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ON GALOIS COMODULES

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Generalizing the notion of Galois corings, Galois comodules were introduced as comodules P over an A-coring $\mathcal{C}$ for which P_A is finitely generated and projective and the evaluation map $\mu_{\mathcal{C}} : \mathbf{Hom}^{\mathcal{C}}(P, \mathcal{C}) \otimes_S P \to \mathcal{C}$ is an isomorphism (of corings) where $S = \mathbf{End}^{\mathcal{C}}(P)$. It has been observed that for such comodules the functors $- \otimes_A \mathcal{C}$ and $\mathbf{Hom}_A(P, -) \otimes_S P$ from the category of right A-modules to the category of right $\mathcal{C}$-comodules are isomorphic. In this note we use this isomorphism related to a comodule P to define Galois comodules without requiring P_A to be finitely generated and projective. This generalises the old notion with this name but we show that essential properties and relationships are maintained. Galois comodules are close to being generators and have common properties with tilting (co)modules. Some of our results also apply to generalised Hopf Galois (coalgebra Galois) extensions.

Key Words: Adjoint functors; Galois comodules; Hopf Galois extensions; Static comodules; Strongly $(\mathcal{C}, A)$-injective (equivariantly injective) comodules.

AMS Classification: 16W30; 16D90; 16D80.

1. INTRODUCTION

Let $\mathcal{C}$ be a coring over the ring A and put $S = \mathrm{End}^{\mathcal{C}}(A)$. A grouplike element $g \in \mathcal{C}$ makes A a right $\mathcal{C}$-comodule by the coaction $\varrho^A : A \to A \otimes_A \mathcal{C},\ a \mapsto 1 \otimes ga$. The notion of *Galois corings* $(\mathcal{C}, g)$ was introduced in Brzeziński (2002) by requiring the canonical map,

$$\chi : A \otimes_S A \to \mathcal{C}, \qquad a \otimes a' \mapsto aga',$$

to be an isomorphism (of corings). It was pointed out in Wisbauer (2004) that this can be seen as the evaluation map

$$\mu_{\mathcal{C}} : \mathrm{Hom}^{\mathcal{C}}(A_g, \mathcal{C}) \otimes_S A \to \mathcal{C},$$

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and that it implies bijectivity of

$$\mu_N : \mathrm{Hom}^{\mathscr{C}}(A_g, N) \otimes_S A \to N,$$

for every $(\mathscr{C}, A)$-injective comodule N.

The notion of Galois corings was extended to comodules by El Kaoutit and Gómez-Torrecillas (2003), where to any bimodule ${}_SP_A$ with P_A finitely generated and projective, a coring $P^* \otimes_S P$ was associated and it was shown that the canonical map

$$\tilde{\mu}_A : \mathrm{Hom}_A(P, A) \otimes_S P \to \mathscr{C}$$

is a coring morphism provided P is also a right $\mathscr{C}$-comodule and $S = \mathrm{End}^{\mathscr{C}}(P)$. In Brzeziński and Wisbauer (2003, 18.25) such comodules P were termed *Galois comodules* provided $\tilde{\mu}_A$ was bijective, and it was proven in Brzeziński and Wisbauer (2003, 18.26) that this condition implies that the functors $\mathrm{Hom}_A(P, -) \otimes_S P$ and $- \otimes_A \mathscr{C}$ from the right A-modules to the right $\mathscr{C}$-comodules are isomorphic.

In a recent article, Brzeziński (2005) further investigated these Galois comodules and pointed out their relevance for *descent theory*, *vector bundles*, and *noncommutative geometry*. Related questions are, for example, also considered by Caenepeel et al. (to appear). In this note we concentrate on comodule properties and we want to free the notion from the condition that P_A has to be finitely generated and projective. This is done by taking the above mentioned isomorphism of functors as definition. Although some symmetry is lost those properties which to us seem to be essential, are preserved. From this point of view Galois comodules are somehow similar to tilting (co)modules, or modules M for which all M-generated modules are M-static: they all share the property that they are generators in their respective categories provided they are flat over their endomorphism rings. Hence the presentation is partly motivated by Wisbauer (1998, 2000) on tilting and static modules. Some results from Brzeziński (2005) and Caenepeel et al. (to appear) are obtained in a more general setting (e.g., 5.10, 5.8).

Relative injectivity of comodules is of special interest in the context of our investigations and leads to category equivalences. In particular, a strongly $(\mathscr{C}, A)$-injective (equivariantly injective) Galois comodule P that is finitely generated and projective as A-module, induces an equivalence between the category of comodules and the $\mathrm{End}^{\mathscr{C}}(P)$-modules (see 5.7).

Given a commutative ring R, an *entwining structure* (A, C, ψ) consists of an R-algebra A, an R-coalgebra C, and an R-linear map $\psi : C \otimes_R A \to A \otimes_R C$ satisfying certain compatibility conditions which ensure that $A \otimes_R C$ allows for an A-coring structure (see Brzeziński and Wisbauer, 2003, Section 32). If A is a right $A \otimes_R C$-comodule (equivalently, there exists a grouplike element in $A \otimes_R C$), then $\mathrm{Hom}^{A \otimes_R C}(A, A \otimes_R C) \simeq A$ and A is a Galois $A \otimes_R C$-comodule if and only if A is a C-Galois extension over $\mathrm{End}^{A \otimes_R C}(A) = A^{coC}$ (see Brzeziński and Wisbauer, 2003, 34.10). Hence a number of results on generalized Hopf Galois (coalgebra Galois) extensions in Schauenburg and Schneider (2005) can be seen as special cases of our results (see 5.10).

The symmetry for Galois comodules that are finitely generated and projective A-modules mentioned above can be maintained for comodules which are direct sums

of comodules of this type. In this context, the *infinite comatrix corings*, as introduced by El Kaoutit and Gómez-Torrecillas (2004), find a natural application (Section 6).

2. PRELIMINARIES

Throughout we will essentially follow the notation in Brzeziński and Wisbauer (2003). For convenience we recall some basic notions.

2.1. Corings. Let A be an associative ring with unit and $\mathcal{C}$ an A-coring with coproduct and counit

$$\Delta : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}, \qquad \varepsilon : \mathcal{C} \to A.$$

Associated to this, there are the right and left dual rings $\mathcal{C}^* = \mathrm{Hom}_A(\mathcal{C}, A)$ and ${}^*\mathcal{C} = {}_A\mathrm{Hom}(\mathcal{C}, A)$ with the convolution products.

2.2. Comodules. A right A-module M is a right $\mathcal{C}$-comodule provided there is an A-linear $\mathcal{C}$-coaction

$$\varrho^M : M \to M \otimes_A \mathcal{C}, \qquad \text{written as } \varrho^M(m) = \sum m_{\underline{0}} \otimes m_{\underline{1}} \text{ for } m \in M,$$

satisfying the coassociativity and counital condition.

We denote the category of right A-modules by $\mathbf{M}_A$ and the category of right $\mathcal{C}$-comodules by $\mathbf{M}^{\mathcal{C}}$. The corresponding left versions are denoted by ${}_A\mathbf{M}$ and ${}^{\mathcal{C}}\mathbf{M}$, respectively. The category $\mathbf{M}^{\mathcal{C}}$ is additive, has coproducts and cokernels, and epimorphisms are surjective maps. The functor $- \otimes_A \mathcal{C} : \mathbf{M}_A \to \mathbf{M}^{\mathcal{C}}$ is right adjoint to the forgetful functor by the isomorphisms, for $M \in \mathbf{M}^{\mathcal{C}}$, $X \in \mathbf{M}_A$,

$$\varphi : \mathrm{Hom}^{\mathcal{C}}(M, X \otimes_A \mathcal{C}) \to \mathrm{Hom}_A(M, X), \qquad f \mapsto (I_X \otimes \underline{\varepsilon}) \circ f,$$

with inverse map $h \mapsto (h \otimes I_{\mathcal{C}}) \circ \varrho^M$.

Notice that for any monomorphism (injective map) $f : X \to Y$ in $\mathbf{M}_A$, the colinear map $f \otimes I_{\mathcal{C}} : X \otimes_A \mathcal{C} \to Y \otimes_A \mathcal{C}$ is a monomorphism in $\mathbf{M}^{\mathcal{C}}$ but need not be injective. In case ${}_A\mathcal{C}$ is flat, monomorphisms in $\mathbf{M}^{\mathcal{C}}$ are injective maps and in this case $\mathbf{M}^{\mathcal{C}}$ is a Grothendieck category (see Brzeziński and Wisbauer, 2003, 18.14).

2.3. The Subcategory $\sigma[\mathbf{M}]$. Let $M \in \mathbf{M}^{\mathcal{C}}$. Homomorphic images of direct sums of copies of M are called *M-generated* comodules. The full subcategory of $\mathbf{M}^{\mathcal{C}}$, whose objects are subcomodules K of M-generated comodules N (i.e., there is an injective colinear map $K \to N$), is denoted by $\sigma[M]$. Notice that this does not imply that morphisms in $\sigma[M]$ have kernels unless ${}_A\mathcal{C}$ is flat.

Cokernels of morphisms $M^{(\Lambda')} \to M^{(\Lambda)}$, with any sets Λ', Λ, are called *M-presented comodules*. Notice that the image of the functor $- \otimes_S M : \mathbf{M}_S \to \mathbf{M}^{\mathcal{C}}$ lies in $\sigma[M]$, in fact, comodules of the form $X \otimes_S M$ with $X \in \mathbf{M}_S$ are M-presented.

2.4. The α-Condition. Defining the convolution product on ${}^*\mathcal{C} = {}_A\mathrm{Hom}(\mathcal{C}, A)$ as in Brzeziński and Wisbauer (2003), any right $\mathcal{C}$-comodule (M, ϱ^M) allows a left ${}^*\mathcal{C}$-module structure by putting $f \rightharpoonup m = (I_M \otimes f) \circ \varrho^M(m)$, for any $f \in {}^*\mathcal{C}$, $m \in M$.

This yields a faithful functor $\Phi : \mathbf{M}^{\mathscr{C}} \to {}_{*\mathscr{C}}\mathbf{M}$ which is a full embedding if and only if the map

$$\alpha_K : K \otimes_A \mathscr{C} \to \mathrm{Hom}_A({}^*\mathscr{C}, K), \qquad n \otimes c \mapsto [f \mapsto nf(c)],$$

is injective for any $K \in \mathbf{M}_A$. This holds if and only if ${}_A\mathscr{C}$ is locally projective and is called *left* α*-condition* on $\mathscr{C}$. In this case $\mathbf{M}^{\mathscr{C}}$ can be identified with $\sigma[{}_{*\mathscr{C}}\mathscr{C}]$, the full subcategory of ${}_{*\mathscr{C}}\mathbf{M}$ whose objects are subgenerated by $\mathscr{C}$.

Symmetrically the *right* α*-condition* is defined and if it holds ${}^{\mathscr{C}}\mathbf{M}$ can be identified with the category $\sigma[\mathscr{C}_{\mathscr{C}^*}]$ where $\mathscr{C}^* = \mathrm{Hom}_A(\mathscr{C}, A)$.

2.5. Morphism Groups. The comodule morphisms between $M, N \in \mathbf{M}^{\mathscr{C}}$ is characterised by the exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathrm{Hom}^{\mathscr{C}}(M, N) \to \mathrm{Hom}_A(M, N) \xrightarrow{\gamma} \mathrm{Hom}_A(M, N \otimes_A \mathscr{C}),$$

where $\gamma(f) = \varrho^N \circ f - (f \otimes I_{\mathscr{C}}) \circ \varrho^M$.

2.6. Cotensor Product. For two comodules $M \in {}^{\mathscr{C}}\mathbf{M}$ and $L \in {}^{\mathscr{C}}\mathbf{M}$ the cotensor product is defined by the exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow M \otimes^{\mathscr{C}} L \longrightarrow M \otimes_A N \xrightarrow{\omega_{M,L}} M \otimes_A \mathscr{C} \otimes_A L$$

where $\omega_{M,L} = \varrho^M \otimes I_L - I_M \otimes^L \varrho$.

2.7. M_A Finitely Generated Projective. Let $M \in \mathbf{M}^{\mathscr{C}}$ such that M_A is finitely generated and projective. Then for $M^* = \mathrm{Hom}_A(M, A)$ the map

$$\varphi : \mathscr{C} \otimes_A M^* \to \mathrm{Hom}_A(M, \mathscr{C}), \qquad c \otimes h \mapsto c \otimes h(-),$$

is an isomorphism and induces a left $\mathscr{C}$-comodule structure on M^* (see Brzeziński and Wisbauer, 2003, 19.19). With a dual basis $m_1, \ldots, m_n \in M$, $\pi_1, \ldots, \pi_n \in M^*$, the inverse map of φ is given by sending $g \in M^*$ to $\sum_i g(m_i) \otimes \pi_i$, and the coaction on M^* is

$$\varrho^{M^*} : M^* \to \mathscr{C} \otimes_A M^*, \qquad g \mapsto (g \otimes I_{\mathscr{C}})\varrho^M \mapsto \sum_i (g \otimes I_{\mathscr{C}})\varrho^M(m_i) \otimes \pi_i.$$

There is a canonical anti-isomorphism between ${}^{\mathscr{C}}\mathrm{End}(M^*)$ and $S = \mathrm{End}^{\mathscr{C}}(M)$ and by this M^* is a right S-module.

