeq A$.

For more details on (Sweedler's) corings the reader is referred to [6].

5. Tensor product of algebras and coalgebras

In this section R denotes again a commutative ring and hence we will have the canonical twist map tw for R-modules (see Section 4).

5.1. **Tensor product of algebras.** Consider two R-algebras (A,m_A,e_A) and (B,m_B,e_B). Given any R-linear map

$$\tau: B\otimes_R A\to A\otimes_R B,$$

the tensor product $A\otimes_R B$ allows for a product

$$m_{AB}:\ A\otimes_R B\otimes_R A\otimes_R B\xrightarrow{I\otimes\tau\otimes I} A\otimes_R A\otimes_R B\otimes_R B\xrightarrow{m_A\otimes m_B} A\otimes_R B.$$

Taking $\tau=\mathsf{tw}_{B,A}$, the resulting product is associative and $A\otimes_R B$ becomes an R-algebra with unit $e_A\otimes e_B$.

The question arises if there are other such linear maps τ making $A\otimes_R B$ an R-algebra. This was intensively studied for algebras and led to various notions of smash products. The reader is referred to [7] (and the references given there) for investigations in this direction.

Considering the related functors $A \otimes_R -$ and $B \otimes_R -$, the lifting conditions from 2.1 are concerned with the diagram

$$\begin{tikzcd} {}_B\mathbb{M} \arrow[r, "\overline{A\otimes_R -}"] \arrow[d, "U_B"'] & {}_B\mathbb{M} \arrow[d, "U_B"] \\ {}_R\mathbb{M} \arrow[r, "A\otimes_R -"] & {}_R\mathbb{M}, \end{tikzcd}$$

and the conditions on $\tau \otimes - : B \otimes_R A \otimes_R - \to A \otimes_R B \otimes_R -$ to be a monad entwining (given in 2.2) are precisely the conditions on τ to provide $A \otimes_R B$ with an R-algebra structure.

For a long time this was hardly realised by algebraists and the conditions needed had to be rediscovered in the particular cases.

5.2. **Tensor product of coalgebras.** Consider two R-coalgebras $(C, \delta_C, \varepsilon)$ and $(D, \delta_D, \varepsilon_D)$. Dual to the algebra case, to obtain a coproduct on the tensor product $C \otimes_R D$ (smash coproduct, e.g. [7]), an R-linear map (in the opposite direction)

$$\tau : C \otimes_R D \to D \otimes_R C$$

is needed to define a coproduct

$$\delta_{CD} : \; C \otimes_R D \xrightarrow{\delta_C \otimes \delta_D} C \otimes_R D \otimes_R D \otimes_R D \xrightarrow{I \otimes \tau \otimes I} C \otimes_R D \otimes_R C \otimes_R D.$$

Again the twist map $\mathsf{tw}_{C,D}$ has the necessary properties to make $C \otimes_R D$ a coassociative R-coalgebra with counit $\varepsilon_C \otimes \varepsilon_D$.

In functorial language, the problem is to find a comonad structure for the composition $C \otimes_R D \otimes_R -$ of the comonads $C \otimes_R -$ and $D \otimes_R -$. This defines the *comonad entwinings* which were mentioned in a remark after 2.2.

5.3. **Tensor product of algebras and coalgebras.** While for algebras and coalgebras it is clear which structures are to be expected from the corresponding tensor products (see 5.1, 5.2), a new structure is needed to describe the product of an algebra and a coalgebra. This was worked out by Breziński and Majid by introducing *entwining structures* in the paper [5] appearing 1998, a notion which turned out to be closely related to comonads (corings). In categorical language this is described by lifting monads to comodule categories as outlined in 2.3 (mixed entwinings) and this goes back to van Osdol's paper [15] from 1971.

