HOM-TENSOR RELATIONS FOR TWO-SIDED HOPF MODULES OVER QUASI-HOPF ALGEBRAS

SAEID BAGHERI[1] AND ROBERT WISBAUER[2]

Abstract. For a *Hopf algebra* H over a commutative ring k, the category $\mathbb{M}_H^H$ of right Hopf modules is equivalent to the category $\mathbb{M}_k$ of k-modules, that is, the comparison functor $-\otimes_k H : \mathbb{M}_k \to \mathbb{M}_H^H$ is an equivalence (Fundamental theorem of Hopf modules). This was proved by Larson and Sweedler via the notion of *coinvariants* M^{coH} for any $M \in \mathbb{M}_H^H$. The coinvariants functor $(-)^{coH} : \mathbb{M}_H^H \to \mathbb{M}_k$ is right adjoint to the comparison functor and can be understood as the Hom-functor $\mathrm{Hom}_H^H(H, -)$ (without referring to an antipode).

For a *quasi-Hopf algebra* H, the category ${}_H\mathbb{M}_H^H$ of *quasi-Hopf H-bimodules* has been introduced by Hausser and Nill and *coinvariants* are defined to show that the functor $-\otimes_k H : \mathbb{M}_k \to {}_H\mathbb{M}_H^H$ is an equivalence. It is the purpose of this paper to show that the related coinvariants functor, right adjoint to the comparison functor, can be seen as the functor ${}_H\mathrm{Hom}_H^H(H \otimes_k H, -)$

More generally, let H be a *quasi-bialgebra* and $\mathcal{A}$ an H-comodule algebra $\mathcal{A}$ (as introduced by Hausser and Nill). Then $-\otimes_k H$ is a comonad on the category ${}_\mathcal{A}\mathbb{M}_H$ of $(\mathcal{A}, H)$-bimodules and defines the Eilenberg-Moore comodule category $({}_\mathcal{A}\mathbb{M}_H)^{-\otimes H}$ which is just the category ${}_\mathcal{A}\mathbb{M}_H^H$ of two-sided Hopf modules. Following ideas of Hausser, Nill, Bulacu, Caenepeel and others, two types of coinvariants are defined to describe right adjoints of the comparison functor $-\otimes_k H : {}_\mathcal{A}\mathbb{M} \to {}_\mathcal{A}\mathbb{M}_H^H$ and to establish an equivalence between the categories ${}_\mathcal{A}\mathbb{M}$ and ${}_\mathcal{A}\mathbb{M}_H^H$ provided H has a quasi-antipode. As our main results we show that these coinvariants functors are isomorphic to the functor ${}_\mathcal{A}\mathrm{Hom}_H^H(\mathcal{A} \otimes_k H, -) : {}_\mathcal{A}\mathbb{M}_H^H \to {}_\mathcal{A}\mathbb{M}$ and give explicit formulas for these isomorphisms.

1. Introduction

For a commutative ring k, the category $\mathbb{M}_k$ of k-modules is monoidal: the tensor product of two k-modules has again a natural k-module structure and for k-modules V, M, N, the canonical map

$$a_{V,M,N} : (V \otimes_k M) \otimes_k N \to V \otimes_k (M \otimes_k N), \quad (v \otimes m) \otimes n \mapsto v \otimes (m \otimes n), \tag{1.1}$$

is an isomorphism. This means that the composition of the endofunctors $V \otimes_k -$, $M \otimes_k -$ on $\mathbb{M}_k$ is the same as the functor $(V \otimes_k M) \otimes_k -$. It is known well-known that the endofunctors $(V \otimes_k -, \mathrm{Hom}_k(V, -))$ form an adjoint pair of functors with unit and counit

$$\eta_M : M \to \mathrm{Hom}_k(V, V \otimes_k M), \quad m \mapsto [v \mapsto v \otimes m],$$
$$\varepsilon_M : V \otimes \mathrm{Hom}_k(V, M) \to M, \quad v \otimes f \mapsto f(v).$$

For a *k-bialgebra* $(H, \mu, \iota, \Delta, \varepsilon)$, denote the category of left H-modules by ${}_H\mathbb{M}$ and the category of right H-comodules by $\mathbb{M}^H$. For two modules $M, N \in {}_H\mathbb{M}$, the tensor product $M \otimes_k N$ is again a left H-module by the action $h \cdot (m \otimes n) = \Delta h(m \otimes n)$ (componentwise action). This turns ${}_H\mathbb{M}$ into a monoidal category. To make this work, coassociativity of the coproduct Δ is needed, since it is to show that for V, M and $N \in {}_H\mathbb{M}$, the k-linear isomorphism 1.1 is also H-linear, that is - using the Sweedler notation -

$$h \cdot ((v \otimes m) \otimes n) = \sum (h_{11}v \otimes h_{12}m) \otimes h_2 n = \sum h_1 v \otimes (h_{21}m \otimes h_{22}n) = h \cdot (v \otimes (m \otimes n)),$$

2000 *Mathematics Subject Classification.* Primary 16W30; 18A40, 18D10.
Key words and phrases. Quasi-Hopf algebra, Coinvariants, Two-sided Hopf module, Monoidal category.

were the middle identity is just the coassociativity condition. In this case, the composition of the functors $H \otimes_k (H \otimes_k -)$ can be identified with the functor $(H \otimes_k H) \otimes_k -$. This is an essential property in the theory of bialgebras and Hopf algebras.

For a bialgebra H, a *right H-Hopf module* M is a right H-module $\rho_M : M \otimes_k H \to M$ as well as a right H-comodule $\rho^M : M \to M \otimes_k H$ such that $\rho^M(mh) = \rho^M(m)\Delta(h)$ for $m \in M$, $h \in H$.

The endomorphism ring $\mathrm{End}_k(H)$ has a second k-algebra structure with the convolution product $*$ and an $S \in \mathrm{End}_k(H)$ is an *antipode* if it is an inverse of the identity map with respect to the convolution product, that is, $id * S = \iota \circ \varepsilon = S * id$. A *Hopf algebra* is a bialgebra which has an antipode and the latter condition is equivalent to the fact that

$$- \otimes_k H : \mathbb{M}_k \to \mathbb{M}_H^H, \quad M \mapsto (M \otimes_k H, id \otimes \mu, id \otimes \Delta),$$

is an equivalence of categories (*Fundamental Theorem for Hopf algebras*, e.g. [4, 15.5]). The adjoint (inverse) to this functor was initially defined in terms of *coinvariants* (see [16, Proposition 1]) and it can be seen as the functor $\mathrm{Hom}_H^H(H, -)$ (e.g. [4, 14.8]).

This paper is concerned with *quasi-bialgebras* as defined in Drinfeld [10] by requiring the same axioms as for bialgebras except for the coassociativity condition of the coproduct which is modified by a normalised 3-cocycle $\phi \in H \otimes H \otimes H$ in such a way that the module categories over H are yet monoidal (even rigid monoidal in the finite case). The map $a_{V,M,N}$ considered in 1.1 is no longer H-linear and the theory of Hopf algebras cannot be transferred to the new situation immediately. For example, the convolution algebra $(\mathrm{End}_k(H), *)$ is no longer associative. However, the $a_{V,M,N}$ may be replaced by non-trivial associativity constraints in the monoidal category ${}_H\mathbb{M}$ and this leads the way to the necessary modification of the classical notions. The notion of an antipode was adapted to a *quasi-antipode* in Drinfeld [10]. The Fundamental Theorem corresponds to the comparison functor

$$- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H, \quad N \mapsto (N \otimes H, \varrho_{N \otimes H}, \varrho^{N \otimes H}),$$

being an equivalence (see 3.4, 3.8 and 5.10). This was first shown by Hausser and Nill [14] by defining a projection $E : M \to M$ which leads to a *coinvariant functor* $(-)^{coH} : {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}$. Another projection $\overline{E} : M \to M$ was defined by Bulacu and Caenepeel [5] leading to a distinct (but isomorphic) coinvariant functor $(-)^{\overline{coH}}$.

For a *quasi-bialgebra* H and a right H-comodule algebra $(\mathcal{A}, \rho, \phi_\rho)$, following Bulacu-Caenepeel [6], we consider the category ${}_\mathcal{A}\mathbb{M}_H^H$ of left two-sided Hopf modules and this category can be considered as the Eilenberg-Moore comodule category $({}_\mathcal{A}\mathbb{M}_H)^{-\otimes H}$ over the comonad $- \otimes_k H : {}_\mathcal{A}\mathbb{M}_H \to {}_\mathcal{A}\mathbb{M}_H$ (see 2.3). Adopting the arguments of Hausser-Nill [14] and Bulacu-Caenepeel [5], over a quasi-Hopf algebra H, we define two (isomorphic) types of coinvariants functors $(-)^{coH}$ and $(-)^{\overline{coH}} : {}_\mathcal{A}\mathbb{M}_H^H \to {}_\mathcal{A}\mathbb{M}$. Each of them defines an inverse to the comparison functor $- \otimes H : {}_\mathcal{A}\mathbb{M} \to {}_\mathcal{A}\mathbb{M}_H^H$ (see 5.3, 5.9). Showing that the ${}_\mathcal{A}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, -) : {}_\mathcal{A}\mathbb{M}_H^H \to {}_\mathcal{A}\mathbb{M}$ is also right adjoint to the comparison functor (see 4.4) implies that it has to be isomorphic to the coinvariants functors. An explicit description of these isomorphisms is given in 5.11.

As corollaries, for the case $\mathcal{A} = H$, we obtain that the functor ${}_H\mathrm{Hom}_H^H(H \otimes H, -) : {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}$ is right adjoint to the comparison functor $- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$ (see 3.10) and, as a consequence, both the coinvariants functors defined by Hausser-Nill in [14] and by Bulacu-Caenepeel [5] are isomorphic to this Hom-functor.

2. Preliminaries

In this section we recall definitions and lemmas to be referred to later in this paper. For more details about module theory we refer to [24], about Hopf algebras, to [4], [15], and [22] and about category theory to [2], [17], and [21].

Throughout k will denote a commutative ring with identity. All (co)algebras, bialgebras, Hopf algebras etc. will be over k; unadorned $\otimes$ and Hom mean $\otimes_k$ and Hom_k, respectively. For k-modules M, N, we denote by $\mathrm{Hom}_k(M, N)$ all k-module homomorphisms from M to

N, $M^* := \mathrm{Hom}_k(M,k)$ and $\mathrm{End}_k(M) := \mathrm{Hom}_k(M,M)$. By $\tau_{M,N} : M \otimes N \to N \otimes M$ we denote the twist map which carries $m \otimes n$ to $n \otimes m$.

2.1. Adjoint Functors. A pair (L,R) of functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ between categories $\mathbb{A}$ and $\mathbb{B}$ is called an **adjoint pair** if there exists a natural isomorphism

$$\mathrm{Mor}_{\mathbb{B}}(L(-),-) \to \mathrm{Mor}_{\mathbb{A}}(-,R(-)),$$

which can be described by natural transformations, the unit $\eta : id_{\mathbb{A}} \to RL$ and the counit $\varepsilon : LR \to id_{\mathbb{B}}$, with

$$\varepsilon L \circ L\eta = 1_L, \quad R\varepsilon \circ \eta R = 1_R.$$

2.2. Comonads. A **comonad** $\mathbf{G} = (G,\delta,\varepsilon)$ on a category $\mathbb{A}$ consists of an endofunctor $G : \mathbb{A} \to \mathbb{A}$ and two natural transformations, the comultiplication $\delta : G \to G^2$ and the counit $\varepsilon : G \to id_{\mathbb{A}}$, such that

$$\delta G \circ \delta = G\delta \circ \delta, \quad \varepsilon G \circ \delta = 1_G = G\varepsilon \circ \delta.$$

2.3. Comonads and their comodules. Given a comonad $\mathbf{G} = (G,\delta,\varepsilon)$ on a category $\mathbb{A}$, a **G-comodule** (A,ρ^A) consists of an object $A \in \mathbb{A}$ and an arrow $\rho^A : A \to G(A)$ in $\mathbb{A}$ such that

$$\delta_A \circ \rho^A = G(\rho^A) \circ \rho^A, \quad \varepsilon_A \circ \rho^A = id_A.$$

The class of all **G**-comodules together with **G**-comodule maps form the **Eilenberg-Moore comodule category** over the comonad **G** and is denoted by $\mathbb{A}^{\mathbf{G}}$. The forgetful functor $U^G : \mathbb{A}^{\mathbf{G}} \to \mathbb{A}$ is left adjoint to the free functor $\phi^{\mathbf{G}} : \mathbb{A} \to \mathbb{A}^{\mathbf{G}}$ (e.g. [11]).