For any $N \in \mathbf{M}^{\mathscr{C}}$, there exists an isomorphism (natural in M)

$$N \otimes^{\mathscr{C}} M^* \xrightarrow{\simeq} \mathrm{Hom}^{\mathscr{C}}(M, N).$$

This follows from the proof of Brzeziński and Wisbauer (2003, 10.11). With the defining sequences for $\mathrm{Hom}^{\mathscr{C}}$ and $\otimes^{\mathscr{C}}$ we have the commutative diagram with

exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & N \otimes^{\mathscr{C}} M^* & \longrightarrow & N \otimes_A M^* & \xrightarrow{\omega_{N,M^*}} & N \otimes_A \mathscr{C} \otimes_A M^* \\ & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(M,N) & \longrightarrow & \mathrm{Hom}_A(M,N) & \xrightarrow{\gamma} & \mathrm{Hom}_A(M, N \otimes_A \mathscr{C}), \end{array}$$

where $\omega_{N,M^*} = \varrho^N \otimes I_{M^*} - I_N \otimes \varrho_{M^*}$ and $\gamma(f) := \varrho^N \circ f - (f \otimes I_{\mathscr{C}}) \circ \varrho^M$. From this diagram lemmata imply the existence and bijectivity of the required morphism.

Notice that this isomorphism is also proven in Caenepeel et al. (to appear, Proposition 2.4).

2.8. Cointegrals. An (A, A)-bilinear map $\delta : \mathscr{C} \otimes_A \mathscr{C} \to A$ is called a *cointegral in* $\mathscr{C}$ if

$$(I_{\mathscr{C}} \otimes \delta) \circ (\Delta \otimes I_{\mathscr{C}}) = (\delta \otimes I_{\mathscr{C}}) \circ (I_{\mathscr{C}} \otimes \Delta).$$

Cointegrals are characterised by the fact that for any $M \in \mathbf{M}^{\mathscr{C}}$, the map

$$v_M = (I_M \otimes \delta) \circ (\varrho^M \otimes I_{\mathscr{C}}) : M \otimes_A \mathscr{C} \to M$$

is a comodule morphism, or by the corresponding property for left $\mathscr{C}$-comodules.

This follows from the proof of Brzeziński and Wisbauer (2003, 3.29). In Caenepeel et al. (to appear, Section 5) these maps are related to the counit for the adjoint pair of functors $- \otimes_A \mathscr{C}$ and the forgetful functor. For R-coalgebras C over a commutative ring R with C_R locally projective, a cointegral is precisely a C^*-balanced R-linear map $C \otimes_R C \to R$ (e.g., Brzeziński and Wisbauer, 2003, 6.4).

2.9. Relative Injectivity. Let M be a right $\mathscr{C}$-comodule and $S = \mathrm{End}^{\mathscr{C}}(M)$.

M is $(\mathscr{C}, A)$-*injective* provided the structure map $\varrho^M : M \to M \otimes_A \mathscr{C}$ is split by a $\mathscr{C}$-morphism $\lambda : M \otimes_A \mathscr{C} \to M$.

We call M *strongly* $(\mathscr{C}, A)$-*injective* if this λ is $\mathscr{C}$-colinear and S-linear. Given a subring $B \subseteq S$, M is said to be B-*strongly* $(\mathscr{C}, A)$-*injective* if λ is $\mathscr{C}$-colinear and B-linear.

We call M *fully* $(\mathscr{C}, A)$-*injective* if there exists a cointegral $\delta_M : \mathscr{C} \otimes_A \mathscr{C} \to \mathscr{A}$ such that ϱ^M is split by $(I_M \otimes \delta_M) \circ (\varrho^M \otimes I_{\mathscr{C}})$.

The notions for left $\mathscr{C}$-comodules are defined symmetrically.

Obviously, fully $(\mathscr{C}, A)$-injective comodules are strongly $(\mathscr{C}, A)$-injective and for a B-strongly $(\mathscr{C}, A)$-injective comodule M and any $X \in \mathbf{M}_B$, $X \otimes_B M$ is $(\mathscr{C}, A)$-injective.

For coalgebras B-strongly $(\mathscr{C}, A)$-injective comodules are named B-*equivariantly* $\mathscr{C}$-*injective* (see Schauenburg and Schneider, 2005, Definition 5.1). Cointegrals δ making M fully $(\mathscr{C}, A)$-injective are said to be M-*normalized* in Caenepeel et al. (to appear, Proposition 5.1).

The fact that under projectivity conditions comodule properties may be considered as module properties has the following implication.

2.10. $\mathcal{C}$ Strongly $(\mathcal{C}, A)$-Injective. *Assume $\mathcal{C}_A$ to be locally projective. Then the following are equivalent:*

(a) *$\mathcal{C}$ is strongly $(\mathcal{C}, A)$-injective;*
(b) *$\mathcal{C}$ is a coseparable coring.*

Proof. One implication is obvious. Recall that $\mathrm{End}^{\mathcal{C}}(\mathcal{C}) \simeq \mathcal{C}^*$ and assume $\mathcal{C}$ to be strongly $(\mathcal{C}, A)$-injective with a $\mathcal{C}^*$-splitting right $\mathcal{C}$-colinear map $v : \mathcal{C} \otimes_A \mathcal{C} \to \mathcal{C}$. By the right α-condition this means that v is also left $\mathcal{C}$-colinear and hence $\mathcal{C}$ is coseparable. □

2.11. Properties of Fully $(\mathcal{C}, A)$-Injective Comodules. *Let $M \in \mathbf{M}^{\mathcal{C}}$ with $S = \mathrm{End}^{\mathcal{C}}(M)$.*

(1) *M is fully $(\mathcal{C}, A)$-injective if and only if*

$$(I_M \otimes \tilde{\delta}_M) \circ \varrho^M = I_M \qquad \text{where } \tilde{\delta}_M = \delta_M \circ \Delta : \mathcal{C} \to A.$$

(2) *$\mathcal{C}$ is a fully $(\mathcal{C}, A)$-injective left (right) comodule if and only if $\mathcal{C}$ is a coseparable coring.*
(3) *Let M be fully $(\mathcal{C}, A)$-injective. Then:*
   (i) *Every comodule in $\sigma[M]$ is fully $(\mathcal{C}, A)$-injective;*
   (ii) *If M is a subgenerator in $\mathbf{M}^{\mathcal{C}}$, then $\mathcal{C}$ is a coseparable coring;*
   (iii) *For any subring $B \subset S$ and $X \in \mathbf{M}_B$, $X \otimes_B M$ is fully $(\mathcal{C}, A)$-injective;*
   (iv) *If M_A is finitely generated and projective, then M^* is a fully $(\mathcal{C}, A)$-injective left $\mathcal{C}$-comodule.*

Proof.

(1) The assertion follows from the equalities

$$(I_M \otimes \tilde{\delta}_M) \circ \varrho^M = (I_M \otimes \delta_M) \circ (\varrho^M \otimes I_{\mathcal{C}}) \circ \varrho^M = I_M.$$

(2) This is shown in Brzeziński and Wisbauer (2003, 26.1). In this case $\delta_{\mathcal{C}} = \varepsilon$.
(3) (i) Obviously any direct sum $M^{(\Lambda)}$ is fully $(\mathcal{C}, A)$-injective. For every $\mathcal{C}$-comodule epimorphism $f : M \to N$ and $m \in M$,

$$(I_N \otimes \delta_M)(f(m)_{\underline{0}} \otimes f(m)_{\underline{1}}) = (I_N \otimes \delta_M)(f(m_{\underline{0}}) \otimes m_{\underline{1}}) = f(m_{\underline{0}}\delta_M(m_{\underline{1}})) = f(m).$$

This proves that factor comodules of M are fully $(\mathcal{C}, A)$-injective. For subcomodules similar arguments apply.
   (ii) This follows from (2) and (3)(i).
   (iv) With the dual basis (m_i, π_i) for M, the coaction of $\mathcal{C}$ on $g \in M^*$ is given by $\varrho^{M^*}(g) = \sum_i (g \otimes I_{\mathcal{C}})\varrho^M(m_i) \otimes \pi_i$, and

$$\sum_i (\delta_M \otimes I_{M^*})(g(m_{i\underline{0}})m_{i\underline{1}} \otimes \pi_i) = \sum_i g(m_{i\underline{0}}\delta_M(m_{i\underline{1}})) \otimes \pi_i = \sum_i g(m_i) \otimes \pi_i = g,$$

proving that M^* is a fully $(\mathcal{C}, A)$-injective comodule. □

2.12 Splitting of Hom- and ⊗-Sequences.

(1) *Let* $M \in \mathbf{M}^{\mathscr{C}}$, $B \subseteq \mathrm{End}^{\mathscr{C}}(M)$ *a subring, and assume* M *to be* B*-strongly* $(\mathscr{C}, A)$*-injective. Then for any* $N \in \mathbf{M}^{\mathscr{C}}$*, the sequence*

$$0 \longrightarrow M \otimes^{\mathscr{C}} L \longrightarrow M \otimes_A L \xrightarrow{\omega_{M,L}} M \otimes_A \mathscr{C} \otimes_A L$$

is splitting in ${}_B\mathbf{M}$*, and for any* $L \in {}^{\mathscr{C}}\mathbf{M}$ *the sequence*

$$0 \longrightarrow \mathrm{Hom}^{\mathscr{C}}(N, M) \longrightarrow \mathrm{Hom}_A(N, M) \xrightarrow{\gamma} \mathrm{Hom}_A(N, M \otimes_A \mathscr{C})$$

is also splitting in ${}_B\mathbf{M}$.

(2) *Let* $L \in {}^{\mathscr{C}}\mathbf{M}$, $D \subseteq {}^{\mathscr{C}}\mathrm{End}(L)$ *a subring, and assume* L *to be* D*-strongly* $(\mathscr{C}, A)$*-injective. Then for any* $M \in \mathbf{M}^{\mathscr{C}}$*, the sequence*

$$0 \longrightarrow M \otimes^{\mathscr{C}} L \longrightarrow M \otimes_A L \xrightarrow{\omega_{M,L}} M \otimes_A \mathscr{C} \otimes_A L$$

is splitting in $\mathbf{M}_D$*, and for any* $K \in {}^{\mathscr{C}}\mathbf{M}$*, the sequence*

$$0 \longrightarrow {}^{\mathscr{C}}\mathrm{Hom}(K, L) \longrightarrow \mathrm{Hom}_A(K, L) \xrightarrow{\gamma} \mathrm{Hom}_A(K, \mathscr{C} \otimes_A L)$$

is also splitting in $\mathbf{M}_D$.

Proof. (1) Let $v : M \otimes_A \mathscr{C} \to M$ be a comodule splitting of ϱ^M.

As in the proof of Brzeziński and Wisbauer (2003, 21.5)(4) it is easy to see that

$$\beta = (v \otimes I_L) \circ (I_M \otimes \varrho^L) : M \otimes_A N \to M \otimes^{\mathscr{C}} N$$

is an $\mathrm{End}^{\mathscr{C}}(L)$-linear retraction. If v is B-linear then obviously β is a B-linear retraction and the proof of Brzeziński and Wisbauer (2003, 21.5)(4) applies.

For the second sequence we can follow the proof of Brzeziński and Wisbauer (2003, 3.18). The inclusion is split by

$$\mathrm{Hom}_A(N, M) \to \mathrm{Hom}^{\mathscr{C}}(N, M) : f \mapsto v \circ (f \otimes I_{\mathscr{C}}) \circ \varrho^N,$$

and γ is split modulo $\mathrm{Hom}^{\mathscr{C}}(N, M)$ by

$$\mathrm{Hom}_A(N, M \otimes_A \mathscr{C}) \to \mathrm{Hom}_A(N, M), \qquad g \mapsto v \circ g.$$

This is clearly a right $\mathrm{End}^{\mathscr{C}}(N)$-linear splitting. If v is B-linear, the splitting maps are also left B-linear.

(2) The assertions can be seen by symmetry. □

For convenience we list the following conditions.

2.13. Associativity Conditions for the Cotensor Product. *Consider two comodules* $M \in \mathbf{M}^{\mathscr{C}}$ *and* $L \in {}^{\mathscr{C}}\mathbf{M}$.