Consider an R-algebra (A, m, e) and an R-coalgebra (C, δ, ε). An R-linear map

$$\lambda : A \otimes_R C \to C \otimes_R A$$

is called a *mixed entwining* provided all the diagrams in 2.3 are commutative for $T = A \otimes_R -$ and $G = C \otimes_R -$. Again $\lambda := \mathsf{tw}_{AC}$ provides an example for such a map.

If this holds, then λ leads to a left A-module structure on $C \otimes_R A$,

$$A \otimes_R C \otimes_R A \xrightarrow{\lambda \otimes I} C \otimes_R A \otimes_R A \xrightarrow{I \otimes m} C \otimes_R A,$$

and the coproduct and counit

$$\underline{\delta} : \; C \otimes_R A \xrightarrow{\delta \otimes I} C \otimes_R C \otimes_R A \xrightarrow{\simeq} C \otimes_R A \otimes_A C \otimes_R A, \quad \underline{\varepsilon} : C \otimes_R A \xrightarrow{\varepsilon \otimes I} A,$$

induce a comonad structure on the endofunctor (A-coring)

$$C \otimes_R A \otimes_A - : {}_A\mathbb{M} \to {}_A\mathbb{M}.$$

The comodules for this comonad are also known as $(C,A)_\lambda$-*entwined modules* (see [6, Section 32]).

Similar to the case of R-algebras (see 4.9), the comodules for $C\otimes_R A\otimes_A -$ may also be seen as modules over some ring, which is obtained by endowing the (A,A)-bimodule $\mathrm{Hom}_A(C\otimes_R A, A)$ with a suitable ring structure (see [6, 32.9]).

5.4. **Bialgebras and Hopf algebras.** Let $\mathbf{B}=(B,m,e,\delta,\varepsilon)$ be a quintuple where B is an R-module, (B,m,e) an R-algebra, and (B,δ,ε) an R-coalgebra.

Following 2.4, $\mathbf{B}$ is called a bimonad provided $\varepsilon: B\to R$ is an algebra morphism, $e: R\to B$ is a coalgebra morphism, and there is a mixed entwining

$$\lambda: B\otimes_R B\to B\otimes_R B,$$

that is, the diagrams in 2.3 are commutative for $T=G=B\otimes_R -$, and λ yields commutativity of the diagram in 2.4(iii).

As mentioned in 5.1 and 5.2, $B\otimes_R B$ allows for an algebra and a coalgebra structure induced by the twist map tw, and in this case the compatibility condition for product and coproduct usually is to require that m is an R-coalgebra morphism (equivalently δ is an R-algebra morphism).

The two conditions are brought together in the diagram (writing $\otimes$ for $\otimes_R$)

$$\begin{tikzcd}
B\otimes B \arrow[r, "m"] \arrow[d, "I\otimes\delta"'] & B \arrow[r, "\delta"] & B\otimes B \\
B\otimes B\otimes B \arrow[rr, "\lambda\otimes B"] \arrow[d, "\delta\otimes I\otimes I"'] & & B\otimes B\otimes B \arrow[u, "I\otimes m"'] \\
B\otimes B\otimes B\otimes B \arrow[rr, "I\otimes\mathsf{tw}\otimes I"] & & B\otimes B\otimes B\otimes B, \arrow[u, "m\otimes I\otimes I"']
\end{tikzcd}$$

where commutativity of the upper rectangle is the condition from 2.4(iii), while commutativity of the outer diagram means that m is a coalgebra morphism (or δ is an algebra morphism).

Thus the two compatibility conditions coincide if we choose a specific λ which makes the bottom rectangle commutative,

$$\lambda: B\otimes_R B\xrightarrow{\delta\otimes I_B} B\otimes_R B\otimes_R B\xrightarrow{I_B\otimes\mathsf{tw}} B\otimes_R B\otimes_R B\xrightarrow{m\otimes I_B} B\otimes_R B,$$

and it can be shown that this λ is a mixed entwining (see [11, Proposition 6.3]). Of course, not every mixed entwining has to be of this form.