2.4. Monoidal categories. A category $\mathbb{A}$ is called a **monoidal (or tensor) category** if there exist a bifunctor $-\otimes- : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$, a distinguished neutral object E, and natural isomorphisms, called associativity and unit constraints,

$$a : (-\otimes-)\otimes- \to -\otimes(-\otimes-), \quad \lambda : E\otimes- \to id_{\mathbb{A}}, \quad \rho : -\otimes E \to id_{\mathbb{A}},$$

$$(id_W \otimes a_{X,Y,Z}) \circ a_{W,(X\otimes Y),Z} \circ (a_{W,X,Y} \otimes id_Z) = a_{W,X,Y\otimes Z} \circ a_{W\otimes X,Y,Z},$$

$$(id_X \otimes \lambda_Y) \circ a_{X,E,Y} = \rho_X \otimes id_Y, \quad \text{for all } W,X,Y,Z \in \mathbb{A}.$$

A monoidal category $(\mathbb{A},\otimes,E,a,\lambda,\rho)$ is said to be **strict** if the isomorphisms a,λ, and ρ are the identity morphisms. For a monoidal category $(\mathbb{A},\otimes,E,a,\lambda,\rho)$, we shortly write $(\mathbb{A},\otimes,E)$ or just $\mathbb{A}$ if no confusion arises. For more details see [15].

2.5. Quasi-bialgebras. A quadruple $(H,\Delta,\varepsilon,\phi)$ is called a **quasi-bialgebra** if H is an associative k-algebra with unit, ϕ an invertible element in $H\otimes H\otimes H$, $\Delta : H \to H\otimes H$ and $\varepsilon : H \to k$ are algebra maps, satisfying the identities, for $h \in H$,

$$(id\otimes\varepsilon)\circ\Delta(h) = h\otimes 1, \quad (\varepsilon\otimes id)\circ\Delta(h) = 1\otimes h, \tag{2.1}$$

$$(id\otimes\Delta)\circ\Delta(h) = \phi\cdot(\Delta\otimes id)\circ\Delta(h)\cdot\phi^{-1}, \tag{2.2}$$

$$(id\otimes id\otimes\Delta)(\phi)(\Delta\otimes id\otimes id)(\phi) = (1\otimes\phi)(id\otimes\Delta\otimes id)(\phi)(\phi\otimes 1), \tag{2.3}$$

$$(id\otimes\varepsilon\otimes id)(\phi) = 1\otimes 1. \tag{2.4}$$

The identities (2.1), (2.3) and (2.4) imply also

$$(\varepsilon\otimes id\otimes id)(\phi) = (id\otimes id\otimes\varepsilon)(\phi) = 1\otimes 1. \tag{2.5}$$

For $h \in H$, we use the Sweedler type notation $\Delta(h) = \sum h_1 \otimes h_2$.

ϕ is called the **Drinfeld reassociator**. The equation (2.3) is a 3-cocycle condition on ϕ. The tensor components of ϕ are denoted by capital letters, those of ϕ^{-1} by small letters,

$$\phi = \sum X^1 \otimes X^2 \otimes X^3 \quad \text{and} \quad \phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3.$$

As in the bialgebra case, the (bi-)module categories over a quasi-bialgebra H is monoidal, yet the associativity constraints in this case are not trivial:

2.6. (Bi-)module categories for quasi-bialgebras. For any quasi-bialgebra $(H,\Delta,\varepsilon,\phi)$, the categories ${}_H\mathbb{M}$, $\mathbb{M}_H$ and ${}_H\mathbb{M}_H$ are monoidal categories with the tensor product $\otimes_k$.

(i) The associativity constraint for objects $M, N, L \in {}_H\mathbb{M}$ is given by

$$a_{M,N,L} : (M \otimes_k N) \otimes_k L \to M \otimes_k (N \otimes_k L), \quad (m \otimes n) \otimes l \mapsto \phi \cdot (m \otimes (n \otimes l)).$$

(ii) The associativity constraint for $M, N, L \in \mathbb{M}_H$ is

$$a'_{M,N,L} : (M \otimes_k N) \otimes_k L \to M \otimes_k (N \otimes_k L), \quad (m \otimes n) \otimes l \mapsto (m \otimes (n \otimes l)) \cdot \phi^{-1},$$

(iii) The associativity constraint for $M, N, L \in {}_H\mathbb{M}_H$ is

$$a''_{M,N,L} : (M \otimes_k N) \otimes_k L \to M \otimes_k (N \otimes_k L), \quad (m \otimes n) \otimes l \mapsto \phi \cdot (m \otimes (n \otimes l)) \cdot \phi^{-1}.$$

2.7. Quasi-Hopf algebras. ([10]) A **quasi-antipode** (S, α, β) for a quasi-bialgebra H consists of an algebra anti-morphism $S : H \to H$ and $\alpha, \beta \in H$ with the identities, for $h \in H$,

$$\sum S(h_1)\alpha h_2 = \varepsilon(h)\alpha, \quad \sum h_1 \beta S(h_2) = \varepsilon(h)\beta \tag{2.6}$$

$$\sum X^1 \beta S(X^2) \alpha X^3 = 1, \qquad \sum S(x^1)\alpha x^2 \beta x^3 = 1. \tag{2.7}$$

These axioms imply $\varepsilon(\alpha)\varepsilon(\beta) = 1$ and $\varepsilon \circ S = \varepsilon$. Note that we do not require the quasi-antipode S to be bijective (as it is done in [10]).

A **quasi-Hopf algebra** is a quasi-bialgebra H together with a quasi-antipode (S, α, β).

2.8. Gauge transformations. Given a quasi-bialgebra $H = (H, \Delta, \varepsilon, \phi)$, a **gauge transformation** on H is an invertible element $F \in H \otimes H$ such that

$$(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1.$$

Using a gauge transformation F on H, one can build a new quasi-bialgebra H_F by keeping the multiplication, unit and counit of H and replacing the comultiplication of H by

$$\Delta_F : H \to H \otimes H, \quad h \mapsto F\,\Delta(h)\,F^{-1},$$

and defining a new Drinfeld reassociator ϕ_F by

$$\phi_F := (1 \otimes F)(id \otimes \Delta)(F) \cdot \phi \cdot (\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1) \in H \otimes H \otimes H.$$

In case H is a quasi-Hopf algebra with antipode S, the quasi-Hopf algebra H_F will be again a quasi-Hopf algebra with the same S but α and β are to be replaced by

$$\alpha_F := \sum S(G^1)\alpha G^2, \quad \beta_F := \sum F^1 \beta S(F^2),$$

where we write $F = \sum F^1 \otimes F^2$ and $F^{-1} = \sum G^1 \otimes G^2 \in H \otimes H$ (see [15, p. 373]).

If H happens to be a bialgebra, then H_F in general is not a bialgebra unless F is a 2-cocycle. Thus, in general, the construction provides non-trivial examples of quasi-bialgebras.

2.9. Properties of quasi-antipodes. For a quasi-Hopf algebra H, Drinfeld ([10]) defines a *gauge element* $f \in H \otimes H$ by the conditions, for any $h \in H$,

$$\begin{aligned} f\,\Delta \circ S(h)\,f^{-1} &= (S \otimes S)\Delta^{cop}(h), \\ (S \otimes S \otimes S)(\phi^{321}) &= (1 \otimes f)(id \otimes \Delta)(f)\phi(\Delta \otimes id)(f^{-1})(f^{-1} \otimes 1), \\ (id \otimes \varepsilon)(f) &= (\varepsilon \otimes id)(f) = 1. \end{aligned}$$

Such an f can be obtained explicitly as follows. First put

$$\sum A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (1 \otimes \phi^{-1})(id \otimes id \otimes \Delta)(\phi),$$

$$\sum B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes id \otimes id)(\phi)(\phi^{-1} \otimes 1),$$

and then define γ and δ in $H \otimes H$ by

$$\gamma = \sum S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4, \quad \delta = \sum B^1 \beta S(B^4) \otimes B^2 \beta S(B^3). \tag{2.8}$$

Then f and f^{-1} are given by the formulas

$$f = \sum (S \otimes S)(\Delta^{op}(x^1))\gamma\Delta(x^2 \beta S(x^3)), \;\; f^{-1} = \sum \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{op}(x^3)), \tag{2.9}$$

and f satisfies the relations

$$f\Delta(\alpha)=\gamma,\quad \Delta(\beta)f^{-1}=\delta. \tag{2.10}$$

Writing $f=\sum f^1\otimes f^2$ and $f^{-1}=\sum g^1\otimes g^2$ in (2.9), it can be easily seen that

$$\sum f^1\beta S(f^2)=S(\alpha),\quad \sum S(\beta f^1)f^2=\alpha,\quad \sum g^1S(g^2\alpha)=\beta. \tag{2.11}$$

3. The category ${}_H\mathbb{M}_H^H$ of quasi-Hopf H-bimodules

Although a quasi-bialgebra H is not a coassociative coalgebra, it can be considered as a coalgebra in the monoidal category ${}_H\mathbb{M}_H$. Thus it makes sense to define comodules over this coalgebra in this monoidal category and this was done by Hausser and Nill in [14] calling them **quasi-Hopf H-bimodules** (generalising Hopf bimodules over Hopf algebras).

For any left H-module N, the tensor product $N\otimes H$ is a right quasi-Hopf H-bimodule (see 3.2). If H is a quasi-Hopf algebra, any quasi-Hopf H-bimodule M is isomorphic to such a tensor product $N\otimes H$, where N is a left H-module (the *coinvariants* of M, [14]). This generalises the *the Fundamental Theorem of Hopf modules* over a Hopf algebra. In this section we are concerned with various interpretations of the coinvariants. For convenience we recall some of the related constructions from Hausser and Nill [14] and Bulacu and Caenepeel [5],

Throughout $(H,\Delta,\varepsilon,\phi)$ denotes a quasi-bialgebra.

3.1. Quasi-Hopf bimodules. Let M be an (H,H)-bimodule and $\varrho^M : M\to M\otimes H$ an (H,H)-bimodule homomorphism. Then (M,ϱ^M) is called a right **quasi-Hopf H-bimodule** if, for all $m\in M$,

$$\begin{aligned}(id_M\otimes\varepsilon)\circ\varrho^M &= id_M,\\ \phi\cdot(\varrho^M\otimes id_H)(\varrho^M(m)) &= (id_M\otimes\Delta)(\varrho^M(m))\cdot\phi,\end{aligned}$$

where we consider the diagonal left and right H-module structure on $M\otimes H$.

A **morphism** between such bimodules is an (H,H)-bimodule morphism $f:M\to L$ satisfying $\varrho^L\circ f=(f\otimes id)\circ\varrho^M$. The category of right quasi-Hopf H-bimodules with the above morphisms is denoted by ${}_H\mathbb{M}_H^H$.

By definition of a quasi-bialgebra, taking $M=H$ and $\varrho^M=\Delta$ provides an example of a quasi-Hopf H-bimodule.