(1) *For a subring* $B \subset \mathrm{End}^{\mathscr{C}}(M)$ *and* $X \in \mathbf{M}_B$,

$$X \otimes_B (M \otimes^{\mathscr{C}} L) \simeq (X \otimes_B M) \otimes^{\mathscr{C}} L$$

provided that:

(i) X *is a flat* B*-module, or*
(ii) $-\otimes^{\mathscr{C}} L$ *is right exact or* L *is* $(\mathscr{C}, A)$*-injective, or*
(iii) M *is* B*-strongly* $(\mathscr{C}, A)$*-injective.*

(2) *For a subring* $D \subset {}^{\mathscr{C}}\mathrm{End}(L)$ *and* $Y \in {}_D\mathbf{M}$,

$$M \otimes^{\mathscr{C}} (L \otimes_D Y) \simeq (M \otimes^{\mathscr{C}} L) \otimes_D Y$$

provided that:

(i) Y *is a flat* D*-module, or*
(ii) $M\otimes^{\mathscr{C}}$*-is a right exact or* M *is* $(\mathscr{C}, A)$*-injective, or*
(iii) L *is* B*-strongly* $(\mathscr{C}, A)$*-injective.*

Proof. The conditions (i) and (ii) are sufficient to imply the assertion by Brzeziński and Wisbauer (2003, 21.4 and 21.5). The sufficiency of (iii) follows from 2.12. □

2.14. Hom-Tensor Relation. *For* $M \in \mathbf{M}^{\mathscr{C}}$, $L \in {}^{\mathscr{C}}\mathbf{M}$, *a subring* $B \subseteq \mathrm{End}^{\mathscr{C}}(M)$, *and any right* B*-module* X, *there is a map*

$$\psi_X : X \otimes_B \mathrm{Hom}^{\mathscr{C}}(N, M) \to \mathrm{Hom}^{\mathscr{C}}(N, X \otimes_B M), \qquad x \otimes h \mapsto x \otimes h(-),$$

and this is an isomorphism provided that:

(i) X *is a flat* B*-module and* N_A *is finitely presented, or*
(ii) M *is* B*-strongly* $(\mathscr{C}, A)$*-injective and* N_A *is finitely generated and projective, or*
(iii) N *is projective in* $\mathbf{M}^{\mathscr{C}}$ *and* N_A *is finitely generated.*

Proof. Consider the commutative diagram with canonical maps

$$\begin{array}{ccccccc}
0 & \longrightarrow & X \otimes_B \mathrm{Hom}^{\mathscr{C}}(N, M) & \longrightarrow & X \otimes_B \mathrm{Hom}_A(N, M) & \longrightarrow & X \otimes_B \mathrm{Hom}_A(N, M \otimes_A \mathscr{C}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(N, X \otimes_B M) & \longrightarrow & \mathrm{Hom}_A(N, X \otimes_B M) & \longrightarrow & \mathrm{Hom}_A(N, X \otimes_B M \otimes_A \mathscr{C}),
\end{array}$$

where the bottom sequence is exact. If (i) holds, then the top sequence is also exact, and by 2.12, this is also true if (ii) holds. In both cases the two right vertical maps are isomorphisms, and hence the first one is also an isomorphism.

Now assume (iii) and consider an exact sequence $F_1 \to F_2 \to X \to 0$ in ${}_B\mathbf{M}$ where F_1, F_2 are free B-modules. With $-\otimes_B M$ and $\mathrm{Hom}^{\mathscr{C}}(N, -)$ we construct

the commutative diagram with exact rows

$$\begin{array}{ccccccc} F_1 \otimes_B \mathrm{Hom}^{\mathscr{C}}(N,M) & \longrightarrow & F_2 \otimes_B \mathrm{Hom}^{\mathscr{C}}(N,M) & \longrightarrow & X \otimes_B \mathrm{Hom}^{\mathscr{C}}(N,M) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi_{F_1}} & & \downarrow{\scriptstyle \psi_{F_2}} & & \downarrow{\scriptstyle \psi_X} & & \\ \mathrm{Hom}^{\mathscr{C}}(N, F_1 \otimes_B M) & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(N, F_2 \otimes_B M) & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(N, X \otimes_B M) & \longrightarrow & 0, \end{array}$$

where ψ_{F_1} and ψ_{F_2} are isomorphisms (since $\mathrm{Hom}^{\mathscr{C}}(N,-)$ commutes with direct sums) and hence ψ_X is an isomorphism.

Notice that projectivity of the comodule N implies projectivity of N_A (see Brzeziński and Wisbauer, 2003, 18.20). Hence $\mathrm{Hom}^{\mathscr{C}}(N,M) \simeq M \otimes^{\mathscr{C}} N^*$ and N^* is coflat. So the assertion also follows from 2.13(1)(ii). □

3. ADJOINT FUNCTORS AND STATIC COMODULES

3.1. Adjoint Pair of Functors. For any right $\mathscr{C}$-comodule P with endomorphism ring $S = \mathrm{End}^{\mathscr{C}}(P)$, the functors (see Brzeziński and Wisbauer, 2003, 18.21).

$$- \otimes_S P : \mathbf{M}_S \to \mathbf{M}^{\mathscr{C}}, \qquad \mathrm{Hom}^{\mathscr{C}}(P,-) : \mathbf{M}^{\mathscr{C}} \to \mathbf{M}_S,$$

form an adjoint pair by the functorial isomorphism (for $N \in \mathbf{M}^{\mathscr{C}}$ and $X \in \mathbf{M}_S$),

$$\mathrm{Hom}^{\mathscr{C}}(X \otimes_S P, N) \to \mathrm{Hom}_S(X, \mathrm{Hom}^{\mathscr{C}}(P,N)), \qquad g \mapsto [x \mapsto g(x \otimes -)],$$

with inverse map $h \mapsto [x \otimes p \mapsto h(x)(p)]$. Counit and unit of this adjunction are given by

$$\mu_N : \mathrm{Hom}^{\mathscr{C}}(P,N) \otimes_S P \to N, \quad f \otimes p \mapsto f(p),$$
$$\nu_X : X \to \mathrm{Hom}^{\mathscr{C}}(P, X \otimes_S P), \quad x \mapsto [p \mapsto x \otimes p],$$

and each of the following compositions of maps yields the identity,

$$\mathrm{Hom}^{\mathscr{C}}(P,N) \xrightarrow{\nu_{\mathrm{Hom}(P,N)}} \mathrm{Hom}^{\mathscr{C}}(P, \mathrm{Hom}^{\mathscr{C}}(P,N) \otimes_S P) \xrightarrow{\mathrm{Hom}(P,\mu_N)} \mathrm{Hom}^{\mathscr{C}}(P,N),$$

$$X \otimes_S P \xrightarrow{\nu_X \otimes I_P} \mathrm{Hom}^{\mathscr{C}}(P, X \otimes_S P) \otimes_S P \xrightarrow{\mu_{X \otimes P}} X \otimes_S P.$$

A $\mathscr{C}$-comodule N is called *P-static* if μ_N is an isomorphism, and an S-module X is called *P-adstatic* if ν_X is an isomorphism. Clearly, P is P-static and for any index set Λ,

$$\mu_{P^{(\Lambda)}} : \mathrm{Hom}^{\mathscr{C}}(P, P^{(\Lambda)}) \otimes_S P \to P^{(\Lambda)},$$

is a comodule epimorphism with splitting map $q \mapsto \sum_\Lambda \epsilon_\lambda \otimes \pi_\lambda(q)$, where ϵ_λ and π_λ denote the canonical inclusions and projections of the coproduct $P^{(\Lambda)}$.

3.2. P_A Finitely Generated and Projective. *Let $P \in \mathbf{M}^{\mathscr{C}}$ with P_A finitely generated and projective and $S = \mathrm{End}^{\mathscr{C}}(P)$.*

(1) *For any* $N \in \mathbf{M}^{\mathscr{C}}$, $\mathrm{Hom}^{\mathscr{C}}(P, N) \simeq N \otimes^{\mathscr{C}} P^*$, *in particular* $S \simeq P \otimes^{\mathscr{C}} P^*$.
(2) *A module* $X \in \mathbf{M}_S$ *is* P*-adstatic, provided*

$$(X \otimes_S P) \otimes^{\mathscr{C}} P^* \simeq X \otimes_S (P \otimes^{\mathscr{C}} P^*).$$

(3) (i) *Every flat* $X \in \mathbf{M}_S$ *is* P*-adstatic.*
(ii) *If* P^* *is coflat or* $(\mathscr{C}, A)$*-injective, or* P *is strongly* $(\mathscr{C}, A)$*-injective, then every* $X \in \mathbf{M}_S$ *is* P*-adstatic.*
(4) *For a subring* $B \subset S$ *and* $Y \in \mathbf{M}_B$,

$$\mathrm{Hom}^{\mathscr{C}}(P, Y \otimes_B P) \simeq Y \otimes_B S$$

provided Y_B *is flat, or* P *is* B*-strongly* $(\mathscr{C}, A)$*-injective, or* P^* *is coflat or* $(\mathscr{C}, A)$*-injective.*

Proof. (1) This is shown by the proof of Brzeziński and Wisbauer (2003, 21.8).

(2) Recall that $X \in \mathbf{M}_S$ is P-adstatic if $v_X : X \to \mathrm{Hom}^{\mathscr{C}}(P, X \otimes_S P)$ is an isomorphism. Under the given condition, (1) implies

$$\mathrm{Hom}^{\mathscr{C}}(P, X \otimes_S P) \simeq X \otimes_S (P \otimes^{\mathscr{C}} P^*) \simeq X.$$

(3) As shown in 2.13, each of the conditions implies the isomorphism required.

(4) From (1) we get $\mathrm{Hom}^{\mathscr{C}}(P, Y \otimes_B P) \simeq (Y \otimes_B P) \otimes^{\mathscr{C}} P^*$, and by 2.13, under each of the conditions required,

$$(Y \otimes_B P) \otimes^{\mathscr{C}} P^* \simeq Y \otimes_B (P \otimes^{\mathscr{C}} P^*) \simeq Y \otimes_B S. \qquad \square$$

3.3. *P* as Generator in $\mathbf{M}^{\mathscr{C}}$. Recall that P is a generator in $\mathbf{M}^{\mathscr{C}}$ if and only if the functor $\mathrm{Hom}^{\mathscr{C}}(P, -) : \mathbf{M}^{\mathscr{C}} \to \mathbf{M}_S$ is faithful and that faithful functors reflect epimorphisms (e.g., Wisbauer, 1991, 11.3). Since, for any $N \in \mathbf{M}^{\mathscr{C}}$,

$$\mathrm{Hom}^{\mathscr{C}}(P, \mu_N) : \mathrm{Hom}^{\mathscr{C}}(P, \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P) \to \mathrm{Hom}^{\mathscr{C}}(P, N)$$

is an epimorphism (surjective), we conclude that μ_N is an epimorphism (that is surjective) in $\mathbf{M}^{\mathscr{C}}$, provided P is a generator in $\mathbf{M}^{\mathscr{C}}$. Taking $\Lambda = \mathrm{Hom}^{\mathscr{C}}(P, N)$, the canonical epimorphism $\varphi_N : P^{(\Lambda)} \to N$ remains an epimorphism under $\mathrm{Hom}^{\mathscr{C}}(P, -)$.

Now assume ${}_A\mathscr{C}$ to be flat. Then $K = \mathrm{Ke}\,\varphi$ is a comodule, and we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \mathrm{Hom}^{\mathscr{C}}(P, K) \otimes_S P & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(P, P^{(\Lambda)}) \otimes_S P & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_K} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mu_N} & & \\
0 & \longrightarrow & K & \longrightarrow & P^{(\Lambda)} & \longrightarrow & N & \longrightarrow & 0,
\end{array}$$

where μ_K is surjective and the central vertical map is an isomorphism (e.g., Wisbauer, 2000, Corollary 3.4). By diagram lemmata, this implies that μ_N is injective (hence an isomorphism).

3.4. Properties of Generators. *Assume ${}_A\mathscr{C}$ to be flat.*

(1) *If P generates the (finitely generated) subcomodules of $P^{(\mathbb{N})}$, then, for every P-generated $\mathscr{C}$-comodule L, μ_L is an isomorphism, and for every $N \in \mathbf{M}^{\mathscr{C}}$, μ_N is injective.*
(2) *P is a generator in $\mathbf{M}^{\mathscr{C}}$ if and only if μ_N is an isomorphism for any $N \in \mathbf{M}^{\mathscr{C}}$, i.e., every right $\mathscr{C}$-comodule is P-static.*

Proof. (1) Clearly, the condition implies that P generates the subcomodules of any direct sum of copies of P and bijectivity of μ_L follows from the considerations above. The image of μ_N is the trace $\mathrm{Tr}(P, N)$ of P in N (sum of all P-generated subcomodules) and $\mathrm{Hom}^{\mathscr{C}}(P, N) = \mathrm{Hom}^{\mathscr{C}}(P, \mathrm{Tr}(P, N))$. Since $\mu_{\mathrm{Tr}(P,N)}$ is bijective, μ_N has to be injective.

(2) is a special case of (1). □

Semisimple right comodules P are defined by the fact that any monomorphism $U \to P$ is a coretraction, that is subcomdules are direct summands. If ${}_A\mathscr{C}$ is flat this is equivalent to P being a (direct) sum of simple subcomodules and then any direct sum of copies of P is semisimple.