It follows from 5.3 that for a bialgebra $\mathbf{B}$, a mixed entwining induces a comonad $B\otimes_R B\otimes_B - : {}_B\mathbb{M}\to {}_B\mathbb{M}$ (B-coring) and the comodules for this (entwined modules) coincide with the Hopf modules for the bimonad $\mathbf{B}$ as defined in 2.5 (see [6, 33.1]).

$\mathbf{B}$ is a Hopf algebra if (and only if) the comparison functor

$$K_B: {}_R\mathbb{M}\to \mathbb{M}_B^B,\quad X\mapsto (B\otimes_R X, m\otimes I_X, \delta\otimes I_X),$$

induces an equivalence. As pointed out in 2.5, K_B is full and faithful and hence to get an equivalence one has to find a (right) adjoint functor which is also full and faithful. This is, for example, the case when the functor $\mathrm{Hom}_B^B(B,-):\mathbb{M}_B^B\to {}_R\mathbb{M}$ is full and faithful. More characterisations of these conditions are given in [6, 15.5].

It follows from 2.5 that an R-module B is a Hopf algebra if and only if $B\otimes_R -$, or equivalently the adjoint functor $\mathrm{Hom}_R(B,-)$, is a Hopf monad. Thus for an infinite dimensional Hopf algebra B over a field k, the dual space $\mathrm{Hom}_k(B,k)$ need not be a Hopf algebra but the (adjoint) functor $\mathrm{Hom}_k(B,-)$ is a Hopf monad on the category of k-vector spaces.

We have seen how the tensor product of two (co-) algebras can be made to (co-) algebras. Given three algebras A, B and C with mutual entwinings, that is $A\otimes_R B$, $A\otimes_R C$ and $B\otimes_R C$ have algebra structures, the question arises if $A\otimes_R B\otimes_R C$ also allows for an algebra structure. To achieve this the entwinings involved have to satisfy a kind of compatibility condition.

5.5. **Yang-Baxter equation.** Let A, B, C be R-modules with R-linear maps

$$\varphi_{BC}: B\otimes_R C\to C\otimes_R B,\quad \varphi_{AB}: A\otimes_R B\to B\otimes_R A,\quad \varphi_{AC}: A\otimes_R C\to C\otimes_R A.$$

The triple $(\varphi_{BC},\varphi_{AB},\varphi_{AC})$ is said to satisfy the *Yang-Baxter equation* if it yields commutativity of the diagram (writing $\otimes$ for $\otimes_R$)

$$\begin{array}{ccccc} A\otimes B\otimes C & \xrightarrow{\varphi_{AB}\otimes C} & B\otimes A\otimes C & \xrightarrow{B\otimes\varphi_{AC}} & B\otimes C\otimes A \\ {\scriptstyle A\otimes\varphi_{BC}}\downarrow & & & & \downarrow{\scriptstyle \varphi_{BC}\otimes A} \\ A\otimes C\otimes B & \xrightarrow{\varphi_{AC}\otimes B} & C\otimes A\otimes B & \xrightarrow{C\otimes\varphi_{AB}} & C\otimes B\otimes A. \end{array}$$

It is well-known that the twist map tw satisfies the Yang-Baxter equations for any R-modules A, B, C.

If A, B and C are R-algebras and the φ's are algebra entwinings, then a canonical R-algebra structure is induced on $A\otimes_R B\otimes_R C$ if and only if the entwinings satisfy the Yang-Baxter equation.

Similar conditions are needed to make the tensor product of three coalgebras or bialgebras again a coalgebra or a bialgebra, respectively. They also have to be satisfied if one requires the 3-fold (or n-fold) tensor product $A\otimes_R A\otimes_R A$ of an algebra (coalgebra, bialgebra) A to be of the same type again.

We mention that the conditions considered can also be formulated for tensor products over non-commutative rings and the related functors. For more details the reader is referred to [18].

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