3.2. (H,H)-bimodules and quasi-Hopf bimodules. For any (H,H)-bimodule N, $N\otimes H$ becomes a right quasi-Hopf H-bimodule by the structures, for any $a,b,h\in H, n\in N$,

$$a\cdot(n\otimes h)\cdot b:=\sum a_1nb_1\otimes a_2hb_2=\Delta(a)(n\otimes h)\Delta(b), \tag{3.1}$$

and a coaction $\varrho^{N\otimes H}: N\otimes H\to(N\otimes H)\otimes H$,

$$\varrho^{N\otimes H}(n\otimes h):=\phi^{-1}\cdot(id\otimes\Delta)(n\otimes h)\cdot\phi=\sum x^1\,n\,X^1\otimes x^2h_1X^2\otimes x^3h_2X^3. \tag{3.2}$$

For any (epi-)morphism $g:N_1\to N_2$ in ${}_H\mathbb{M}_H$, $g\otimes id_H : N_1\otimes H\to N_2\otimes H$ is an (epi-)morphism in ${}_H\mathbb{M}_H^H$. This gives rise to a functor

$$-\otimes_k H : {}_H\mathbb{M}_H\to{}_H\mathbb{M}_H^H,\quad N\mapsto(N\otimes H,\varrho_{N\otimes H},\varrho^{N\otimes H}),$$

where $\varrho_{N\otimes H}$ is denotes the diagonal (H,H)-bimodule structure map given in (3.1) and $\varrho^{N\otimes H}$ is the coaction of $N\otimes H$ defined in (3.2).

In particular, $H\otimes H$ belongs to ${}_H\mathbb{M}_H^H$ with the structures, for $h,a,b\in H$,

$$h\cdot(a\otimes b)=\Delta(h)(a\otimes b),\ (a\otimes b)\cdot h=(a\otimes b)\Delta(h),\ \varrho^{H\otimes H}(a\otimes b)=\phi^{-1}\cdot(id\otimes\Delta)(a\otimes b)\cdot\phi.$$

Any left H-module N may be considered as an (H,H)-bimodule with the trivial right H-module structure, that is, $n\cdot b:=\varepsilon(b)n$. Then, in 3.2, the right H-module structure on $N\otimes H$ comes out as $(n\otimes h)\cdot b=\sum\varepsilon(b_1)n\otimes hb_2=n\otimes hb$. This leads to the following important special case:

3.3. Left H-modules and quasi-Hopf bimodules. Let $N \in {}_H\mathbb{M}$ and $a, b, h \in H$, $n \in N$.

(1) $N \otimes H$ is a right quasi-Hopf H-bimodule with the bimodule structure the coaction,

$$a \cdot (n \otimes h) \cdot b := \Delta(a)(n \otimes hb), \tag{3.3}$$

$$\varrho^{N\otimes H} : N \otimes H \to (N \otimes H) \otimes H, \quad n \otimes h \mapsto \phi^{-1} \cdot (id \otimes \Delta)(n \otimes h). \tag{3.4}$$

(2) If $g : N_1 \to N_2$ is an (epi-)morphism in ${}_H\mathbb{M}$, then $g \otimes id_H : N_1 \otimes H \to N_2 \otimes H$ is an (epi-)morphism in ${}_H\mathbb{M}^H_H$.

(3) In particular, $H \otimes H$ belongs to ${}_H\mathbb{M}^H_H$ with the structures

$$h \cdot (a \otimes b) \cdot h' = \Delta(h)(a \otimes b)(1 \otimes h'), \quad \varrho^{H\otimes H}(a \otimes b) = \phi^{-1} \cdot (id \otimes \Delta)(a \otimes b). \tag{3.5}$$

3.4. Comparison functor. For any $N \in {}_H\mathbb{M}$, $N \otimes H \in {}_H\mathbb{M}^H_H$ with the (H,H)-bimodule structure given in (3.3) and the H-comodule structure map given in (3.4). This gives rise to the **comparison functor**

$$- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}^H_H, \quad N \mapsto (N \otimes H, \varrho_{N\otimes H}, \varrho^{N\otimes H}),$$

where $\varrho_{N\otimes H}$ denotes the (H,H)-bimodule structure map from (3.3) and $\varrho^{N\otimes H}$ the right H-comodule structure of $N \otimes H$ defined in (3.4).

In [19, Proposition 3.6], Schauenburg showed that $({}_H\mathbb{M}^H_H, \otimes_H, H)$ is a monoidal category and with this monoidal structure on ${}_H\mathbb{M}^H_H$, the comparison functor $- \otimes_k H$ is monoidal.

We now want to find right adjoints for the comparison functor.

3.5. Hausser-Nill and Bulacu-Caenepeel coinvariants in ${}_H\mathbb{M}^H_H$. Let H be a quasi-Hopf algebra. For any $M \in {}_H\mathbb{M}^H_H$, Hausser and Nill consider the projection map

$$E : M \to M, \quad m \mapsto \sum X^1\, m_0\, \beta S(X^2 m_1)\alpha X^3. \tag{3.6}$$

and define as covariants $M^{coH} := E(M)$, we call these **HN-coinvariants**. They form a left H-module by the action, for $h \in H$, $m \in M^{coH}$,

$$h \blacktriangleright m := E(h\, m) \tag{3.7}$$

Bulacu and Caenepeel in [5], gave an alternative definition for coinvariants, by considering a different projection map

$$\overline{E} : M \to M, \quad m \mapsto \sum m_0 \beta S(m_1),$$

and putting $M^{\overline{coH}} := \overline{E}(M)$, we call them **BC-coinvariants**. They can be characterised by

$$\begin{aligned} M^{\overline{coH}} &= \{m \in M | \overline{E}(m) = m\} \\ &= \{m \in M | \varrho^M(m) = \textstyle\sum x^1\, m\, S(x^3_2 X^3) f^1 \otimes x^2 X^1 \beta S(x^3_1 X^2) f^2\}, \end{aligned} \tag{3.8}$$

where $f = \sum f^1 \otimes f^2 \in H \otimes H$ is the gauge element from (2.9) (see [5, Lemma 3.6]).

$M^{\overline{coH}}$ forms a left H-module with the *left adjoint action* of $h \in H$ (see [5, Lemma 3.6]),

$$h \triangleright m = \sum h_1\, m\, S(h_2).$$

For any morphism $f : M \to L$ in ${}_H\mathbb{M}^H_H$, $f(M^{coH}) \subseteq L^{coH}$ and $f(M^{\overline{coH}}) \subseteq L^{\overline{coH}}$.

These notions yield functors $(-)^{coH}$ and $(-)^{\overline{coH}} : {}_H\mathbb{M}^H_H \to {}_H\mathbb{M}$.

3.6. Relation between the projections E and $\overline{E}$. Let H be a quasi-Hopf algebra, $M \in {}_H\mathbb{M}^H_H$ and $E, \overline{E} : M \to M$ be the projections defined in (3.6) and (3.7). Then (as shown in [5]) for all $m \in M$,

(i) $\overline{E}(m) = \sum E(x^1\, m)\, x^2 \beta S(x^3)$,

(ii) $E(m) = \sum X^1\, \overline{E}(m)\, S(X^2)\alpha X^3$,

(iii) $E : M^{\overline{coH}} \to M^{coH}$ is an H-module isomorphism with inverse $\overline{E} : M^{coH} \to M^{\overline{coH}}$.

3.7. Coinvariants as right adjoints. *Let H be a quasi-Hopf algebra, $N \in {}_H\mathbb{M}$ and $M \in {}_H\mathbb{M}_H^H$.*

(1) $\psi_{N,M} : {}_H\mathrm{Hom}_H^H(N \otimes H, M) \to {}_H\mathrm{Hom}(N, M^{coH}),\ f \mapsto [n \mapsto f(n \otimes 1)]$,
is a functorial isomorphism with inverse map $g \mapsto [n \otimes h \mapsto g(n)\, h)]$.
Thus, the functors $(- \otimes_k H, (-)^{coH})$ form an adjoint pair with unit and counit

$$\eta_N : N \to (N \otimes H)^{coH},\ n \mapsto n \otimes 1_H; \quad \varepsilon_M : M^{coH} \otimes_k H \to M,\ m \otimes h \mapsto m\, h,$$

(2) ${}_H\mathrm{Hom}_H^H(N \otimes H, M) \xrightarrow{\psi_{N,M}} {}_H\mathrm{Hom}(N, M^{\overline{coH}}), \quad f \mapsto [n \mapsto \overline{E}(f(n \otimes 1_H))]$,
is a functorial isomorphism with inverse map $g \mapsto [n \otimes h \mapsto \sum X^1\, g(n)\, S(X^2)\alpha X^3 h]$.
So the functors $(- \otimes H, (-)^{\overline{coH}})$ form an adjoint pair with unit and counit

$$\eta_N : N \to (N \otimes H)^{\overline{coH}}, \quad n \mapsto \sum x^1\, n \otimes x^2 \beta S(x^3),$$
$$\varepsilon_M : M^{\overline{coH}} \otimes_k H \to M, \quad m \otimes h \mapsto \sum X^1\, m\, S(X^2)\alpha X^3 h.$$

This is shown in [6] and [14]. From there we also get:

3.8. Fundamental Theorem of quasi-Hopf bimodules. (see [14, Theorem 3.8]) Let H be a quasi-Hopf algebra and $M \in {}_H\mathbb{M}_H^H$. Referring to the H-module structures defined in 3.5 we get:

(1) $\varepsilon_M : M^{coH} \otimes H \to M, \quad m \otimes h \mapsto m\, h$, is an isomorphism in ${}_H\mathbb{M}_H^H$ with inverse map $\varepsilon_M^{-1}(m) = \sum E(m_0) \otimes m_1$.

(2) $\bar{\nu} : M^{\overline{coH}} \otimes H \to M, \quad n \otimes h \mapsto = \sum X^1\, n\, S(X^2)\alpha X^3 h$, is an isomorphism in ${}_H\mathbb{M}_H^H$ with inverse map $\bar{\nu}^{-1}(m) = \sum \overline{E}(m_0) \otimes m_1$.

The isomorphism $M^{\overline{coH}} \cong M^{coH}$ (see 3.6) implies $(N \otimes H)^{coH} \cong (N \otimes H)^{\overline{coH}}$ as left H-modules. Both $(-)^{coH}$ and $(-)^{\overline{coH}}$ are inverses – hence right adjoints – to the comparison functor $- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$. We can describe these also by a Hom functor.

3.9. The functor ${}_H\mathrm{Hom}_H^H(V \otimes_k H, -)$. *Let $V \in {}_H\mathbb{M}_H$.*

(1) *For $M \in {}_H\mathbb{M}_H$, ${}_H\mathrm{Hom}_H(V \otimes H, M) \in {}_H\mathbb{M}$ with the left H-module structure given for $h, h' \in H$ and $v \in V$, by*

$$(h' \cdot f)(v \otimes h) = f(v\, h' \otimes h).$$

This yields a functor ${}_H\mathrm{Hom}_H(V \otimes H, -) : {}_H\mathbb{M}_H \to {}_H\mathbb{M}$, and by corestriction, a functor

$${}_H\mathrm{Hom}_H^H(V \otimes H, -) : {}_H\mathbb{M}_H^H \to {}_H\mathbb{M}.$$

(2) *Let $N \in {}_H\mathbb{M}$ and consider it as an (H,H)-bimodule with the trivial right H-module structure. Then*

(i) $\psi : {}_H\mathrm{Hom}_H^H(V \otimes H, N \otimes H) \to {}_H\mathrm{Hom}_H(V \otimes H, N),\ f \mapsto (id \otimes \varepsilon) \circ f$,
is an isomorphism in ${}_H\mathbb{M}$ with inverse map $g \mapsto (g \otimes id_H) \circ \varrho^{V \otimes H}$.

(ii) $\theta : {}_H\mathrm{Hom}_H(V \otimes H, N) \to {}_H\mathrm{Hom}(V, N),\ f \mapsto f(- \otimes 1_H)$,
is an isomorphism in ${}_H\mathbb{M}$ with inverse map $g \mapsto [v \otimes h \mapsto \varepsilon(h) g(v)]$.