3.5. Semisimple Comodules. *Let ${}_A\mathscr{C}$ be flat, $P \in \mathbf{M}^{\mathscr{C}}$ and $S = \mathrm{End}^{\mathscr{C}}(P)$.*

(1) *The following are equivalent:*

(a) *P is semisimple;*
(b) *for any set Λ, $\mathrm{End}^{\mathscr{C}}(P^{(\Lambda)})$ is a von Neumann regular ring and, for any $N \in \mathbf{M}^{\mathscr{C}}$,*

$$\mu_N : \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P \to N \qquad \text{is injective;}$$

(c) *for any set Λ, $\mathrm{End}^{\mathscr{C}}(P^{(\Lambda)})$ is a regular ring, and for any $L \in \sigma[P]$,*

$$\mu_L : \mathrm{Hom}^{\mathscr{C}}(P, L) \otimes_S P \to L \qquad \text{is an isomorphism.}$$

(2) *If P is finitely generated (in $\mathbf{M}^{\mathscr{C}}$), then the following are equivalent:*

(a) *P is semisimple;*
(b) *S is a right (left) semisimple ring, and for any $N \in \mathbf{M}^{\mathscr{C}}$,*

$$\mu_N : \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P \to N \qquad \text{is injective;}$$

(c) *S is a right (left) semisimple ring, and for any $L \in \sigma[P]$,*

$$\mu_L : \mathrm{Hom}^{\mathscr{C}}(P, L) \otimes_S P \to L \qquad \text{is an isomorphism.}$$

(3) *The following are equivalent:*

(a) *P is simple;*

(b) *S is a division ring, and for any $N \in \mathbf{M}^{\mathscr{C}}$,*

$$\mu_N : \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P \to N \qquad \textit{is injective;}$$

(c) *S is a division ring, and for any $L \in \sigma[P]$,*

$$\mu_L : \mathrm{Hom}^{\mathscr{C}}(P, L) \otimes_S P \to L \qquad \textit{is an isomorphism.}$$

Proof. (1) (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) For any $s \in S$, the image and the kernel are direct summands in P. This implies that S is von Neumann regular (e.g., Wisbauer, 1991, 37.7). Since $P^{(\Lambda)}$ is also semisimple the same argument shows that $\mathrm{End}^{\mathscr{C}}(P^{(\Lambda)})$ is von Neumann regular. Since P generates all subcomodules of any $P^{(\Lambda)}$ the remaining assertions follow from 3.4(1).

(b) $\Rightarrow$ (a) Let $N \subset P$ be any subcomodule and construct the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(P,N) \otimes_S P & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(P,P) \otimes_S P & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(P,P/N) \otimes_S P & & \\
 & & \downarrow{\scriptstyle \mu_N} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mu_{P/N}} & & \\
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & P/N & \longrightarrow & 0,
\end{array}$$

in which the top row is exact by regularity of S (${}_SP$ is flat). Clearly, $\mu_{P/N}$ is an epimorphism and is injective by assumption. This implies that μ_N is an epimorphism and hence N is P-generated. So there is some epimorphism $h : P^{(\Lambda)} \to N$. Considering h as an endomorphism of $P^{(\Lambda)}$, the fact that $\mathrm{End}^{\mathscr{C}}(P^{(\Lambda)})$ is regular implies that the image of h is a direct summand in $P^{(\Lambda)}$ and hence in P (see Wisbauer, 1991, 37.7). This shows that P is semisimple.

(2) The endomorphism ring of a finite direct sum of simple comodules is right (left) semisimple, and this implies that $\mathrm{End}^{\mathscr{C}}(P^{(\Lambda)})$ is von Neumann regular. Hence the proof of (1) applies.

(3) By Schur's Lemma, the endomorphism ring of a simple comodule is a division ring and again the proof of (1) applies. □

Note that assertion (3) is also proven in Brzeziński (2005, Theorem 3.1).

If ${}_A\mathscr{C}$ is flat, a generator in $\mathbf{M}^{\mathscr{C}}$ is characterized by the fact that all comodules are P-static. This suggests the study of comodules P by the classes of P-static modules. Transferring observations from module theory, we may consider the following cases.

3.6. Some Classes *P*-Static. Consider the following conditions for $P \in \mathbf{M}^{\mathscr{C}}$:

(1) All comodules in $\mathbf{M}^{\mathscr{C}}$ are P-static;
(2) The class of P-generated modules is P-static;
(3) The class of P-presented comodules is P-static;
(4) The class of injective comodules in $\mathbf{M}^{\mathscr{C}}$ is P-static;
(5) The class of $(\mathscr{C}, A)$-injective comodules in $\mathbf{M}^{\mathscr{C}}$ is P-static.

The first case was handled in 3.4 for ${}_A\mathscr{C}$ flat. In module categories, the second case describes an important property of *self-tilting* modules; for those, an additional projectivity condition is required (see Wisbauer, 1998, 4.2; Wisbauer, 2000, 4.4). The third case generalizes tilting modules (see Wisbauer, 2000, 4.3). For a module P, the corresponding property (4) essentially means that all P-injective modules in $\sigma[P]$ are P-static and—if P is a balanced bimodule—this can be seen as descending chain condition on certain matrix subgroups of P (see Wisbauer, 1998, 5.4; Zimmermann, 1997). In all these cases, the functor $\mathrm{Hom}^{\mathscr{C}}(P, -)$ induces equivalences between the P-static classes and the corresponding adstatic classes. Properties of these classes correspond to properties of the module P. For example, if the class of P-adstatic comodules is closed under infinite coproducts, then P has to be self-small, i.e., $\mathrm{Hom}^{\mathscr{C}}(P, P^{(\Lambda)}) \simeq \mathrm{Hom}^{\mathscr{C}}(P, P)^{(\Lambda)}$.

If ${}_A\mathscr{C}$ is flat, monomorphisms are injective maps and kernels exist in $\mathbf{M}^{\mathscr{C}}$, and hence most of the proofs for module categories can be transferred to $\mathbf{M}^{\mathscr{C}}$. In particular, if ${}_A\mathscr{C}$ is locally projective, $\mathbf{M}^{\mathscr{C}}$ can be identified with $\sigma[{}_{*\mathscr{C}}\mathscr{C}]$ and the results mentioned immediately apply to comodules. Without such restrictions, all the properties listed are also of interest and deserve to be investigated elsewhere. Here we will investigate the comodules characterized by the condition required in (5).

4. GALOIS COMODULES

Throughout this section, let $\mathscr{C}$ be an A-coring, $P \in \mathbf{M}^{\mathscr{C}}$, and put $S = \mathrm{End}^{\mathscr{C}}(P)$, $T = \mathrm{End}_A(P)$. With the evaluation map $\mu : \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P \to N$, there is a functorial morphism between the functors $- \otimes_A \mathscr{C}$ and $\mathrm{Hom}_A(P, -) \otimes_S P$ from $\mathbf{M}_A$ to $\mathbf{M}^{\mathscr{C}}$,

$$\tilde{\mu} : \mathrm{Hom}_A(P, -) \otimes_S P \xrightarrow{\simeq} \mathrm{Hom}^{\mathscr{C}}(P, - \otimes_A \mathscr{C}) \otimes_S P \xrightarrow{\mu} - \otimes_A \mathscr{C}.$$

4.1 Galois Comodules. *We call P a Galois comodule if the following equivalent conditions hold:*

(a) $\tilde{\mu} : \mathrm{Hom}_A(P, -) \otimes_S P \to - \otimes_A \mathscr{C}$ *is an isomorphism;*
(b) *for any $K \in \mathbf{M}_A$ there is a functorial isomorphism of comodules*

$$\tilde{\mu}_K : \mathrm{Hom}_A(P, K) \otimes_S P \to K \otimes_A \mathscr{C}, \qquad g \otimes p \mapsto (g \otimes I_{\mathscr{C}})\varrho^P(p);$$

(c) *every $(\mathscr{C}, A)$-injective $N \in \mathbf{M}^{\mathscr{C}}$ is P-static, i.e.,*

$$\mu_N : \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P \to N, \qquad f \otimes p \mapsto f(p),$$

is an isomorphism (in $\mathbf{M}^{\mathscr{C}}$).

Proof. We prove the equivalence of the conditions.

(a) $\Leftrightarrow$ (b) is clear from the definition.

(b) $\Rightarrow$ (c) (see proof of Brzeziński and Wisbauer, 2003, 18.26) Assume $N \in \mathbf{M}^{\mathscr{C}}$ to be $(\mathscr{C}, A)$-injective. Then, by Brzeziński and Wisbauer (2003, 18.18), the

canonical sequence

$$0 \longrightarrow \mathrm{Hom}^{\mathscr{C}}(P, N) \xrightarrow{i} \mathrm{Hom}_A(P, N) \xrightarrow{\gamma} \mathrm{Hom}_A(P, N \otimes_A \mathscr{C})$$

is (split and hence) pure in $\mathbf{M}_S$, where $\gamma(f) = \varrho^N \circ f - (f \otimes I_{\mathscr{C}}) \circ \varrho^P$. Hence tensoring with ${}_SP$ yields the commutative diagram with exact rows,

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P & \longrightarrow & \mathrm{Hom}_A(P, N) \otimes_S P & \longrightarrow & \mathrm{Hom}_A(P, N \otimes_A \mathscr{C}) \otimes_S P \\
 & & \downarrow \mu_N & & \downarrow \tilde{\mu}_N & & \downarrow \tilde{\mu}_{N \otimes C} \\
0 & \longrightarrow & N & \longrightarrow & N \otimes_A \mathscr{C} & \longrightarrow & N \otimes_A \mathscr{C} \otimes_A \mathscr{C},
\end{array}$$

where the $\tilde{\mu}$'s are isomorphisms and so is μ_N.

(c) $\Rightarrow$ (b) Since $K \otimes_A \mathscr{C}$ is $(\mathscr{C}, A)$-injective, the assertion follows from the commutative diagram of right $\mathscr{C}$-comodule maps

$$\begin{array}{ccc}
\mathrm{Hom}^{\mathscr{C}}(P, K \otimes_A \mathscr{C}) \otimes_S P & \xrightarrow{\mu_{K \otimes \mathscr{C}}} & K \otimes_A \mathscr{C} \\
\simeq \downarrow & & \downarrow = \\
\mathrm{Hom}_A(P, K) \otimes_S P & \xrightarrow{\tilde{\mu}_K} & K \otimes_A \mathscr{C}
\end{array}
\qquad
\begin{array}{ccc}
f \otimes p & \longmapsto & f(p) \\
\downarrow & & \downarrow = \\
(I \otimes \underline{\varepsilon}) \circ f \otimes p & \longmapsto & \sum (I \otimes \underline{\varepsilon}) \circ f(p_{\underline{0}}) \otimes p_{\underline{1}}.
\end{array}$$

□

4.2. Properties of Galois Comodules. *Let $P \in \mathbf{M}^{\mathscr{C}}$ be a Galois comodule. Then:*

(1) *For any $(\mathscr{C}, A)$-injective $N \in \mathbf{M}^{\mathscr{C}}$, there is an isomorphism*

$$v_{\mathrm{Hom}^{\mathscr{C}}(P,N)} : \mathrm{Hom}^{\mathscr{C}}(P, N) \to \mathrm{Hom}^{\mathscr{C}}(P, \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P),$$

that is, $\mathrm{Hom}^{\mathscr{C}}(P, N)$ is P-adstatic.

(2) *For any $K \in \mathbf{M}_A$, there is an isomorphism*

$$v_{\mathrm{Hom}_A(P,K)} : \mathrm{Hom}_A(P, K) \to \mathrm{Hom}^{\mathscr{C}}(P, \mathrm{Hom}_A(P, K) \otimes_S P).$$

(3) *There are right $\mathscr{C}$-comodule isomorphisms*

$$\mathrm{Hom}^{\mathscr{C}}(P, \mathscr{C}) \otimes_S P \simeq \mathscr{C} \simeq \mathrm{Hom}_A(P, A) \otimes_S P.$$

(4) *Since $T = \mathrm{End}^{\mathscr{C}}(P, P \otimes_A \mathscr{C})$, there is a T-linear isomorphism*

$$T \otimes_S P \to P \otimes_A \mathscr{C}, \quad t \otimes p \mapsto (t \otimes I_{\mathscr{C}})\varrho^P(p), \qquad \text{and}$$
$$P^* \otimes_T P \otimes_A \mathscr{C} \simeq P^* \otimes_T T \otimes_S P \simeq P^* \otimes_S P \simeq \mathscr{C}.$$

(5) *For any $K \in \mathbf{M}_A$ and index set Λ,*

$$\mathrm{Hom}^{\mathscr{C}}(P, (K \otimes_A \mathscr{C})^{\Lambda}) \otimes_S P \simeq \mathrm{Hom}_A(P, K)^{\Lambda} \otimes_S P \simeq K^{\Lambda} \otimes_A \mathscr{C}.$$

(6) *There are isomorphisms*

$$\mathrm{Hom}_A(\mathscr{C}, A) \simeq \mathrm{Hom}_A(P^* \otimes_S P, A) \simeq \mathrm{End}_S(P^*),$$

$$\mathrm{Hom}^{\mathscr{C}}(\mathscr{C}, P) \simeq \mathrm{Hom}^{\mathscr{C}}(P^* \otimes_S P, P) \simeq \mathrm{Hom}_S(P^*, S), \qquad \textit{and}$$
$$\mathrm{Hom}^{\mathscr{C}}(P \otimes_A \mathscr{C}, P) \simeq \mathrm{Hom}^{\mathscr{C}}(T \otimes_S P, P) \simeq \mathrm{Hom}_S(T, S).$$

Proof. (1), (2) follow from the fact that the composition $\mathrm{Hom}^{\mathscr{C}}(P, \mu_N) \circ \nu_{\mathrm{Hom}^{\mathscr{C}}(P,N)}$ yields the identity.

(3), (4) Put $N = \mathscr{C}$ or $N = P \otimes_A \mathscr{C}$ in the characterizing relations.

(5) This follows from the fact that the product of Λ copies of $K \otimes_A \mathscr{C}$ in $\mathbf{M}^{\mathscr{C}}$ is isomorphic to $K^{\Lambda} \otimes_A \mathscr{C}$.

(6) Apply isomorphisms from (3), (4) and properties of adjoint functors (see 3.1). □

4.3. $(\mathscr{C}, A)$-Injective Modules.

Let P be a Galois comodule.

(1) *For $N \in \mathbf{M}^{\mathscr{C}}$ the following are equivalent:*

(a) *N is $(\mathscr{C}, A)$-injective;*
(b) *$\mathrm{Hom}^{\mathscr{C}}(P, \varrho^N) : \mathrm{Hom}^{\mathscr{C}}(P, N) \to \mathrm{Hom}^{\mathscr{C}}(P, N \otimes_A \mathscr{C})$ is a coretraction in $\mathbf{M}_S$.*

(2) *For P the following are equivalent:*

(a) *P is $(\mathscr{C}, A)$-injective;*
(b) *the inclusion $i : S \to T$ is split by a right S-linear map.*

(3) *For P the following are equivalent:*

(a) *P is strongly $(\mathscr{C}, A)$-injective;*
(b) *the inclusion $i : S \to T$ is split by an (S, S)-bilinear map.*

In this case every P-static comodule is $(\mathscr{C}, A)$-injective.

(4) *For P the following are equivalent:*

(a) *P is fully $(\mathscr{C}, A)$-injective;*
(b) *$\mathscr{C}$ is a coseparable A-coring.*

In this case every comodule in $\mathbf{M}$ is P-static and fully $(\mathscr{C}, A)$-injective.

Proof. (1) By (a), ϱ^N splits in $\mathbf{M}^{\mathscr{C}}$ and hence $\mathrm{Hom}^{\mathscr{C}}(P, \varrho^N)$ splits in $\mathbf{M}_S$. In turn, (b) yields a splitting of ϱ^N by tensoring with $- \otimes_S P$.

(2) follows from (1) since $T \simeq \mathrm{Hom}^{\mathscr{C}}(P, P \otimes_A \mathscr{C})$ as right S-module.

(3) By 2.12, the inclusion $\mathrm{Hom}^{\mathscr{C}}(P, P) \to \mathrm{Hom}^{\mathscr{C}}(P, P \otimes_A \mathscr{C}) \simeq T$ is split as an (S, S)-bilinear map.

If P is strongly $(\mathscr{C}, A)$-injective, then for any $X \in \mathbf{M}_S$, the tensor product is $(\mathscr{C}, A)$-injective. So in particular P-static comodules are $(\mathscr{C}, A)$-injective.

(4) (a) $\Rightarrow$ (b) Since P is a subgenerator this follows from 2.11.

(b) $\Rightarrow$ (a) Over a coseparable coring all comodules are fully $(\mathscr{C}, A)$-injective. □

Notice that the Assertions (2) and (3) in 4.3 are shown in Brzeziński (2005, Theorem 7.2) for f.g. projective A-modules. The arguments in Brzeziński (2005) can also be adapted to general Galois comodules.

4.4. Galois Comodules Under the α-Condition. If $\mathscr{C}$ satisfies the left α-condition, $\mathbf{M}^{\mathscr{C}}$ can be identified with the ${}^*\mathscr{C}$-module category $\sigma[{}_{{}^*\mathscr{C}}\mathscr{C}]$ (see 2.4) and Galois comodules may be explained in these terms.

By the ring antimorphism $A \to {}^*\mathscr{C}$ (see Brzeziński and Wisbauer, 2003, 17.7) any left ${}^*\mathscr{C}$-module has a right A-module structure. It follows from the functorial isomorphisms on ${}_{{}^*\mathscr{C}}\mathbf{M}$ for $K \in \mathbf{M}_A$,

$$\mathrm{Hom}_A(-, K) \simeq \mathrm{Hom}_A({}^*\mathscr{C} \otimes_{{}^*\mathscr{C}} -, K) \simeq {}_{{}^*\mathscr{C}}\mathrm{Hom}(-, \mathrm{Hom}_A({}^*\mathscr{C}, K)),$$

that $\mathrm{Hom}_A({}^*\mathscr{C}, K)$ is $({}^*\mathscr{C}, A)$-injective, that is, injective with respect to short exact sequences in ${}_{{}^*\mathscr{C}}\mathbf{M}$ which split in $\mathbf{M}_A$. Moreover, since the canonical map

$$\gamma_K : K \to \mathrm{Hom}_A({}^*\mathscr{C}, K), \qquad k \mapsto [f \mapsto fk],$$

is A-split by $f \mapsto f(\varepsilon)$, it follows that a left ${}^*\mathscr{C}$-module K is $({}^*\mathscr{C}, A)$-injective if and only if γ_K splits in ${}_{{}^*\mathscr{C}}\mathbf{M}$. For any $P \in \mathbf{M}^{\mathscr{C}}$ and $K \in \mathbf{M}_A$, there are morphisms

$$\mathrm{Hom}_A(P, K) \simeq \mathrm{Hom}^{\mathscr{C}}(P, K \otimes_A \mathscr{C}) \xrightarrow{\ i\ } {}_{{}^*\mathscr{C}}\mathrm{Hom}(P, K \otimes_A \mathscr{C})$$
$$\xrightarrow{\ \mathrm{Hom}(P,\alpha)\ } {}_{{}^*\mathscr{C}}\mathrm{Hom}(P, \mathrm{Hom}_A({}^*\mathscr{C}, K)) \simeq \mathrm{Hom}_A(P, K),$$

where i is the inclusion and α is the canonical map from 2.4. It is straightforward to prove that the composition of these maps yields the identity on $\mathrm{Hom}_A(P, K)$. Hence injectivity of α implies that $\mathrm{Hom}(P, \alpha)$ is an isomorphism and leads to the following statement.

4.5. Proposition. *Let $P \in \mathbf{M}^{\mathscr{C}}$ be a Galois comodule, assume $\mathscr{C}$ to satisfy the left α-condition, and put $S = \mathrm{End}^{\mathscr{C}}(P) = {}_{{}^*\mathscr{C}}\mathrm{End}(P)$. Then for any $K \in \mathbf{M}_A$,*

$${}_{{}^*\mathscr{C}}\mathrm{Hom}(P, \mathrm{Hom}_A({}^*\mathscr{C}, K)) \otimes_S P \simeq \mathrm{Hom}^{\mathscr{C}}(P, K \otimes_A \mathscr{C}) \otimes_S P \simeq K \otimes_A \mathscr{C},$$

implying $K \otimes_A \mathscr{C} \simeq \mathrm{Tr}(P, \mathrm{Hom}_A({}^*\mathscr{C}, K))$.

Proof. Combine the observations above with isomorphisms for Galois comodules. Notice that $\mathrm{Hom}_A({}^*\mathscr{C}, K)$ need not be a $\mathscr{C}$-comodule but the trace of P yields a ${}^*\mathscr{C}$-submodule lying in $\mathbf{M}^{\mathscr{C}}$. The last isomorphism is a special case of the corresponding observation for modules in Wisbauer (1996, 20.4). □

4.6. Semisimple Base Ring. If the ring A is left semisimple (Artinian semisimple), then all A-modules are projective, and injective, and $(\mathscr{C}, A)$-injective

comodules are in fact $\mathscr{C}$-injective. Moreover, the α-condition is satisfied, and $\mathbf{M}^{\mathscr{C}}$ corresponds to the category $\sigma[{}_{*_{\mathscr{C}}}\mathscr{C}]$. In this case, Galois comodules are just the comodules P for which all injectives in $\sigma[{}_{*_{\mathscr{C}}}\mathscr{C}]$ are P-static. Such modules were considered in Wisbauer (2000).

4.7. Remarks. The ideas outlined in 4.4 can be used as guideline to study modules M of Galois type for module categories over ring extensions $B \to A$ by the condition that all (A, B)-injective A-modules are M-static.

Notice that so far we did not make any assumptions either on the A-module or on the S-module structure of P. Of course, properties of this type influence the behaviour of Galois comodules and we look at the S-module structure first.

4.8. Module Properties of ${}_SP$. *Let $P \in \mathbf{M}^{\mathscr{C}}$ be a Galois comodule.*

(1) *If ${}_SP$ is finitely generated, then ${}_A\mathscr{C}$ is finitely generated.*
(2) *If ${}_SP$ is Mittag-Leffler, then ${}_A\mathscr{C}$ is Mittag-Leffler.*
(3) *If ${}_SP$ is finitely presented, then ${}_A\mathscr{C}$ is finitely presented.*
(4) *If ${}_SP$ is projective, then ${}_A\mathscr{C}$ is projective.*
(5) *If ${}_TP$ is finitely generated and ${}_SP$ is locally projective, then ${}_A\mathscr{C}$ is locally projective.*
(6) *If ${}_SP$ is flat, then ${}_A\mathscr{C}$ is flat and P is a generator in $\mathbf{M}^{\mathscr{C}}$.*
(7) *If ${}_SP$ is faithfully flat, then ${}_A\mathscr{C}$ is flat and P is a projective generator in $\mathbf{M}^{\mathscr{C}}$.*

Proof. (1), (2), (3) Putting $K = A$ in 4.2(6) we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(P, A)^{\Lambda} \otimes_S P & \xrightarrow{\simeq} & A^{\Lambda} \otimes_A \mathscr{C} \\ \downarrow{\scriptstyle \varphi_P} & & \downarrow{\scriptstyle \varphi_{\mathscr{C}}} \\ (\mathrm{Hom}_A(P, A) \otimes_S P)^{\Lambda} & \xrightarrow{\simeq} & \mathscr{C}^{\Lambda}, \end{array}$$

where the φ's denote the canonical maps. Then (e.g., Wisbauer, 1991, 12.9)

$$\begin{array}{rcccccl} {}_SP \text{ is fin. gen.} & \Rightarrow & \varphi_P \text{ surjective} & \Rightarrow & \varphi_{\mathscr{C}} \text{ surjective} & \Leftrightarrow & {}_A\mathscr{C} \text{ fin. gen.}, \\ {}_SP \text{ is ML} & \Rightarrow & \varphi_P \text{ injective} & \Rightarrow & \varphi_{\mathscr{C}} \text{ injective} & \Leftrightarrow & {}_A\mathscr{C} \text{ ML}, \\ {}_SP \text{ is fin. pres.} & \Rightarrow & \varphi_P \text{ bijective} & \Rightarrow & \varphi_{\mathscr{C}} \text{ bijective} & \Leftrightarrow & {}_A\mathscr{C} \text{ fin. pres.} \end{array}$$

Recall that by definition $\mathscr{C}$ is Mittag–Leffler (ML) if $\varphi_{\mathscr{C}}$ is injective.

(4) Let ${}_SP$ be projective. Then $T \otimes_S P \simeq P \otimes_A \mathscr{C}$ is projective as left T-module. Consider any epimorphism $F \to \mathscr{C}$ where F is a free module in ${}_A\mathbf{M}$. Then $I_P \otimes f$ is a splitting epimorphism in ${}_T\mathbf{M}$, and in the commutative diagram with exact rows

$$\begin{array}{ccccccc} P^* \otimes_T P \otimes_A F & \xrightarrow{I \otimes I \otimes f} & P^* \otimes_T P \otimes_A \mathscr{C} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \simeq} & & \\ F & \xrightarrow{\quad f \quad} & \mathscr{C} & \longrightarrow & 0, \end{array}$$

where the first vertical map is the evaluation and the right isomorphism is from 4.2(4), the top row is splitting in ${}_A\mathbf{M}$ and hence f also splits showing that ${}_A\mathscr{C}$ is projective.

(5) Let ${}_SP$ be locally projective. To check local projectivity of ${}_A\mathscr{C}$ consider the diagram in ${}_A\mathbf{M}$ with $k \in \mathbb{N}$ and exact bottom row,

$$\begin{array}{ccccc} A^k & \xrightarrow{i} & \mathscr{C} & & \\ & & \downarrow g & & \\ L & \xrightarrow{f} & N & \longrightarrow & 0. \end{array}$$

Applying $P\otimes_A$—we obtain the diagram

$$\begin{array}{ccccc} P^k & \xrightarrow{I_P\otimes i} & P\otimes_A \mathscr{C} & & \\ & & \downarrow I\otimes g & & \\ P\otimes_A L & \xrightarrow{I\otimes f} & P\otimes_A N & \longrightarrow & 0. \end{array}$$

Since $P\otimes_A \mathscr{C} \simeq T\otimes_S P$ is a locally projective T-module (by Brzeziński and Wisbauer, 2003, 42.11) and ${}_TP^k$ is finitely generated by assumption, there is some T-morphism $h : P\otimes_A \mathscr{C} \to P\otimes_A L$ with

$$(I\otimes f)\circ h\circ (I_P\otimes i) = I\otimes g.$$

Applying $P^*\otimes_T -$ and the evaluation map, we obtain $f\circ (I_{P^*}\otimes h)\circ i = g$. This shows that ${}_A\mathscr{C}$ is locally projective.