(iii) ${}_H\mathrm{Hom}(V, N) \to {}_H\mathrm{Hom}_H^H(V \otimes H, N \otimes H),\ g \mapsto g \otimes id_H$,
is a left H-module isomorphism with the inverse map $f \mapsto (id \otimes \varepsilon) \circ f(- \otimes 1_H)$.
Thus the comparison functor $- \otimes_k H : {}_H\mathbb{M} \to {}_H\mathbb{M}_H^H$ is full and faithful.

Let $V = H$ and consider $H \otimes H$ with the structures given in (3.5). Then, for any $M \in {}_H\mathbb{M}_H^H$ we have a left H-module structure on ${}_H\mathrm{Hom}_H^H(H \otimes H, M)$, for $h, a, b \in H$ and $f \in {}_H\mathrm{Hom}_H^H(H \otimes H, M)$,

$$(h \cdot f)(a \otimes b) = f(a\, h \otimes b).$$

This structure leads to a right adjoint for the comparison functor (see also [4, 18.10]).

3.10. ${}_H\mathrm{Hom}^H_H(H\otimes H,-)$ **as right adjoint to the comparison-functor.** *For* $M\in{}_H\mathbb{M}^H_H$ *and* $N\in{}_H\mathbb{M}$, *there is a functorial isomorphism*

$${}_H\mathrm{Hom}^H_H(N\otimes H,M)\to{}_H\mathrm{Hom}(N,{}_H\mathrm{Hom}^H_H(H\otimes H,M)),\ f\mapsto\{n\mapsto[a\otimes b\mapsto f(a\,n\otimes b)]\},$$

with inverse map $g\mapsto[n\otimes h\mapsto g(n)(1_H\otimes h)]$.

Thus the comparison functor $-\otimes_k H$ *(from 3.4) is left adjoint to the functor*

$${}_H\mathrm{Hom}^H_H(H\otimes H,-):{}_H\mathbb{M}^H_H\to{}_H\mathbb{M},$$

with unit and counit

$$\begin{aligned}\eta_N:N\to{}_H\mathrm{Hom}^H_H(H\otimes H,N\otimes H),\quad & n\mapsto[a\otimes b\mapsto a\,n\otimes b],\\ \varepsilon_M:{}_H\mathrm{Hom}^H_H(H\otimes H,M)\otimes H\to M,\quad & f\otimes h\mapsto f(1_H\otimes h).\end{aligned}$$

Proof. The proof will follow from more general assertions in 4.9. □

Of course the three adjoint versions of the comparison functor have to be isomorphic and explicitly this reads as follows.

3.11. Coinvariants as ${}_H\mathrm{Hom}^H_H$**-functor.** *Let* M *be a right quasi-Hopf* H*-bimodule.*

(1) *There is a functorial isomorphism in* ${}_H\mathbb{M}$

$$\bar{\psi}_M:{}_H\mathrm{Hom}^H_H(H\otimes_k H,M)\to M^{coH},\quad f\mapsto f(1\otimes 1),$$

with inverse map $m\mapsto[a\otimes b\mapsto E(a\,m)\,b]$.

(2) *There is a functorial isomorphism in* ${}_H\mathbb{M}$,

$$\bar{\theta}_M:{}_H\mathrm{Hom}^H_H(H\otimes_k H,M)\to M^{\overline{coH}},\quad f\mapsto\sum f(x^1\otimes x^2\beta S(x^3)),$$

with inverse map $m\mapsto[a\otimes b\mapsto\overline{E}(a\,m)\,b]$.

Proof. This will follow from the more general results proved in 5.11. □

4. Two-sided Hopf modules

Again $(H,\Delta,\varepsilon,\phi)$ will denote a quasi-bialgebra. Hausser and Nill [12] gave a definition of H**-comodule (co)algebras** taking care of the non-coassociativity of the coproduct.

4.1. Comodule algebras. A unital associative algebra $\mathcal{A}$ is called a **right** H**-comodule algebra** if there exist an algebra morphism $\rho:\mathcal{A}\to\mathcal{A}\otimes H$ and an invertible element $\phi_\rho\in\mathcal{A}\otimes H\otimes H$ such that

(R1) $\phi_\rho\cdot(\rho\otimes id_H)\circ\rho(a)=(id_H\otimes\Delta)\circ\rho(a)\cdot\phi_\rho$ for all $a\in\mathcal{A}$.

(R2) $(1_{\mathcal{A}}\otimes\phi)(id\otimes\Delta\otimes id)(\phi_\rho)\cdot(\phi_\rho\otimes 1_H)=(id\otimes id\otimes\Delta)(\phi_\rho)\cdot(\rho\otimes id\otimes id)(\phi_\rho)$

(R3) $(id_{\mathcal{A}}\otimes\varepsilon)\circ\rho=id_{\mathcal{A}}$

(R4) $(id_{\mathcal{A}}\otimes\varepsilon\otimes id_H)(\phi_\rho)=1_{\mathcal{A}}\otimes 1_H$.

These conditions also imply $(id\otimes id\otimes\varepsilon)(\phi_\rho)=1_{\mathcal{A}}\otimes 1_H$.

Any quasi-bialgebra H is a right H-comodule algebra with $\mathcal{A}=H$, $\rho=\Delta$ and $\phi_\rho=\phi$.

As for the reassociator ϕ of a quasi-bialgebra H, we use capital letters for the components of ϕ_ρ and small letters for the components of ϕ_ρ^{-1}, that is,

$$\phi_\rho=\sum\tilde{X}^1_\rho\otimes\tilde{X}^2_\rho\otimes\tilde{X}^3_\rho\quad\text{and}\quad\phi_\rho^{-1}=\sum\tilde{x}^1_\rho\otimes\tilde{x}^2_\rho\otimes\tilde{x}^3_\rho. \tag{4.1}$$

Although a quasi-bialgebra is not coassociative one can associate monoidal categories to quasi-bialgebras in which they induce comonads. This point of view has been taken in [7], [14], [19], and [6] in order to define relative Hopf modules, quasi-Hopf bimodules, and two-sided two-cosided Hopf modules.

For a right H-comodule algebra $(\mathcal{A},\rho,\phi_\rho)$, we show that the tensor functor $-\otimes_k H$ is a comonad on the category ${}_{\mathcal{A}}\mathbb{M}_H$ and we consider the category of two-sided Hopf modules ${}_{\mathcal{A}}\mathbb{M}^H_H$ as the Eilenberg-Moore comodule category over this comonad. Furthermore, we show that the Hom-functor ${}_{\mathcal{A}}\mathrm{Hom}^H_H(\mathcal{A}\otimes H,-)$ is right adjoint to the comparison functor $-\otimes_k H$.

Other forms of adjoint functors to $-\otimes_k H$ are obtained by defining Hausser-Nill and Bulacu-Caenepeel type coinvariants for this category (following [6, 5], [9]). The relationship between these is explicitly described.

4.2. Category ${}_{\mathcal{A}}\mathbb{M}_H^H$ of two-sided Hopf modules. Let $(\mathcal{A}, \rho, \phi_\rho)$ be a right H-comodule algebra. A left **two-sided** $(\mathcal{A}, H)$**-Hopf module** is an $(\mathcal{A}, H)$-bimodule M, together with a k-linear map

$$\varrho^M : M \to M \otimes H, \quad \varrho^M(m) = \sum m_0 \otimes m_1,$$

satisfying the relations

$$(id_M \otimes \varepsilon) \circ \varrho^M = id_M, \tag{4.2}$$

$$(id_M \otimes \Delta) \circ \varrho^M(m) = \phi_\rho \cdot (\varrho^M \otimes id_H) \circ \varrho^M(m) \cdot \phi^{-1}, \tag{4.3}$$

$$\varrho^M(a\, m) = \sum a_{(0)}\, m_0 \otimes a_{(1)} m_1, \tag{4.4}$$

$$\varrho^M(m\, h) = \sum m_0\, h_1 \otimes m_1 h_2, \tag{4.5}$$

for $m \in M$, $h \in H$ and $a \in \mathcal{A}$, where $\rho(a) = \sum a_{(0)} \otimes a_{(1)}$.

The category of left two-sided $(\mathcal{A}, H)$-Hopf modules and right H-linear, left $\mathcal{A}$-linear, and right H-colinear maps is denoted by ${}_{\mathcal{A}}\mathbb{M}_H^H$.

For the special case $\mathcal{A} = H$, the category of two-sided (H, H)-Hopf modules is nothing but the category of right quasi-Hopf H-bimodules (see section 3.1).

4.3. Subgenerator for ${}_{\mathcal{A}}\mathbb{M}_H^H$. *Let $(\mathcal{A}, \rho, \phi_\rho)$ be a right H-comodule algebra.*

(1) *For any $N \in {}_{\mathcal{A}}\mathbb{M}$, $N \otimes H \in {}_{\mathcal{A}}\mathbb{M}_H^H$ with structure maps, for $h, h' \in H$, $n \in N$, $a \in \mathcal{A}$,*

$$a \cdot (n \otimes h) = \sum a_{(0)}\, n \otimes a_{(1)} h, \qquad (n \otimes h) \cdot h' = n \otimes hh'. \tag{4.6}$$

$$\varrho^{N\otimes H}(n \otimes h) = \sum \tilde{x}_\rho^1\, n \otimes \tilde{x}_\rho^2 h_1 \otimes \tilde{x}_\rho^3 h_2 = \phi_\rho^{-1} \cdot (id \otimes \Delta)(n \otimes h), \tag{4.7}$$

(2) *If $g : N_1 \to N_2$ is an (epi-)morphism in ${}_{\mathcal{A}}\mathbb{M}$, then $g \otimes id_H : N_1 \otimes H \to N_2 \otimes H$ is an (epi-)morphism in ${}_{\mathcal{A}}\mathbb{M}_H^H$.*

(3) *With the structure maps, for $h, h' \in H$ and $a, a' \in \mathcal{A}$,*

$$a' \cdot (a \otimes h') = \sum a'_{(0)} a \otimes a'_{(1)} h, \quad (a \otimes h)\, h' = a \otimes hh', \quad \varrho^{\mathcal{A}\otimes H}(a \otimes h) = \phi_\rho^{-1} \cdot (\sum a \otimes h_1 \otimes h_2),$$

$\mathcal{A} \otimes H \in {}_{\mathcal{A}}\mathbb{M}_H^H$ and it is a subgenerator *for this category.*

Proof. The parts (1) and (2) are straightforward to see.

(3) Using a similar approach as in section 3.1, we see that for any $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, the left $\mathcal{A}$-module M is a homomorphic image of $\mathcal{A}^{(\Lambda)}$, for some cardinal Λ. Therefore $M \otimes H$ is a homomorphic image of

$$\mathcal{A}^{(\Lambda)} \otimes H \cong (\mathcal{A} \otimes H)^{(\Lambda)}.$$

For any $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, the coaction $\varrho^M : M \to M \otimes H$ is a (mono-)morphism in the category ${}_{\mathcal{A}}\mathbb{M}_H^H$, so we can consider M as a subobject of $M \otimes H$, the latter being generated by $\mathcal{A} \otimes H$ in ${}_{\mathcal{A}}\mathbb{M}_H^H$. □

The parts (1) and (2) in the above assertion give rise to

4.4. The comparison functor $- \otimes_k H : {}_{\mathcal{A}}\mathbb{M} \to {}_{\mathcal{A}}\mathbb{M}_H^H$. Let $(\mathcal{A}, \varrho, \phi_\varrho)$ be a right H-comodule algebra. For any $N \in {}_{\mathcal{A}}\mathbb{M}$, $N \otimes H \in {}_{\mathcal{A}}\mathbb{M}_H^H$ with the $(\mathcal{A}, H)$-bimodule structure from (4.6) and the H-comodule structure map from (4.7). This leads to the **comparison functor**

$$- \otimes_k H : {}_{\mathcal{A}}\mathbb{M} \to {}_{\mathcal{A}}\mathbb{M}_H^H, \quad N \mapsto (N \otimes H, \varrho_{N\otimes H}, \varrho^{N\otimes H}),$$

where $\varrho_{N\otimes H}$ denotes the $(\mathcal{A}, H)$-bimodule and $\varrho^{N\otimes H}$ the right H-comodule structure of $N \otimes H$.

4.5. $-\otimes_k V$ **as endofunctor of** ${}_{\mathcal{A}}\mathbb{M}_H$**.** Let $(\mathcal{A},\rho,\phi_\rho)$ be a right H-comodule algebra, $N \in {}_{\mathcal{A}}\mathbb{M}_H$ and $V \in {}_H\mathbb{M}_H$. Then the coaction

$$\rho : \mathcal{A} \to \mathcal{A} \otimes_k H, \quad \rho(a) = \sum a_{(0)} \otimes a_{(1)},$$

induces an $(\mathcal{A}, H)$-bimodule structure on $N \otimes_k V$, for $h \in H$, $a \in \mathcal{A}$, $v \in V$, and $n \in N$,

$$a \cdot (n \otimes v) \cdot h = \sum a_{(0)}\, n\, h_1 \otimes a_{(1)}\, v\, h_2 = \rho(a)\,(n \otimes v)\,\Delta(h).$$

With this structure we obtain an endofunctor $-\otimes_k V : {}_{\mathcal{A}}\mathbb{M}_H \to {}_{\mathcal{A}}\mathbb{M}_H$, and the special case $V = H$ yields

$$G := -\otimes_k H : {}_{\mathcal{A}}\mathbb{M}_H \to {}_{\mathcal{A}}\mathbb{M}_H, \quad N \mapsto N \otimes H,$$

with the $(\mathcal{A}, H)$-bimodule structure on $N \otimes H$ given as above. This is a comonad.

4.6. $-\otimes_k H$ **as a comonad on** ${}_{\mathcal{A}}\mathbb{M}_H$**.** *Let $(\mathcal{A},\rho,\phi_\rho)$ a right H-comodule algebra.*

(1) *$-\otimes_k H : {}_{\mathcal{A}}\mathbb{M}_H \to {}_{\mathcal{A}}\mathbb{M}_H$ is a comonad on ${}_{\mathcal{A}}\mathbb{M}_H$ with the comultiplication, on $N \in {}_{\mathcal{A}}\mathbb{M}_H$,*

$$\delta_N : N \otimes H \to (N \otimes H) \otimes H, \quad n \otimes h \mapsto \phi_\rho^{-1} \cdot (id \otimes \Delta)(n \otimes h) \cdot \phi,$$

and counit ϵ defined by $\epsilon_N = id_N \otimes \varepsilon : N \otimes H \to N$.

(2) *The category of two-sided Hopf modules ${}_{\mathcal{A}}\mathbb{M}_H^H$ is isomorphic to the Eilenberg-Moore comodule category $({}_{\mathcal{A}}\mathbb{M}_H)^{-\otimes H}$.*

Proof. (1) First we show the coassociativity of δ, i.e., for $N \in {}_{\mathcal{A}}\mathbb{M}_H$, $n \in N$ and $h \in H$,

$$\delta_{N\otimes H} \circ \delta_N(n \otimes h) = (\delta_N \otimes id_H) \circ \delta_N(n \otimes h). \tag{4.8}$$

For this, using the definition of δ_N, we compute

$$\begin{aligned}
\text{L.H.S} &= (\phi_\rho^{-1} \otimes 1) \cdot \{(id \otimes \Delta \otimes id)(\phi_\rho^{-1} \cdot [(id \otimes \Delta)(n \otimes h)] \cdot \phi)\} \cdot (\phi \otimes 1) \\
&= (\phi_\rho^{-1} \otimes 1) \cdot (id \otimes \Delta \otimes id)(\phi_\rho^{-1}) \cdot [(id \otimes \Delta \otimes id) \circ (id \otimes \Delta)(n \otimes h)] \\
&\quad \cdot (id \otimes \Delta \otimes id)(\phi) \cdot (\phi \otimes 1) \\
\text{by (2.2)} \quad &= (\phi_\rho^{-1} \otimes 1) \cdot (id \otimes \Delta \otimes id)(\phi_\rho^{-1}) \cdot (1_{\mathcal{A}} \otimes \phi^{-1}) \cdot [(id \otimes id \otimes \Delta) \circ (id \otimes \Delta)(n \otimes h)] \\
&\quad \cdot (1_H \otimes \phi) \cdot (id \otimes \Delta \otimes id)(\phi) \cdot (\phi \otimes 1).
\end{aligned}$$

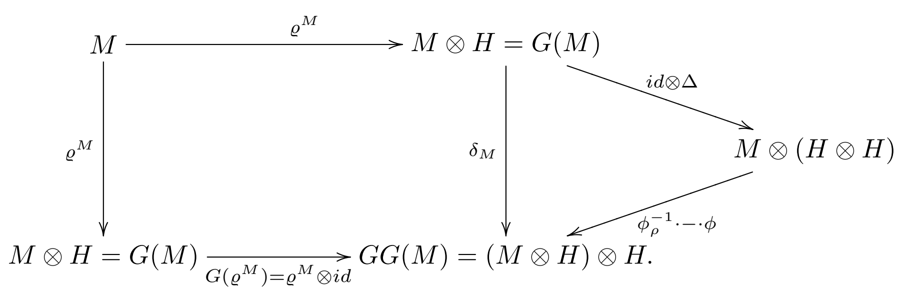
On the other hand,

$$\begin{aligned}
\text{R.H.S} &= (\rho \otimes id \otimes id)(\phi_\rho^{-1}) \cdot \{(id \otimes \otimes id\Delta)(\phi_\rho^{-1} \cdot [(id_N \otimes \Delta)(n \otimes h)] \cdot \phi)\} \cdot (\Delta \otimes id \otimes id)(\phi) \\
&= (\rho \otimes id \otimes id)(\phi_\rho^{-1}) \cdot (id_N \otimes id_H \otimes \Delta)(\phi_\rho^{-1}) \cdot [(id \otimes id \otimes \Delta) \circ (id \otimes \Delta)(n \otimes h)] \\
&\quad \cdot (id \otimes id \otimes \Delta)(\phi) \cdot (\Delta \otimes id \otimes id)(\phi).
\end{aligned}$$

By (2.3) and 4.1, both sides of (4.8) are equal to each other. Thus, δ is coassociative.

It can be easily seen that $\epsilon_N = id_N \otimes \varepsilon : N \otimes H \to N$ is a counit for δ.

(2) To prove the isomorphism $({}_{\mathcal{A}}\mathbb{M}_H)^{-\otimes H} \cong {}_{\mathcal{A}}\mathbb{M}_H^H$, we take an object $M \in ({}_{\mathcal{A}}\mathbb{M}_H)^{-\otimes H}$ and note that we have a G-comodule structure morphism $\varrho^M : M \to M \otimes H = G(M)$ in ${}_{\mathcal{A}}\mathbb{M}_H$ inducing commutativity of the diagram



The commutativity of the outer diagram is precisely the condition (4.3) on M to be a two-sided Hopf module. It is easy to see that the condition (4.2) is equivalent to the counitality of ϵ. □

The following helps to find a right adjoint to the comparison functor (from 4.4).

4.7. The functor ${}_{\mathcal{A}}\mathrm{Hom}_H^H(V\otimes H,-)$. *Let $(\mathcal{A},\rho,\phi_\rho)$ be a right H-comodule algebra, $V\in {}_{\mathcal{A}}\mathbb{M}_{\mathcal{A}}$.*

(1) *If $M\in {}_{\mathcal{A}}\mathbb{M}_H$, then ${}_{\mathcal{A}}\mathrm{Hom}_H(V\otimes H,M)\in {}_{\mathcal{A}}\mathbb{M}$ with the left $\mathcal{A}$-action, for $h\in H$, $a\in\mathcal{A}$ and $v\in V$,*

$$(a\cdot f)(v\otimes h)=f(v\,a\otimes h).$$

This leads to the functor ${}_{\mathcal{A}}\mathrm{Hom}_H(V\otimes H,-):{}_{\mathcal{A}}\mathbb{M}_H\to {}_{\mathcal{A}}\mathbb{M}$ and, by corestriction, to

$${}_{\mathcal{A}}\mathrm{Hom}_H^H(V\otimes H,-):{}_{\mathcal{A}}\mathbb{M}_H^H\to {}_{\mathcal{A}}\mathbb{M}.$$

(2) *Let $N\in {}_{\mathcal{A}}\mathbb{M}$.*

(i) $\psi:{}_{\mathcal{A}}\mathrm{Hom}_H^H(V\otimes H,N\otimes H)\to {}_{\mathcal{A}}\mathrm{Hom}_H(V\otimes H,N),\ f\mapsto (id\otimes\varepsilon)\circ f,$
is an isomorphism in ${}_{\mathcal{A}}\mathbb{M}$ with inverse map $g\mapsto (g\otimes id_H)\circ\varrho^{V\otimes H}$.

(ii) $\theta:{}_{\mathcal{A}}\mathrm{Hom}_H(V\otimes H,N)\to {}_{\mathcal{A}}\mathrm{Hom}(V,N),\ f\mapsto f(-\otimes 1_H),$
is an isomorphism in ${}_{\mathcal{A}}\mathbb{M}$ with inverse map $g\mapsto [v\otimes h\mapsto \varepsilon(h)g(v)]$.

(iii) ${}_{\mathcal{A}}\mathrm{Hom}(V,N)\to {}_{\mathcal{A}}\mathrm{Hom}_H^H(V\otimes H,N\otimes H),\ g\mapsto g\otimes id_H,$
is an isomorphism in ${}_{\mathcal{A}}\mathbb{M}$ with inverse map $f\mapsto (id\otimes\varepsilon)\circ f(-\otimes 1_H)$.
Thus the comparison functor $-\otimes_k H$ is full and faithful.

Note that here we consider the right H-module structure of N to be the trivial one.

Proof. (1) For all $a\in\mathcal{A}$ and $f\in {}_{\mathcal{A}}\mathrm{Hom}_H(V\otimes H,M)$, it is easy to see that $a\cdot f$ is an $(\mathcal{A},H)$-bilinear map. In this way, we have ${}_{\mathcal{A}}\mathrm{Hom}_H(V\otimes H,M)\in {}_{\mathcal{A}}\mathbb{M}$. In case $M\in {}_{\mathcal{A}}\mathbb{M}_H^H$ and $f\in {}_{\mathcal{A}}\mathrm{Hom}_H^H(V\otimes H,M)$, the H-colinearity of of $a\cdot f$ follows from the H-colinearity of f itself. Thus, ${}_{\mathcal{A}}\mathrm{Hom}_H^H(V\otimes H,M)\in {}_{\mathcal{A}}\mathbb{M}$ and we obtain a functor ${}_{\mathcal{A}}\mathrm{Hom}_H^H(V\otimes H,-):{}_{\mathcal{A}}\mathbb{M}_H^H\to {}_{\mathcal{A}}\mathbb{M}$.

(2) (i) As seen in 4.5, the functor $-\otimes_k H:{}_{\mathcal{A}}\mathbb{M}_H\to {}_{\mathcal{A}}\mathbb{M}_H$ is a comonad and the category ${}_{\mathcal{A}}\mathbb{M}_H^H$ of two-sided Hopf modules is the Eilenberg-Moore comodule category $({}_{\mathcal{A}}\mathbb{M}_H)^{-\otimes H}$. Now, considering the functor $-\otimes H:{}_{\mathcal{A}}\mathbb{M}_H\to {}_{\mathcal{A}}\mathbb{M}_H^H$ as the free functor which is right adjoint to the forgetful functor (by 2.3), we obtain the isomorphism of part (i).