(6) We have $-\otimes_A \mathscr{C} \simeq \mathrm{Hom}_A(P,-)\otimes_S P$. Clearly, $\mathrm{Hom}_A(P,-)$ (always) preserves injective maps. If ${}_SP$ is flat, then $-\otimes_S P$ also preserves injectivity of morphisms and hence $-\otimes_A \mathscr{C}$ preserves injective maps, i.e., ${}_A\mathscr{C}$ is flat.

For any $M \in \mathbf{M}^{\mathscr{C}}$ we have an exact sequence of comodules

$$0 \longrightarrow M \xrightarrow{\varrho^M} M\otimes_A \mathscr{C} \longrightarrow M\otimes_A \mathscr{C}\otimes_A \mathscr{C}.$$

By left exactness of $\mathrm{Hom}^{\mathscr{C}}(P,-)$ and $-\otimes_S P$, we obtain the exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}^{\mathscr{C}}(P,M)\otimes_S P & \longrightarrow & \mathrm{Hom}_A(P,M)\otimes_S P & \longrightarrow & \mathrm{Hom}_A(P, M\otimes_A\mathscr{C})\otimes_S P \\ & & \downarrow & & \downarrow\simeq & & \downarrow\simeq \\ 0 & \longrightarrow & M & \xrightarrow{\varrho^M} & M\otimes_A\mathscr{C} & \longrightarrow & M\otimes_A\mathscr{C}\otimes_A\mathscr{C} \end{array}$$

from which we see that the first vertical map is also an isomorphism. This shows that P is a generator.

(7) By (6), ${}_A\mathscr{C}$ is flat and P is a generator. Consider any epimorphism $f: M \to N$ in $\mathbf{M}^{\mathscr{C}}$. From this we obtain the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}^{\mathscr{C}}(P,M)\otimes_S P & \xrightarrow{\mathrm{Hom}(P,f)\otimes I_P} & \mathrm{Hom}^{\mathscr{C}}(P,N)\otimes_S P & \longrightarrow & 0 \\ \downarrow\simeq & & \downarrow\simeq & & \\ M & \longrightarrow & N & \longrightarrow & 0, \end{array}$$

where the vertical maps are isomorphisms by 3.4 and hence the exactness of the bottom row implies exactness of the top row. Now faithfulness of the functor $-\otimes_S P$ implies that $\mathrm{Hom}^{\mathscr{C}}(P, f)$ is an epimorphism and hence P is projective in $\mathbf{M}^{\mathscr{C}}$. □

4.9. Remark. Notice that the condition ${}_TP$ finitely generated is satisfied if P is a generator in $\mathbf{M}_A$. For Galois comodules P, this is the case provided $\varepsilon: \mathscr{C} \to A$ is surjective.

4.10. Corollary. *Let $P \in \mathbf{M}^{\mathscr{C}}$ be a Galois comodule.*

(1) *If* (i) ${}_SP$ *is projective or*

(ii) ${}_SP$ *is locally projective and* ${}_TP$ *is finitely generated,*

then $\mathbf{M}^{\mathscr{C}}$ is equivalent to the full category of ${}_{*\mathscr{C}}\mathbf{M}$ subgenerated by the ${}^*\mathscr{C}$-module P, i.e., $\mathbf{M}^{\mathscr{C}} = \sigma[{}_{*\mathscr{C}}P]$.

(2) *If ${}_SP$ is finitely generated and projective, then $\mathbf{M}^{\mathscr{C}} = {}_{*\mathscr{C}}\mathbf{M}$.*

Proof. (1) Under the given conditions, ${}_A\mathscr{C}$ is locally projective (see 4.8(3), (4)) and $\mathbf{M}^{\mathscr{C}} = \sigma[{}_{*\mathscr{C}}\mathscr{C}]$. Since P subgenerates $\mathscr{C}$, it is a subgenerator in $\mathbf{M}^{\mathscr{C}}$ and hence the assertion follows.

(2) The condition implies that ${}_A\mathscr{C}$ is finitely generated and projective and hence $\mathbf{M}^{\mathscr{C}} = {}_{*\mathscr{C}}\mathbf{M}$ (by Brzeziński and Wisbauer, 2003, 19.6). □

4.11. Semisimple Galois Comodules. *Assume ${}_A\mathscr{C}$ to be flat. For a semisimple right $\mathscr{C}$-comodule P, the following are equivalent:*

(a) P is a Galois comodule;
(b) P is a generator in $\mathbf{M}^{\mathscr{C}}$;
(c) $\mu_{\mathscr{C}}: \mathrm{Hom}^{\mathscr{C}}(P, \mathscr{C})\otimes_S P \to \mathscr{C}$ is surjective.

In this case, $\mathscr{C}$ is a right semisimple coring (and ${}_A\mathscr{C}$ is projective).

Proof. Since P is semisimple it is a generator in $\sigma[P]$ (see 3.5).

(a) ⇒ (c) This is trivial.

(c) ⇒ (b) Surjectivity of $\mu_{\mathscr{C}}$ means that $\mathscr{C}$ is P-generated. Since $\mathscr{C}$ is a subgenerator in $\mathbf{M}^{\mathscr{C}}$ (see Brzeziński and Wisbauer, 2003, 18.13(1)) this implies $\sigma[P] = \mathbf{M}^{\mathscr{C}}$.

(b) ⇒ (a) follows from 3.4(2). □

4.12. Simple Galois Comodules. *If ${}_A\mathcal{C}$ is flat the following are equivalent:*

(a) *There is a simple Galois comodule in $\mathbf{M}^{\mathcal{C}}$;*
(b) *every nonzero comodule in $\mathbf{M}^{\mathcal{C}}$ is a Galois comodule;*
(c) *$\mathcal{C}$ is homogeneously semisimple as right comodule;*
(d) *$\mathcal{C}$ is right semisimple and all simple right comodules are isomorphic.*

Proof. This follows by the characterizations of simple right semisimple corings in Brzeziński and Wisbauer (2003, 19.15) and the fact that each nonzero comodule is a generator in this case. □

5. GALOIS COMODULES f.g. PROJECTIVE AS *A*-MODULES

Some of the results in the preceding section were proven in Brzeziński and Wisbauer (2003, 18.27) for the special case when P_A is finitely generated and projective. As already observed (in 3.2) the latter condition provides nice properties of the functor $\mathrm{Hom}^{\mathcal{C}}(P, -)$ which will lead to a left right symmetry of the Galois comodules. If P_A will be finitely generated and projective, we denote a dual basis of P by $p_1, \ldots, p_n \in P$ and $\pi_1, \ldots, \pi_n \in P^*$.

5.1. $\mu_{\mathcal{C}}$ Splitting in $\mathbf{M}^{\mathcal{C}}$. *Let $P \in \mathbf{M}^{\mathcal{C}}$ with P_A finitely generated and projective and $S = \mathrm{End}^{\mathcal{C}}(P)$. Assume that*

$$(P^* \otimes_S P) \otimes^{\mathcal{C}} P^* \simeq P^* \otimes_S (P \otimes^{\mathcal{C}} P^*)$$

canonically. Then the following are equivalent:

(a) *The map $\mu_C : \mathrm{Hom}^{\mathcal{C}}(P, \mathcal{C}) \otimes_S P \to \mathcal{C}$ is a splitting epimorphism in $\mathbf{M}^{\mathcal{C}}$;*
(b) *$\mu_{\mathcal{C}}$ is an isomorphism.*

The condition is satisfied provided P is strongly $(\mathcal{C}, A)$-injective, or P_S^* is flat, or P^* is coflat or (C, A)-injective.

Proof. We only have to prove (a) ⇒ (b). It follows from 3.2 that P^* is P-adstatic. By assumption, there is a splitting exact sequence in $\mathbf{M}^{\mathcal{C}}$,

$$0 \longrightarrow K \longrightarrow \mathrm{Hom}^{\mathcal{C}}(P, \mathcal{C}) \otimes_S P \longrightarrow \mathcal{C} \longrightarrow 0.$$

Since P^* is P-adstatic by 3.2(3), applying $\mathrm{Hom}^{\mathcal{C}}(P, -)$ yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}^{\mathcal{C}}(P, K) \longrightarrow \mathrm{Hom}^{\mathcal{C}}(P, \mathrm{Hom}^{\mathcal{C}}(P, \mathcal{C}) \otimes_S P) \xrightarrow{\simeq} \mathrm{Hom}^{\mathcal{C}}(P, \mathcal{C}).$$

From this we see $\mathrm{Hom}^{\mathcal{C}}(P, K) = 0$ and—since K is a P-generated comodule—this implies $K = 0$. □

5.2. Lemma. *Let $P \in \mathbf{M}^{\mathcal{C}}$ with P_A finitely generated and projective. Then the following are equivalent:*

(a) *$\mathcal{C}$ is P-static as right $\mathcal{C}$-comodule;*
(b) *$\mathcal{C}$ is P^*-static as left $\mathcal{C}$-comodule.*

Proof. The canonical map $\phi : P \to {}^*(P^*), \phi(p)(f) = f(p)$ for $p \in P, f \in P^*$, is bijective and the diagram

$$\begin{array}{ccc} P^* \otimes_S P & \longrightarrow & \mathscr{C} \\ \downarrow{\scriptstyle I\otimes\phi} & & \downarrow{\scriptstyle =} \\ P^* \otimes_S {}^*(P^*) & \longrightarrow & \mathscr{C} \end{array} \qquad \begin{array}{ccc} g \otimes p & \longmapsto & \sum g(p_{\underline{0}})p_{\underline{1}} \\ \downarrow & & \downarrow \\ g \otimes h & \longmapsto & (I_{\mathscr{C}} \otimes h)\varrho^{P^*}(g), \end{array}$$

is commutative by the equalities

$$\begin{aligned}(I_{\mathscr{C}} \otimes \phi(p))\varrho^{P^*}(g) &= \sum_i (g \otimes I_{\mathscr{C}})\varrho^P(p_i)\phi(p)(\pi_i) \\ &= (g \otimes I_{\mathscr{C}})\varrho^P\left(\sum_i p_i\pi_i(p)\right) = \sum g(p_{\underline{0}})p_{\underline{1}}.\end{aligned}$$

By definition, $\mathscr{C}$ is P-static provided the map in the top row is an isomorphism of right $\mathscr{C}$-comodules, and $\mathscr{C}$ is P^*-static as left $\mathscr{C}$-comodule provided the map in the bottom row of the diagram is an isomorphism of left $\mathscr{C}$-comodules. □

Recall that for any bimodule ${}_BP_A$ with P_A finitely generated and projective (with dual basis as above), the (A, A)-bimodule $P^* \otimes_B P$ is an A-coring with coproduct and counit defined by

$$\underline{\Delta} : P^* \otimes_B P \to (P^* \otimes_B P) \otimes_A (P^* \otimes_B P), \qquad f \otimes p \mapsto \sum f \otimes p_i \otimes \pi_i \otimes p,$$
$$\underline{\varepsilon} : P^* \otimes_B P \to A, \qquad f \otimes p \mapsto f(p).$$

For a Galois comodule this coring is isomorphic to $\mathscr{C}$.

5.3. Galois Comodules with P_A f.g. Projective. *Let $P \in \mathbf{M}^{\mathscr{C}}$ with P_A finitely generated and projective and $S = \mathrm{End}^{\mathscr{C}}(P)$. Then the following are equivalent:*

(a) *P is a Galois right $\mathscr{C}$-comodule;*
(b) *$\mathscr{C}$ is P-static as right $\mathscr{C}$-comodule;*
(c) *P^* is a Galois left $\mathscr{C}$-comodule;*
(d) *$\mathscr{C}$ is P^*-static as left $\mathscr{C}$-comodule;*
(e) *$\tilde{\mu}_A : P^* \otimes_S P \to \mathscr{C}$ is an A-coring isomorphism.*

Proof. (a) ⇔ (b) This is shown in Brzeziński and Wisbauer (2003, 18.26).

(c) ⇔ (d) The assertion is the left hand version of (a) ⇔ (b).

(b) ⇔ (d) This is proven in 5.2.

(b) ⇔ (e) It remains to show that $\tilde{\mu}_A$ is a coring morphism. Proofs for this are given in El Kaoutit and Gómez-Torrecillas (2003, Proposition 2.7) and Brzeziński and Wisbauer (2003, 18.26). With our notation it is seen by the following argument.