(ii) First we note that for $f\in {}_{\mathcal{A}}\mathrm{Hom}_H(V\otimes H,N)$, $h\in H$, $a\in\mathcal{A}$ and $v\in V$,

$$\begin{aligned}
a\,[\theta(f)(v)] &= a\,[f(v\otimes 1_H)]\\
\text{f is left $\mathcal{A}$-linear}\quad &= \sum f(a_{(0)}\,v\otimes a_{(1)})\\
\text{f is right H-linear}\quad &= \sum f(a_{(0)}\,v\otimes 1_H)\,a_{(1)}\\
\text{N is trivial right H-module}\quad &= \sum f(a_{(0)}\,v\otimes 1_H)\,\varepsilon(a_{(1)}) \;=\; f(a\,v\otimes 1_H) \;=\; \theta(f)(a\,v).
\end{aligned}$$

This means that $\theta(f)\in {}_{\mathcal{A}}\mathrm{Hom}(V,N)$. It is straightforward to show that, for $g\in {}_{\mathcal{A}}\mathrm{Hom}(V,N)$, we have $\theta'(g)\in {}_{\mathcal{A}}\mathrm{Hom}_H(V\otimes H,N)$. Bijectivity and left $\mathcal{A}$-linearity of θ follow from direct computations.

(iii) This follows from the composition of the isomorphisms in parts (i) and (ii). □

4.8. Corollary. *Let $(\mathcal{A},\rho,\phi_\rho)$ be a right H-comodule algebra.*

(1) *For $M\in {}_{\mathcal{A}}\mathbb{M}_H^H$, we have a left $\mathcal{A}$-module structure on ${}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A}\otimes H,M)$ given, for $h\in H$, $a,a'\in\mathcal{A}$ and $f\in {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A}\otimes H,M)$, by $(a'\cdot f)(a\otimes h)=f(aa'\otimes h)$.*

(2) *For $N\in {}_{\mathcal{A}}\mathbb{M}$, the morphism*

$$\eta_N:N\to {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A}\otimes H,N\otimes H),\quad n\mapsto [a\otimes h\mapsto a\,n\otimes h],$$

is an isomorphism with inverse map $f\mapsto (id\otimes\varepsilon)\circ f(1_{\mathcal{A}}\otimes 1_H)$.

Proof. (1) Follows directly from 4.7 by taking $V = \mathcal{A}$.

(2) Composition of the isomorphisms ψ^{-1} and θ^{-1} gives rise to the isomorphisms

$$N \cong {}_{\mathcal{A}}\mathrm{Hom}(\mathcal{A}, N) \cong {}_{\mathcal{A}}\mathrm{Hom}_H(\mathcal{A} \otimes H, N) \cong {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, N \otimes H).$$

Using 4.7, we see that this composition gives the isomorphism η_N. □

The Hom-functor from 4.7 is right adjoint to the comparison functor $- \otimes_k H$ from 4.4:

4.9. Hom-tensor adjunction for ${}_{\mathcal{A}}\mathbb{M}_H^H$. *Let $(\mathcal{A}, \varrho, \phi_\varrho)$ be a right H-comodule algebra, $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, and $N \in {}_{\mathcal{A}}\mathbb{M}$. Then there is a functorial isomorphism*

$$\Omega : {}_{\mathcal{A}}\mathrm{Hom}_H^H(N \otimes H, M) \longrightarrow {}_{\mathcal{A}}\mathrm{Hom}(N, {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M)),\ f \mapsto \{n \mapsto [a \otimes h \mapsto f(a\, n \otimes h)]\},$$

with inverse map Ω' given by $g \mapsto \{n \otimes h \mapsto g(n)(1_{\mathcal{A}} \otimes h)\}$.

Thus the functors $(- \otimes_k H, {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, -))$ form an adjoint pair with unit and counit

$$\eta_N : N \to {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, N \otimes H), \quad n \mapsto [a \otimes h \mapsto a\, n \otimes h],$$
$$\varepsilon_M : {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M) \otimes H \to M, \quad f \otimes h \mapsto f(1_{\mathcal{A}} \otimes h).$$

Proof. First we show that for any $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(N \otimes H, M)$, $\Omega(f)$ is left $\mathcal{A}$-linear. For $h \in H, a, a' \in \mathcal{A}$ and $n \in N$,

$$[a' \cdot (\Omega(f)(n))](a \otimes h) = \Omega(f)(n)(aa' \otimes h) = f(n\, aa' \otimes h) = [\Omega(f)(a'\, n))](a \otimes h).$$

Thus, we have $\Omega(f) \in {}_{\mathcal{A}}\mathrm{Hom}(N, {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M))$.

For any $g \in {}_{\mathcal{A}}\mathrm{Hom}(N, {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M))$, we show that $\Omega'(g) \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(N \otimes H, M)$.

(i) $\Omega'(g)$ is left $\mathcal{A}$-linear. For $a \in \mathcal{A}$ and $n \in N$,

$$\begin{aligned} \Omega'(g)((n \otimes h) \cdot a) &= \sum \Omega'(g)(a_{(0)}\, n \otimes a_{(1)} h) = \sum g(a_{(0)}\, n)(1_{\mathcal{A}} \otimes a_{(1)} h) \\ {\scriptstyle g \text{ is right } \mathcal{A}\text{-linear}} &= \sum (a_{(0)} \cdot g(n))(1_{\mathcal{A}} \otimes a_{(1)} h) = \sum g(n)(a_{(0)} \otimes a_{(1)} h) \\ &= g(n)(\rho(a)\,(1 \otimes h)) = a\,[g(n)(1 \otimes h)] \;=\; a\,[\Omega'(g)(n \otimes h)]. \end{aligned}$$

(ii) It can be easily seen that $\Omega'(g)$ is right H-linear.

(iii) For the right H-colinearity of $\Omega'(g)$ we show that

$$(\varrho^M \circ \Omega'(g))(n \otimes h) = \sum (\Omega'(g) \otimes id)(\tilde{x}_\rho^1\, n \otimes \tilde{x}_\rho^2 h_1 \otimes \tilde{x}_\rho^3 h_2).$$

By the colinearity of $g(n)$,

$$(\varrho^M \circ \Omega'(g))(n \otimes h) = \varrho^M(g(n)(1_{\mathcal{A}} \otimes h)) = g(n)(\tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2 h_1) \otimes \tilde{x}_\rho^3 h_2.$$

On the other hand,

$$\begin{aligned} (\Omega'(g) \otimes id)(\sum \tilde{x}_\rho^1\, n \otimes \tilde{x}_\rho^2 h_1 \otimes \tilde{x}_\rho^3 h_2) &= \sum g(\tilde{x}_\rho^1\, n)(1 \otimes \tilde{x}_\rho^2 h_1) \otimes \tilde{x}_\rho^3 h_2 \\ {\scriptstyle g \text{ is } \mathcal{A}\text{-linear}} &= \sum [\tilde{x}_\rho^1 \cdot g(n)](1 \otimes \tilde{x}_\rho^2 h_1) \otimes \tilde{x}_\rho^3 h_2 \\ &= \sum g(n)(\tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2 h_1) \otimes \tilde{x}_\rho^3 h_2. \end{aligned}$$

This shows the H-colinearity of $\Omega'(g)$.

Ω and Ω' are inverse to each other: For $n \in N, h \in H$ and $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M)$,

$$(\Omega' \circ \Omega(f))(n \otimes h) = (\Omega(f))(n)(1_{\mathcal{A}} \otimes h) = f(1_{\mathcal{A}}\, n \otimes h) = f(n \otimes h).$$

Conversely, for any $h \in H, n \in N, a \in \mathcal{A}$ and $g \in {}_{\mathcal{A}}\mathrm{Hom}(N, {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A} \otimes H, M))$,

$$\begin{aligned} \{[(\Omega \circ \Omega')(g)](n)\}(a \otimes h) &= (\Omega'(g))(a\, n \otimes h) = g(a\, n)(1_{\mathcal{A}} \otimes h) \\ {\scriptstyle g \text{ is } \mathcal{A}\text{-linear}} &= [a \cdot g(n)](1_{\mathcal{A}} \otimes h) = g(n)(a \otimes h). \end{aligned}$$

i.e. $\Omega \circ \Omega'(g) = g$. It is easy to see that Ω is functorial in both components M and N. □

Remark. Taking $\mathcal{A} = H$, 3.10 is a special case of 4.9 above.

5. Coinvariants for ${}_{\mathcal{A}}\mathbb{M}_H^H$

In this section we show that right adjoints for the comparison functor from 4.4 can also be described by coinvariants.

Throughout this section, we assume $(H, \Delta, \varepsilon, \phi)$ to be a quasi-Hopf algebra with quasi-antipode (S, α, β). For a right H-comodule algebra $\mathcal{A}$, by [12, Lemma 9.1], we have for all $a \in \mathcal{A}$,

$$\begin{aligned}\sum a_{(0)_{(0)}}\tilde{x}^1_\rho \otimes a_{(0)_{(1)}}\tilde{x}^2_\rho S(a_{(1)}) &= \sum \tilde{x}^1_\rho a \otimes \tilde{x}^2_\rho \beta S(\tilde{x}^3_\rho) \\ \sum \tilde{X}^1_\rho a_{(0)_{(0)}} \otimes S(\tilde{X}^2_\rho a_{(0)_{(1)}})\alpha \tilde{X}^3_\rho a_{(1)} &= \sum a\tilde{X}^1_\rho \otimes S(\tilde{X}^2_\rho)\alpha\tilde{X}^3_\rho\end{aligned}$$

$$\sum (\tilde{X}^1_\rho)_{(0)}\tilde{x}^1_\rho \otimes (\tilde{X}^1_\rho)_{(1)}\tilde{x}^2_\rho\beta S(\tilde{X}^2_\rho\tilde{x}^3_\rho)\alpha\tilde{X}^3_\rho = 1_{\mathcal{A}} \otimes 1_H \tag{5.1}$$

$$\sum \tilde{X}^1_\rho(\tilde{x}^1_\rho)_{(0)} \otimes S(\tilde{X}^2_\rho(\tilde{x}^1_\rho)_{(1)})\alpha\tilde{X}^3_\rho\tilde{x}^2_\rho\beta S(\tilde{x}^3_\rho) = 1_{\mathcal{A}} \otimes 1_H \tag{5.2}$$

5.1. Hausser-Nill-type coinvariants for ${}_{\mathcal{A}}\mathbb{M}_H^H$. Let $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra. For $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, define a projection $\mathcal{E} : M \to M$, for $m \in M$, by

$$\mathcal{E}(m) := \sum \tilde{X}^1_\rho m_0 \beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho,$$

and define **HN-type coinvariants** of M by $M^{coH} := \mathcal{E}(M)$. For $m \in M, a \in \mathcal{A}$ put

$$a \blacktriangleright m := \mathcal{E}(a\, m)$$

Similar to 3.5 (see also [14, Proposition 3.4]), we have the following properties:

5.2. Properties of HN-type coinvariants. For $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$, $a \in \mathcal{A}$, $h \in H$ and $m \in M$ we have, with the above notations,

(i) $\mathcal{E}(m\,h) = \varepsilon(h)\mathcal{E}(m)$,

(ii) $\mathcal{E}^2 = \mathcal{E}$,

(iii) $a \blacktriangleright \mathcal{E}(m) = \mathcal{E}(a\,m) = a \blacktriangleright m$,

(iv) $(ab) \blacktriangleright m = a \blacktriangleright (b \blacktriangleright m)$,

(v) $a\,\mathcal{E}(m) = \sum [a_{(0)} \blacktriangleright \mathcal{E}(m)]\, a_{(1)}$,

(vi) $\sum \mathcal{E}(m_0)\, m_1 = m$,

(vii) $\sum \mathcal{E}(\mathcal{E}(m)_0) \otimes \mathcal{E}(m)_1 = \mathcal{E}(m) \otimes 1$.

Proof.

(i)
$$\begin{aligned}\mathcal{E}(m\,h) &= \sum \tilde{X}^1_\rho (m\,h)_0 \beta S(\tilde{X}^2_\rho (m\,h)_1)\alpha\tilde{X}^3_\rho = \sum \tilde{X}^1_\rho m_0 h_1 \beta S(\tilde{X}^2_\rho m_1 h_2)\alpha\tilde{X}^3_\rho \\ &= \varepsilon(h)\sum \tilde{X}^1_\rho m_0 \beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho = \varepsilon(h)\mathcal{E}(m).\end{aligned}$$

(ii) We use part (i) to compute

$$\begin{aligned}\mathcal{E}^2(m) &= \mathcal{E}(\sum \tilde{X}^1_\rho m_0 \beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho) \\ \overset{\text{by (i)}}{=} &\sum \mathcal{E}(\tilde{X}^1_\rho m_0)\varepsilon(\beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho) \\ &= \sum \mathcal{E}(\tilde{X}^1_\rho m_0)\varepsilon(\beta)\varepsilon(\tilde{X}^2_\rho)\varepsilon(m_1)\varepsilon(\alpha)\varepsilon(\tilde{X}^3_\rho) = \sum \mathcal{E}(m_0\varepsilon(m_1)) = \mathcal{E}(m).\end{aligned}$$

(iii)
$$\begin{aligned}a \blacktriangleright \mathcal{E}(m) &= \mathcal{E}(a\,\mathcal{E}(m)) = \sum \mathcal{E}(a\tilde{X}^1_\rho m_0 \beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho) \\ &= \sum \mathcal{E}(a\tilde{X}^1_\rho m_0)\varepsilon(\beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho) \\ &= \sum \mathcal{E}(a\tilde{X}^1_\rho m_0)\varepsilon(\beta)\varepsilon\circ S(m_1)\varepsilon\circ S(\tilde{X}^2_\rho)\varepsilon(\alpha)\varepsilon(\tilde{X}^3_\rho) \\ &= \sum \mathcal{E}(a\varepsilon(m_1)m_0)\varepsilon(\beta)\varepsilon(\alpha) = \sum \mathcal{E}(a\,m) = a \blacktriangleright m.\end{aligned}$$

(iv) follows immediately from part (iii).

$$\begin{array}{rrcl}
\text{(v)} & a\,\mathcal{E}(m) & = & a\sum \tilde{X}^1_\rho\, m_0\,\beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho = \sum a\tilde{X}^1_\rho\, m_0\,\beta S(m_1)S(\tilde{X}^2_\rho)\alpha\tilde{X}^3_\rho \\
& \text{by (5)} & = & \sum \tilde{X}^1_\rho a_{(0)_{(0)}}\, m_0\,\beta S(m_1)S(a_{(0)_{(1)}})S(\tilde{X}^2_\rho)\alpha\tilde{X}^3_\rho a_{(1)} \\
& & = & \sum \tilde{X}^1_\rho a_{(0)_{(0)}}\, m_0\,\beta S(\tilde{X}^2_\rho a_{(0)_{(1)}} m_1)\alpha\tilde{X}^3_\rho a_{(1)} \\
& & = & \sum \tilde{X}^1_\rho\,(a_{(0)}\,m)_0\,\beta S(\tilde{X}^2_\rho(a_{(0)}\,m)_1)\alpha\tilde{X}^3_\rho a_{(1)} = \sum \mathcal{E}(a_{(0)}\,m)\,a_{(1)} \\
& \text{by (iii)} & = & \sum [a_{(0)} \blacktriangleright \mathcal{E}(m)]\,a_{(1)}.
\end{array}$$

$$\begin{array}{rrcl}
\text{(vi)} & \mathcal{E}(m_0)m_1 & = & \sum \tilde{X}^1_\rho m_{00}\beta S(\tilde{X}^2_\rho m_{01})\alpha\tilde{X}^3_\rho m_1 \\
& \text{by (4.3)} & = & \sum m_0\,X^1\beta S(m_{11}X^2)\alpha m_{12}X^3 \;=\; \sum m_0\,X^1\beta S(X^2)S(m_{11})\alpha m_{12}X^3 \\
& \text{by (2.6)} & = & \sum \varepsilon(m_1)m_0\,(X^1\beta S(X^2)\alpha X^3) = m\,1_H = m.
\end{array}$$

$$\begin{array}{rcl}
\text{(vii)} & & \sum \mathcal{E}(\mathcal{E}(m)_0)\otimes\mathcal{E}(m)_1 \\
& = & \sum \mathcal{E}([\tilde{X}^1_\rho\, m_0\,\beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho]_0)\otimes[(\tilde{X}^1_\rho)\,m_0\,\beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho]_1 \\
& = & \sum \mathcal{E}((\tilde{X}^1_\rho)_{(0)}\,m_{00}\,\beta_1[S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho]_1)\otimes(\tilde{X}^1_\rho)_{(1)}m_{01}\beta_2 S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho)_2 \\
\text{by (i)} & = & \sum \mathcal{E}((\tilde{X}^1_\rho)_{(0)}\,m_{00})\otimes\varepsilon(\beta_1)\varepsilon(S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho)_1)(\tilde{X}^1_\rho)_{(1)}m_{01}\beta_2 S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho)_2 \\
& = & \sum \mathcal{E}((\tilde{X}^1_\rho)_{(0)}\,m_{00})\otimes(\tilde{X}^1_\rho)_{(1)}m_{01}\beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho \\
\text{by (4.3)} & = & \sum \mathcal{E}((\tilde{X}^1_\rho)_{(0)}\tilde{x}^1_\rho\,m_0\,X^1)\otimes(\tilde{X}^1_\rho)_{(1)}\tilde{x}^2_\rho m_{11}X^2\beta S(\tilde{X}^2_\rho\tilde{x}^3_\rho m_{12}X^3)\alpha\tilde{X}^3_\rho \\
\text{by (i)} & = & \sum \mathcal{E}((\tilde{X}^1_\rho)_{(0)}\tilde{x}^1_\rho\,m_0)\varepsilon(X^1)\otimes(\tilde{X}^1_\rho)_{(1)}\tilde{x}^2_\rho m_{11}X^2\beta S(\tilde{X}^2_\rho\tilde{x}^3_\rho m_{12}X^3)\alpha\tilde{X}^3_\rho \\
& = & \sum \mathcal{E}((\tilde{X}^1_\rho)_{(0)}\tilde{x}^1_\rho\,m_0)\otimes(\tilde{X}^1_\rho)_{(1)}\tilde{x}^2_\rho m_{11}\beta S(\tilde{X}^2_\rho\tilde{x}^3_\rho m_{12})\alpha\tilde{X}^3_\rho \\
\text{by (2.6)} & = & \sum \mathcal{E}((\tilde{X}^1_\rho)_{(0)}\tilde{x}^1_\rho\,m_0)\otimes(\tilde{X}^1_\rho)_{(1)}\tilde{x}^2_\rho\varepsilon(m_1)\beta S(\tilde{X}^2_\rho\tilde{x}^3_\rho)\alpha\tilde{X}^3_\rho \\
& = & \sum \mathcal{E}((\tilde{X}^1_\rho)_{(0)}\tilde{x}^1_\rho\,m)\otimes(\tilde{X}^1_\rho)_{(1)}\tilde{x}^2_\rho\beta S(\tilde{X}^2_\rho\tilde{x}^3_\rho)\alpha\tilde{X}^3_\rho \\
\text{by (5.1)} & = & (\mathcal{E}\otimes id)([(1_{\mathcal{A}}\otimes 1_H)\,(m\otimes 1_H)]) = \mathcal{E}(m)\otimes 1_H.
\end{array}$$

□

By (ii), (vi) and (vii), we get characterisations of HN-type coinvariants:

$$\begin{array}{rcl}
M^{coH} := \mathcal{E}(M) & = & \{n\in M|\mathcal{E}(n)=n\} \\
& = & \{n\in M|\sum\mathcal{E}(n_0)\otimes n_1 = \mathcal{E}(n)\otimes 1\} \\
& = & Ke((\mathcal{E}\otimes id)\circ[\varrho^M - (-\otimes 1_H)]).
\end{array}$$

M^{coH} with the left $\mathcal{A}$-action $\blacktriangleright$ is a left $\mathcal{A}$-module and for any morphism $f: M\to L$ in ${}_{\mathcal{A}}\mathbb{M}^H_H$, it is not hard to show that $f(M^{coH})\subseteq L^{coH}$. This gives rise to a functor $(-)^{coH}$ which is right adjoint to the comparison functor.

5.3. The adjoint pair $(-\otimes_k H, (-)^{coH})$ for HN-type coinvariants. *Let $(\mathcal{A},\rho,\phi_\rho)$ be a right H-comodule algebra, $N\in{}_{\mathcal{A}}\mathbb{M}$ and $M\in{}_{\mathcal{A}}\mathbb{M}^H_H$. There is a functorial isomorphism*

$$\psi_{N,M}: {}_{\mathcal{A}}\mathrm{Hom}^H_H(N\otimes H, M)\to{}_{\mathcal{A}}\mathrm{Hom}(N, M^{coH}),\quad f\mapsto[n\mapsto f(n\otimes 1)],$$

with inverse map $\psi'_{N,M}$ given by $g\mapsto[n\otimes h\mapsto g(n)\,h)]$.

Thus, the functors

$$-\otimes_k H: {}_{\mathcal{A}}\mathbb{M}\to{}_{\mathcal{A}}\mathbb{M}^H_H,\quad (-)^{coH}: {}_{\mathcal{A}}\mathbb{M}^H_H\to{}_{\mathcal{A}}\mathbb{M},$$

form an adjoint pair with unit and counit

$$\eta_N: N\to(N\otimes H)^{coH},\ n\mapsto n\otimes 1;\quad \varepsilon_M: M^{coH}\otimes_k H\to M,\ m\otimes h\mapsto m\,h.$$

Proof. First, we show that $f(n\otimes 1)\in M^{coH}$: Since f is H-colinear,

$$\varrho^M(f(n\otimes 1))=\sum f(\tilde{x}^1_\rho n\otimes \tilde{x}^2_\rho)\otimes \tilde{x}^3_\rho,$$

so we have

$$\begin{array}{rcl}\mathcal{E}(f(n\otimes 1)) &=& \sum \tilde{X}^1_\rho\, f(\tilde{x}^1_\rho n\otimes \tilde{x}^2_\rho)\,\beta S(\tilde{X}^2_\rho \tilde{x}^3_\rho)\alpha \tilde{X}^3_\rho\\ {\scriptstyle f \text{ is } (\mathcal{A},H)\text{-bilinear}} &=& \sum f(\rho(\tilde{X}^1_\rho)\,(\tilde{x}^1_\rho n\otimes \tilde{x}^2_\rho\beta S(\tilde{X}^2_\rho \tilde{x}^3_\rho)\alpha \tilde{X}^3_\rho\\ &=& \sum f([(\tilde{X}^1_\rho)_{(0)}\,\tilde{x}^1_\rho\otimes (\tilde{X}^1_\rho)_{(1)}\tilde{x}^2_\rho\beta S(\tilde{X}^2_\rho \tilde{x}^3_\rho)\alpha \tilde{X}^3_\rho]\,(n\otimes 1))\\ {\scriptstyle \text{by } (5.1)} &=& f(n\otimes 1).\end{array}$$

$\psi:=\psi_{N,M}$ and $\psi':=\psi'_{N,M}$ are inverse to each other: For $n\in N, h\in H$, $f\in {}_{\mathcal{A}}\mathrm{Hom}^H_H(N\otimes H,M)$,

$$[(\psi'\circ\psi)(f)](n\otimes h)=\psi(f)(n)\,h=f(n\otimes 1)\,h=f(n\otimes h).$$

Conversely, for $n\in N$ and $g\in {}_{\mathcal{A}}\mathrm{Hom}(N,M^{coH})$,

$$[(\psi\circ\psi')(g)](n)=\psi'(g)(n\otimes 1)=g(n)\,1=g(n).$$

□

Remark. For $\mathcal{A}=H$, 5.3 implies 3.7 as a special case.