For any $p \in P$ and $f \in P^*$, $p = \sum_i p_i \pi_i(p)$,

$$\tilde{\mu}_A(f \otimes p) = \sum f(p_{\underline{0}})p_{\underline{1}} = \sum \sum_i f(p_{i\underline{0}})p_{i\underline{1}}\pi_i(p), \qquad \text{and}$$

$$\begin{aligned}(\tilde{\mu}_A \otimes \tilde{\mu}_A) \circ \underline{\Delta}(f \otimes p) &= \sum \sum_i f(p_{i\underline{0}})p_{i\underline{1}} \otimes \pi_i(p_{\underline{0}})p_{\underline{1}} \\ &= \sum f(p_{\underline{00}})p_{\underline{01}} \otimes p_{\underline{1}} \\ &= \sum f(p_{\underline{0}})p_{\underline{11}} \otimes p_{\underline{12}} \\ &= \Delta \circ \tilde{\mu}_A(f \otimes p),\end{aligned}$$

and it is easy to see that $\varepsilon \circ \tilde{\mu}_A = \underline{\varepsilon}$. □

5.4. Remark. It was shown in 4.8(5) that for a Galois comodule $P \in \mathbf{M}^{\mathscr{C}}$, ${}_SP$ locally projective and ${}_TP$ finitely generated, implies that ${}_A\mathscr{C}$ is locally projective. In case P_A is finitely generated and projective, ${}_SP$ locally projective implies ${}_A\mathscr{C}$ locally projective without the additional assumption that ${}_TP$ is finitely generated (see Brzeziński and Wisbauer, 2003, 19.7).

In the special situation that A is a $\mathscr{C}$-comodule, i.e., there is a grouplike element $g \in \mathscr{C}$, and $S = \mathrm{End}^{\mathscr{C}}(A)$, it is a Galois (right) comodule ($(\mathscr{C}, g)$ is a Galois coring) if and only if the map (compare introduction)

$$A \otimes_S A \to \mathscr{C}, \qquad a \otimes a' \mapsto aga'$$

is an isomorphism. Under the given conditions, $A \otimes_S A$ has a canonical coring structure (Sweedler coring) and the map is a coring isomorphisms (see Brzeziński and Wisbauer, 2003, 28.18).

At various places we have observed nice properties of strongly $(\mathscr{C}, A)$-injective comodules. For Galois comodules this notion is symmetric in the following sense—an observation also proven in Brzeziński (2005, Theorem 7.2).

5.5. Strongly $(\mathscr{C}, A)$-Injective Galois Comodules. *Let P be a Galois comodule with P_A finitely generated and projective and $S = \mathrm{End}^{\mathscr{C}}(P)$. Then the following are equivalent:*

(a) *P is strongly $(\mathscr{C}, A)$-injective;*
(b) *P^* is strongly $(\mathscr{C}, A)$-injective;*
(c) *the inclusion $i : S \to T$ is split by an (S, S)-bilinear map.*

Proof. This follows from 4.3 and symmetry. □

5.6. *P*-Static Comodules. *Let $P \in \mathbf{M}^{\mathscr{C}}$ with P_A finitely generated and projective and assume $\mathscr{C}$ to be P-static. Then $N \in \mathbf{M}^{\mathscr{C}}$ is P-static, provided*

$$(N \otimes^{\mathscr{C}} P^*) \otimes_S P \simeq N \otimes^{\mathscr{C}} (P^* \otimes_S P)$$

canonically. This holds if N is $(\mathscr{C}, A)$*-injective, or* $N\otimes^{\mathscr{C}}$*—is right exact, or P is strongly* $(\mathscr{C}, A)$*-injective, or P is flat as S-module.*

Proof. The first claim follows by the isomorphisms

$$\mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P \simeq (N \otimes^{\mathscr{C}} P^*) \otimes_S P \simeq N \otimes^{\mathscr{C}} (P^* \otimes_S P) \simeq N.$$

The remaining assertions are derived from 2.13. □

The relevance of the isomorphism in 5.6 was also observed in Caenepeel et al. (to appear, Proposition 2.4). Notice that 5.6 shows again - in this special case—that P is a Galois comodule provided $\mathscr{C}$ is P-static (see 5.3), and that Galois comodules are generators provided they are flat over their endomorphism rings (see 4.8).

5.7. Equivalences. *Let* $P \in \mathbf{M}^{\mathscr{C}}$ *be a Galois comodule with* P_A *finitely generated and projective. Then*

$$\mathrm{Hom}^{\mathscr{C}}(P, -) : \mathbf{M}^{\mathscr{C}} \to \mathbf{M}_S$$

is an equivalence with inverse functor $- \otimes_S P$ *provided that*

(i) *P is strongly* $(\mathscr{C}, A)$*-injective, or*
(ii) P^* *is* $(\mathscr{C}, A)$*-injective and* ${}_SP$ *is flat, or*
(iii) P^* *is coflat and* ${}_SP$ *is flat, or*
(iv) $\mathscr{C}$ *is a coseparable coring.*

Proof. Under each of the conditions (i)–(iii), all right S-modules are P-adstatic by 3.2 and the right $\mathscr{C}$-comodules are P-static by 5.6.

(iv) For a coseparable coring, all comodules are strongly $(\mathscr{C}, A)$-injective and hence (i) holds. □

Parts of the preceding theorem are proven in Brzeziński (2005, Proposition 7.3). Here we offer alternative proofs and do not require ${}_A\mathscr{C}$ to be flat in the first case.

5.8. Remarks. In Caenepeel et al. (to appear, Proposition 5.6), the coring $\mathscr{C}$ is required to be coseparable, finitely generated, and projective as right A-module, and $\mu_{\mathscr{C}} : \mathrm{Hom}^{\mathscr{C}}(P, \mathscr{C}) \otimes_S P \to \mathscr{C}$ should be surjective. These conditions immediately imply that $\mu_{\mathscr{C}}$ splits in $\mathbf{M}^{\mathscr{C}}$ and that P^* is coflat. Hence P is a Galois comodule by 5.1 and the claim of Caenepeel et al. (to appear, Proposition 5.6)—namely that $\mathrm{Hom}^{\mathscr{C}}(P, -)$ is an equivalence—follows from 5.7.

5.9. Splitting Over a Subring of $\mathrm{End}^{\mathscr{C}}(P)$. *Let* $P \in \mathbf{M}^{\mathscr{C}}$ *with* P_A *finitely generated and projective,* $S = \mathrm{End}^{\mathscr{C}}(P)$ *and* $B \subseteq S$ *a subring. Assume* P^* *to be flat as a right B-module, or P to be B-strongly* $(\mathscr{C}, A)$*-injective. Then the following are equivalent:*

(a) *The canonical map* $\mu'_{\mathscr{C}} : \mathrm{Hom}^{\mathscr{C}}(P, \mathscr{C}) \otimes_B P \to \mathscr{C}$ *is a splitting epimorphism in* $\mathbf{M}^{\mathscr{C}}$*;*
(b) P^* *is a Galois comodule and is* (S, B)*-projective as right module.*

Proof. (a) $\Rightarrow$ (b) Cotensoring with $-\otimes^C P^*$, $\mu'_{\mathscr{C}}$ yields the splitting epimorphism in $\mathbf{M}_S$,

$$P^* \otimes_B S \simeq (P^* \otimes_B P) \otimes^{\mathscr{C}} P^* \to \mathscr{C} \otimes^{\mathscr{C}} P^* \simeq P^*,$$

where the first isomorphism is due to the conditions on P^* or P (see 2.14). This shows that P^* is (S, B)-projective (see Wisbauer, 1996, 20.3). In particular, P^* is flat as S-module.

Furthermore, $\mu'_{\mathscr{C}}$ factors over a splitting epimorphism $\mu_C : \mathrm{Hom}^{\mathscr{C}}(P, \mathscr{C}) \to \mathscr{C}$ in $\mathbf{M}^{\mathscr{C}}$. By Corollary 5.1, this implies that $\mu_{\mathscr{C}}$ is an isomorphism, i.e., P is a Galois module.

(b) $\Rightarrow$ (a) By assumption, the map $P^* \otimes_B S \to P^*$ splits in $\mathbf{M}_S$. Since P^* (hence P) is a Galois comodule, tensoring with $-\otimes_S P$ yields a splitting comodule epimorphism

$$P^* \otimes_B P \simeq P^* \otimes_B S \otimes_S P \to P^* \otimes_S P \simeq \mathscr{C}. \qquad \square$$

5.10. Remarks. (1) In case A is an algebra over a field (or a commutative von Neumann regular ring) B, then in 5.9, P^* is always a flat B-module and the assertion yields Brzeziński (2005, Theorem 4.4) as a special case.

(2) As pointed out in the introduction, entwining structures can be considered as corings and hence the assertions in 3.2 and 4.3 may be compared with Lemma 4.1 and Remarks 4.2 and 5.3 in Schauenburg and Schneider (2005). Furthermore, the splitting properties considered in 5.9 are related to Remark 4.4, Theorem 2.2, and results of Section 5 in Schauenburg and Schneider (2005).

6. DIRECT SUMS OF f.g. PROJECTIVE *A*-MODULES

For any direct sum $P = \bigoplus_\Lambda P_\lambda$ of finitely generated $\mathscr{C}$-comodules P_λ and any $\mathscr{C}$-comodule N, recall the definition (see Wisbauer, 1991, Section 51)

$$\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, N) = \{f \in \mathrm{Hom}^{\mathscr{C}}(P, N) \mid f(P_\lambda) = 0 \text{ for almost all } \lambda \in \Lambda\},$$

and put $T = \widehat{\mathrm{End}}^{\mathscr{C}}(P)$. Then $\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, N)$ is a right T-module and

$$\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, N) \simeq \bigoplus_\Lambda \mathrm{Hom}^{\mathscr{C}}(P_\lambda, N) \simeq \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S T,$$

where $S = \mathrm{End}^{\mathscr{C}}(P)$ (see Wisbauer, 1991, 51.2).

6.1. Lemma. *With the notation above,*

$$\Big(\bigoplus_\Lambda \mathrm{Hom}^{\mathscr{C}}(P_\lambda, N)\Big) \otimes_T P \simeq (\mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S T) \otimes_T P \simeq \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P.$$

Proof. By associativity of the tensor product we only need to recall the isomorphism $T \otimes_T P \simeq P$. $\square$

With this isomorphism, the special structure of comodules that are finitely generated as A-modules can be extended to direct sums of modules of this type.

6.2. $P^* \otimes_S P$ as Left Comodule. Consider a family $\{P_\lambda\}_\Lambda$ of comodules $P_\lambda \in \mathbf{M}^{\mathscr{C}}$ such that each P_λ is finitely generated and projective as A-module. Then $P = \bigoplus_\Lambda P_\lambda$ is in $\mathbf{M}^{\mathscr{C}}$. Since all P_λ^* are left $\mathscr{C}$-comodules, their direct sum $\bigoplus_\Lambda P_\lambda^*$ is a left $\mathscr{C}$-comodule. For $S = \mathrm{End}^{\mathscr{C}}(P)$, 6.1 yields the identification $P^* \otimes_S P \simeq (\bigoplus_\Lambda P_\lambda^*) \otimes_T P$ which makes $P^* \otimes_S P$ to a left $\mathscr{C}$-comodule.

The following are equivalent:

(a) *$\mathscr{C}$ is P-static as right $\mathscr{C}$-comodule;*
(b) *$\mathscr{C}$ is $\bigoplus_\Lambda P_\lambda^*$-static as left $\mathscr{C}$-comodule.*

Proof. Applying 6.1 repeatedly yields isomorphisms

$$\begin{aligned} P^* \otimes_S P &\simeq \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_T \left(\bigoplus_\Lambda P_\lambda\right) \\ &\simeq \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_T \left(\bigoplus_\Lambda {}^*(P_\lambda^*)\right) \\ &\simeq \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_{\widetilde{S}} {}^*\left(\bigoplus_\Lambda (P_\lambda)^*\right), \end{aligned}$$

where $\widetilde{S} = {}^{\mathscr{C}}\mathrm{End}(\bigoplus_\Lambda P_\lambda^*)$. With this isomorphisms the proof of 5.2 applies. □

The construction of corings for finitely generated projective A-modules can also be extended to direct sums of modules of this type.

6.3. Coring Structure on Direct Sums. Consider a family $\{P_\lambda\}_\Lambda$ of $\mathscr{C}$-comodules with $S_\lambda = \mathrm{End}^{\mathscr{C}}(P_\lambda)$ that are finitely generated and projective as right A-modules with dual basis $p_{\lambda_i} \in P_\lambda$, $\phi_{\lambda_i} \in P_\lambda^*$, $\lambda_i \in I_\lambda$. For each $\lambda \in \Lambda$, we have a coring with coproduct

$$\underline{\Delta}_\lambda : P_\lambda^* \otimes_{S_\lambda} P_\lambda \to P_\lambda^* \otimes_{S_\lambda} P_\lambda \otimes_A P_\lambda^* \otimes_{S_\lambda} P_\lambda, \qquad f_\lambda \otimes p_\lambda \mapsto \sum_i f_\lambda \otimes p_{\lambda_i} \otimes \phi_{\lambda_i} \otimes p_\lambda,$$

and the evaluation as counit.

Put $P = \bigoplus_\Lambda P_\lambda$, $S = \mathrm{End}^{\mathscr{C}}(P)$, and $S_\Lambda = \bigoplus_\Lambda S_\lambda$. Then S_Λ is a ring without unit and P is a left S_Λ module by componentwise multiplication. This means in particular that we can identify

$$P_\lambda^* \otimes_{S_\lambda} P_\lambda = P_\lambda^* \otimes_{S_\Lambda} P,$$

and so, by the universal property of the coproduct, the $\underline{\Delta}_\lambda$ yield an (A, A)-bilinear map

$$\underline{\Delta} : \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_{S_\Lambda} P \to \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_{S_\Lambda} P \otimes_A \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_{S_\Lambda} P.$$

By construction, this is just the coring coproduct of the $P_\lambda^* \otimes_{S_\lambda} P_\lambda$ (e.g., Brzeziński and Wisbauer, 2003, 17.18).

Clearly, $S_\Lambda \subseteq T = \widehat{\mathrm{End}}^{\mathscr{C}}(P)$ and the epimorphism $\beta : (\bigoplus_\Lambda P_\lambda^*) \otimes_{S_\Lambda} P \to (\bigoplus_\Lambda P_\lambda^*) \otimes_T P$ yields the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_{S_\Lambda} P & \xrightarrow{\underline{\Delta}} & \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_{S_\Lambda} P \otimes_A P^* \otimes_{S_\Lambda} P \\ & & \downarrow \beta & & \downarrow \beta\otimes\beta \\ & & \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_T P & \xrightarrow{\underline{\Delta}'} & \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_T P \otimes_A \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_T P, \end{array}$$

where $\underline{\Delta}'$ exists provided $\mathrm{Ke}\,\beta \subseteq \mathrm{Ke}\,(\beta \otimes \beta) \circ \underline{\Delta}$. To show this take any

$$f \otimes_{S_\Lambda} p \in \left(\bigoplus_\Lambda P_\lambda^*\right) \otimes_{S_\Lambda} P \qquad \text{such that } \sum_i f \otimes_T p_i \otimes_A \pi_i \otimes_T p = 0.$$

Then $0 = \sum_i f \otimes_T p_i \pi_i(p) = f \otimes_T p$ proving that $f \otimes_{S_\Lambda} p \in \mathrm{Ke}\,\beta$. It is easy to see that a similar argument works for finite sums of elements of the form $f \otimes_{S_\Lambda} p$. This shows that our condition on $\mathrm{Ke}\,\beta$ is satisfied and that $\underline{\Delta}'$ exists making $(\bigoplus_\Lambda P_\lambda^*) \otimes_S P$ an A-coring with the evaluation map as counit.

By 6.1, $P^* \otimes_S P = (\bigoplus_\Lambda P_\lambda^*) \otimes_T P$ and thus $\underline{\Delta}'$ defines an A-coring structure on $P^* \otimes_S P$ with the evaluation as counit.

The coring structure on $P^* \otimes_S P$ as given here was introduced in El Kaoutit and Gómez-Torrecillas (2004) (along a different line of arguments) and these corings are called *infinite comatrix corings* there.

With this preparation we are now able to extend the characterization of Galois comodules which are finitely generated and projective to those which are direct sums of such comodules.

6.4. P_A Direct Sum of f.g. Projectives. *Consider a family $\{P_\lambda\}_\Lambda$ of $\mathscr{C}$-comodules that are finitely generated and projective as right A-modules (with dual basis as in 6.3) and put $S_\lambda = \mathrm{End}^{\mathscr{C}}(P_\lambda)$. Then for $P = \bigoplus_\Lambda P_\lambda$ and $S = \mathrm{End}^{\mathscr{C}}(P)$, the following are equivalent:*

(a) *P is a Galois right $\mathscr{C}$-comodule;*
(b) *$\mathscr{C}$ is P-static as right $\mathscr{C}$-comodule;*
(c) *$\bigoplus_\Lambda P_\lambda^*$ is a Galois left $\mathscr{C}$-comodule;*
(d) *$\mathscr{C}$ is $\bigoplus_\Lambda P_\lambda^*$-static as left $\mathscr{C}$-comodule;*
(e) *$\tilde{\mu}_A : P^* \otimes_S P \to \mathscr{C}$ is an A-coring isomorphism.*

Proof. (a) $\Leftrightarrow$ (b) One implication is trivial. Assume $\mathscr{C}$ to be P-static. Then for any $K \in \mathbf{M}_A$, the isomorphisms from 6.1 yield

$$\begin{aligned} \mathrm{Hom}_A(P, K) \otimes_S P &\simeq \bigoplus_\Lambda \mathrm{Hom}_A(P_\lambda, K) \otimes_T P \\ &\simeq \bigoplus_\Lambda K \otimes_A P_\lambda^* \otimes_T P \end{aligned}$$

$$\simeq K \otimes_A \left(\bigoplus_\Lambda P_\lambda^* \right) \otimes_T P$$
$$\simeq K \otimes_A P^* \otimes_S P \simeq K \otimes_A \mathscr{C}.$$

Now the assertion follows from 4.1(b).

(c) $\Leftrightarrow$ (d) is the left hand version of (a) $\Leftrightarrow$ (b).

(b) $\Leftrightarrow$ (d) This is shown in 6.2.

(b) $\Leftrightarrow$ (e) It remains to show that $\tilde{\mu}_A$ is a coring morphism. As mentioned in the proof of 5.3, the maps $P_\lambda^* \otimes_{S_\lambda} P_\lambda \to \mathscr{C}$ are coring morphisms. Hence $P^* \otimes_{S_\Lambda} P \to \mathscr{C}$ is a coring morphism and so is the factorisation $P^* \otimes_S P \to \mathscr{C}$. Notice that this is also proven in El Kaoutit and Gómez-Torrecillas (2004, Lemma 3.7). □

6.5. The Functor $\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, -)$. Given $P = \bigoplus_\Lambda P_\lambda$ as a direct sum of finitely generated $\mathscr{C}$-comodules, we put $S = \mathrm{End}^{\mathscr{C}}(P)$ and $T = \widehat{\mathrm{Hom}}^{\mathscr{C}}(P, P)$ (see beginning of this section). T is a ring with enough idempotents and we denote by $\mathbf{M}_T$ the category of all right T-modules X with $XT = X$. Then we have a functor

$$\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, -) : \mathbf{M}^{\mathscr{C}} \to \mathbf{M}_T$$

and for any $N \in \mathbf{M}^{\mathscr{C}}$ the isomorphisms

$$\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, N) \otimes_T P \simeq \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S T \otimes_T P \simeq \mathrm{Hom}^{\mathscr{C}}(P, N) \otimes_S P.$$

Thus N is P-static if and only if $\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, N) \otimes_T P \simeq N$.

If the P_λ are finitely generated and projective as A-modules, then

$$\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, -) \simeq \bigoplus_\Lambda \mathrm{Hom}^{\mathscr{C}}(P_\lambda, -) \simeq - \otimes^{\mathscr{C}} \left(\bigoplus_\Lambda P_\lambda^* \right).$$

So in this case the functor $- \otimes^{\mathscr{C}} (\bigoplus_\Lambda P_\lambda^*)$ is right adjoint to the functor $- \otimes_S P$ (thus yielding El Kaoutit and Gómez-Torrecillas, 2004, Proposition 4.5).

6.6. *P*-Static Comodules. *Let $P = \bigoplus_\Lambda P_\lambda$ where the $P_\lambda \in \mathbf{M}^{\mathscr{C}}$ are finitely generated and projective as A-modules and assume $\mathscr{C}$ to be P-static. Then $N \in \mathbf{M}^{\mathscr{C}}$ is P-static, provided*

$$\left(N \otimes^{\mathscr{C}} \left(\bigoplus_\Lambda P_\lambda^* \right) \right) \otimes_T P \simeq N \otimes^{\mathscr{C}} \left(\left(\bigoplus_\Lambda P_\lambda^* \right) \otimes_T P \right)$$

canonically. This holds if N is $(\mathscr{C}, A)$-injective, or $N \otimes^{\mathscr{C}} -$ is right exact, or the P_λ's are strongly $(\mathscr{C}, A)$-injective, or P is flat as S-module.

Proof. The first claim follows by the isomorphisms

$$\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, N) \otimes_T P \simeq \left(\bigoplus_\Lambda \mathrm{Hom}^{\mathscr{C}}(P_\lambda, N) \right) \otimes_T P$$

$$\simeq \bigoplus_{\Lambda} (N \otimes^{\mathscr{C}} P_\lambda^*) \otimes_T P$$

$$\simeq N \otimes^{\mathscr{C}} \left(\left(\bigoplus_{\Lambda} P_\lambda^* \right) \otimes_T P \right) \simeq N.$$

The remaining assertions are derived from 2.13. □

6.7. Equivalences. *Let $P = \bigoplus_\Lambda P_\lambda$ be a Galois comodule, where the $P_\lambda \in \mathbf{M}^{\mathscr{C}}$ are finitely generated and projective as A-modules. Then*

$$\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, -) : \mathbf{M}^{\mathscr{C}} \to \mathbf{M}_T$$

is an equivalence with inverse functor $- \otimes_T P$ provided that

(i) *P is strongly $(\mathscr{C}, A)$-injective, or*
(ii) *each P_λ^* is $(\mathscr{C}, A)$-injective and ${}_T P$ is flat, or*
(iii) *each P_λ^* is coflat and ${}_T P$ is flat, or*
(iv) *$\mathscr{C}$ is a coseparable coring.*

Proof. The same arguments as for the proof of 5.7 apply. □

By the isomorphisms $\mathrm{Hom}^{\mathscr{C}}(P_\lambda, -) \simeq - \otimes^{\mathscr{C}} P_\lambda^*$, P_λ^* is coflat if and only if P_λ is projective in $\mathbf{M}^{\mathscr{C}}$. Hence, by 4.8, the condition (iii) in 6.7 implies that ${}_A\mathscr{C}$ is flat and that P is a projective generator in the Grothendieck category $\mathbf{M}^{\mathscr{C}}$. With familiar arguments from module theory (see Wisbauer, 1991, 51.11), this situation can be described in the following way.

6.8. Projective Generators in $\mathbf{M}^{\mathscr{C}}$. *Let $P = \bigoplus_\Lambda P_\lambda$, where the $P_\lambda \in \mathbf{M}^{\mathscr{C}}$ are finitely generated and projective as A-modules. Then the following are equivalent:*

(a) *${}_A\mathscr{C}$ is flat and P is a projective generator in $\mathbf{M}^{\mathscr{C}}$;*
(b) *P is a Galois comodule and ${}_T P$ is faithfully flat;*
(c) *${}_A\mathscr{C}$ is flat and $\widehat{\mathrm{Hom}}^{\mathscr{C}}(P, -) : \mathbf{M}^{\mathscr{C}} \to \mathbf{M}_T$ is an equivalence;*
(d) *${}_A\mathscr{C}$ is flat and $- \otimes_T P : \mathbf{M}_T \to \mathbf{M}^{\mathscr{C}}$ is an equivalence.*

Notice that similar characterizations are also proven in El Kaoutit and Gómez-Torrecillas (2004, Theorem 4.7).

Finally, we ask when $\mathscr{C}$ is a right Galois comodule. For this, recall that $\mathrm{End}^{\mathscr{C}}(\mathscr{C}) \simeq \mathscr{C}^*$ and that—in our notation—$\mathscr{C}^*$ acts on $\mathscr{C}$ from the right. Then the evaluation map

$$\mu_{\mathscr{C}} : \mathscr{C} \otimes_{\mathscr{C}^*} \mathrm{Hom}^{\mathscr{C}}(\mathscr{C}, \mathscr{C}) \to \mathscr{C}$$

is an isomorphism and hence we conclude from 4.5, 5.3, and 6.4 the following.

6.9. $\mathscr{C}$ as Galois Comodule.

(1) *If $\mathscr{C}_A$ is finitely generated and projective, then $\mathscr{C}$ is a Galois right $\mathscr{C}$-comodule, $\mathscr{C}^*$ is a Galois left $\mathscr{C}$-comodule, and $K \otimes_A \mathscr{C} \simeq \mathrm{Hom}_A({}^*\mathscr{C}, K)$, for any $K \in \mathbf{M}_A$.*

(2) *If* $\mathcal{C} = \bigoplus_\Lambda \mathcal{C}_\lambda$ *with right subcomodules* $\mathcal{C}_\lambda$ *that are finitely generated and projective as right A-modules, then* $\mathcal{C}$ *is a Galois right* $\mathcal{C}$*-comodule and* $\bigoplus_\Lambda \mathcal{C}_\lambda^*$ *is a Galois left* $\mathcal{C}$*-comodule.*

Notice that in (2), $\mathcal{C}^*$ is not a left $\mathcal{C}$-comodule unless the sum is finite (finiteness theorem, Brzeziński and Wisbauer, 2003, 19.12). For further properties of $\mathcal{C}$ as Galois comodule we refer to Caenepeel et al. (to appear, Section 7).

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