5.4. HN-type coinvariants of $N\otimes H\in {}_{\mathcal{A}}\mathbb{M}^H_H$. *For any $N\in {}_{\mathcal{A}}\mathbb{M}$, the HN-type coinvariants of the two-sided Hopf module $N\otimes H$, come out as*

$$(N\otimes H)^{coH}\simeq N,$$

and for $n\in N$ and $h\in H$, we have $\mathcal{E}(n\otimes h)=n\otimes\varepsilon(h)1_H$.

Proof. The definition of the right H-module structure of $N\otimes H$ implies that $(n\otimes h)=(n\otimes 1)\,h$. Now, by part (i) of 5.2, we have

$$\mathcal{E}(n\otimes h)=\mathcal{E}((n\otimes 1)\,h)=\mathcal{E}(n\otimes 1)\varepsilon(h),$$

thus we are left to show that $\mathcal{E}(n\otimes 1)=n\otimes 1_H$:

$$\begin{array}{rcl}\mathcal{E}(n\otimes 1) &=& \sum \tilde{X}^1_\rho\,(n\otimes 1)_0\,\beta S(\tilde{X}^2_\rho (n\otimes 1)_1)\alpha \tilde{X}^3_\rho\\ &=& \sum \tilde{X}^1_\rho\cdot(\tilde{x}^1_\rho n\otimes \tilde{x}^2_\rho)\,\beta S(\tilde{X}^2_\rho \tilde{x}^3_\rho)\alpha \tilde{X}^3_\rho\\ &=& \sum (\tilde{X}^1_\rho)_{(0)}\tilde{x}^1_\rho n\otimes (\tilde{X}^1_\rho)_{(1)}\tilde{x}^2_\rho\beta S(\tilde{X}^2_\rho \tilde{x}^3_\rho)\alpha \tilde{X}^3_\rho\\ {\scriptstyle \text{by } (5.1)} &=& (1_{\mathcal{A}}\otimes 1_H)\,(n\otimes 1)=n\otimes 1_H.\end{array}$$

□

This means that that the unit $\eta_N: N\to (N\otimes H)^{coH}$ of the adjunction in 5.3 is an isomorphism with inverse map $n\otimes h\mapsto n\varepsilon(h)$ proving (again) the fully faithfulness of the comparison functor $-\otimes_k H: {}_{\mathcal{A}}\mathbb{M}\to {}_{\mathcal{A}}\mathbb{M}^H_H$ (see 2.1 and 4.7).

5.5. Fundamental Theorem for ${}_{\mathcal{A}}\mathbb{M}^H_H$ with HN-type coinvariants. *Let $(\mathcal{A},\rho,\phi_\rho)$ be a right H-comodule algebra and $M\in {}_{\mathcal{A}}\mathbb{M}^H_H$. Consider $M^{coH}=\mathcal{E}(M)$ as a left $\mathcal{A}$-module with left $\mathcal{A}$-action $\blacktriangleright$, defined by*

$$a\blacktriangleright m:=\mathcal{E}(a\,m)=\sum \tilde{X}^1_\rho a_{(0)}\,m_0\,\beta S(\tilde{X}^2_\rho a_{(1)}m_1)\alpha \tilde{X}^3_\rho.$$

Then the map

$$\varepsilon_M: M^{coH}\otimes H\to M,\qquad m\otimes h\mapsto m\,h,$$

is an isomorphism in ${}_{\mathcal{A}}\mathbb{M}^H_H$ with inverse map $\varepsilon'_M(m)=\sum\mathcal{E}(m_0)\otimes m_1$.

Proof. ε_M is an isomorphism of k-modules: for $h \in H$ and $n \in N$,

$$\begin{aligned}
\varepsilon'_M \circ \varepsilon_M(n \otimes h) &= \varepsilon'_M(n\,h) = \sum \mathcal{E}(n_0\,h_1) \otimes n_1 h_2 \\
\text{by (i)} \quad &= \sum \mathcal{E}(n_0)\varepsilon(h_1) \otimes n_1 h_2 \\
&= \sum \mathcal{E}(n_0) \otimes n_1 h = \sum (\mathcal{E}(n_0) \otimes n_1)(1 \otimes h) \\
\text{by (vii)} \quad &= (n \otimes 1)(1 \otimes h) = n \otimes h.
\end{aligned}$$

Conversely, for $m \in M$,

$$\varepsilon_M \circ \varepsilon'_M(m) = \varepsilon_M(\sum \mathcal{E}(m_0) \otimes m_1) = \sum \mathcal{E}(m_0)\, m_1 = m.$$

We are left to show that ε_M is a morphism in ${}_{\mathcal{A}}\mathbb{M}^H_H$. By definition of the $(\mathcal{A}, H)$-bimodule structure of $M^{coH} \otimes H$, for $h \in H$, $a \in \mathcal{A}$ and $n \in M^{coH}$,

$$a \cdot (n \otimes h) \cdot h' = \sum a_{(0)} \blacktriangleright n \otimes a_{(1)} h h' = \sum \mathcal{E}(a_{(0)}\, n) \otimes a_{(1)} h h'.$$

Therefore, we have

$$\begin{aligned}
\varepsilon_M(a \cdot (n \otimes h) \cdot h') &= \sum \mathcal{E}(a_{(0)}\, n)\, a_{(1)} h h' \\
\text{by (iii)} \quad &= \sum [a_{(0)} \blacktriangleright \mathcal{E}(n)]\, a_{(1)} h h' \\
&= a\, \mathcal{E}(n)\, h h' \ = \ a\, n\, h h' \ = \ a\, \varepsilon_M(n \otimes h)\, h'.
\end{aligned}$$

Finally, we show that ε'_M (and therefore ε_M) is H-colinear: for $m \in M$,

$$\begin{aligned}
\varrho^{M^{coH} \otimes H}(\varepsilon'_M(m)) &= \sum \mathcal{E}(\tilde{x}^1_\rho\, m_0) \otimes \tilde{x}^2_\rho m_{11} \otimes \tilde{x}^3_\rho m_{12} \\
&= \sum \mathcal{E}(m_{00}\, X^1) \otimes m_{01} X^2 \otimes m_1 X^3 \\
&= \sum \mathcal{E}(m_{00}) \varepsilon(X^1) \otimes m_{01} X^2 \otimes m_1 X^3 \\
&= \sum \mathcal{E}(m_{00}) \otimes m_{01} \otimes m_1 \\
&= (\mathcal{E} \otimes id) \varrho^M(m_0) = (\varepsilon'_M \otimes id) \varrho^M(m).
\end{aligned}$$

□

The above form of the Fundamental Theorem yields an additional characterisation of coinvariants, for any $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$, as

$$\begin{aligned}
M^{coH} &= \{n \in M | \varrho^M(n) = \sum (\tilde{x}^1_\rho \blacktriangleright n)\, \tilde{x}^2_\rho \otimes \tilde{x}^3_\rho\} \\
&= Ke(\varrho^M - [(\varrho_M \otimes id) \circ (\mathcal{E} \otimes id \otimes id)(\phi^{-1}_\rho\, (- \otimes 1_{\mathcal{A}} \otimes 1_H))]).
\end{aligned}$$

5.6. Bulacu-Caenepeel-type coinvariants for ${}_{\mathcal{A}}\mathbb{M}^H_H$. Let $\mathcal{A}$ be a right H-comodule algebra. With similar arguments as in (3.8) (see also [5]), for any $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$, we consider the projection

$$\overline{\mathcal{E}} : M \to M, \quad m \mapsto \sum m_0\, \beta S(m_1),$$

and define **BC-type coinvariants** for $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$ as

$$M^{\overline{coH}} := \overline{\mathcal{E}}(M) = \{m \in M \mid \overline{\mathcal{E}}(m) = m\}.$$

This generalises the concept of coinvariants of quasi-Hopf bimodules $M \in {}_H\mathbb{M}^H_H$.

5.7. HN versus BC-type projections. *Let $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$ and $\mathcal{E}, \overline{\mathcal{E}} : M \to M$ be defined by*

$$\mathcal{E}(m) = \sum \tilde{X}^1_\rho\, m_0\, \beta S(\tilde{X}^2_\rho m_1) \alpha \tilde{X}^3_\rho, \qquad \overline{\mathcal{E}}(m) = \sum m_0\, \beta S(m_1),$$

for all $m \in M$. Then

(i) $\overline{\mathcal{E}}(m) = \sum \mathcal{E}(\tilde{x}^1_\rho\, m)\, \tilde{x}^2_\rho \beta S(\tilde{x}^3_\rho)$, $\quad \mathcal{E}(m) = \sum \tilde{X}^1_\rho\, \overline{\mathcal{E}}(m)\, S(\tilde{X}^2_\rho) \alpha \tilde{X}^3_\rho$,

(ii) $\overline{\mathcal{E}} : M^{coH} \to M^{\overline{coH}}$ *is an isomorphism in ${}_{\mathcal{A}}\mathbb{M}$ with inverse $\mathcal{E} : M^{\overline{coH}} \to M^{coH}$.*

Proof. (i)

$$\begin{aligned}
\sum \mathcal{E}(\tilde{x}^1_\rho m)\,\tilde{x}^2\beta S(\tilde{x}^3_\rho) &= \sum \tilde{X}^1_\rho(\tilde{x}^1_\rho)_{(0)}\,m_0\,\beta S(\tilde{X}^2_\rho(\tilde{x}^1_\rho)_{(1)}m_1)\alpha\tilde{X}^3_\rho\tilde{x}^2_\rho\beta S(\tilde{x}^3_\rho)\\
&= \sum \tilde{X}^1_\rho(\tilde{x}^1_\rho)_{(0)}\,m_0\,\beta S(m_1)S(\tilde{X}^2_\rho(\tilde{x}^1_\rho)_{(1)})\alpha\tilde{X}^3_\rho\tilde{x}^2_\rho\beta S(\tilde{x}^3_\rho)\\
&= \sum \tilde{X}^1_\rho(\tilde{x}^1_\rho)_{(0)}\,\overline{E}(m)\,S(\tilde{X}^2_\rho(\tilde{x}^1_\rho)_{(1)})\alpha\tilde{X}^3_\rho\tilde{x}^2_\rho\beta S(\tilde{x}^3_\rho)\\
{\scriptstyle \text{by (5.2)}} &= \overline{E}(m).
\end{aligned}$$

The other equality is an easy substitution of $\overline{\mathcal{E}}(m)$.

(ii) For any $m \in M^{coH}$,

$$\begin{aligned}
\mathcal{E}(\overline{\mathcal{E}}(m)) &= \mathcal{E}(\sum m_0\,\beta S(m_1))\\
&= \sum \tilde{X}^1_\rho\,m_{00}\,\beta_1 S(m_1)_1\beta S(\tilde{X}^2_\rho m_{01}\beta_2 S(m_1)_2)\alpha\tilde{X}^3_\rho\\
&= \sum \tilde{X}^1_\rho\,m_{00}\,\beta_1 S(m_1)_1\beta S(S(m_1)_2)S(\beta_2)S(\tilde{X}^2_\rho m_{01})\alpha\tilde{X}^3_\rho\\
&= \sum(\tilde{X}^1_\rho\,m_{00}\,\beta S(\tilde{X}^2_\rho m_{01})\alpha\tilde{X}^3_\rho)\varepsilon(m_1)\varepsilon(\beta)\\
&= \mathcal{E}(m_0)\varepsilon(m_1)\varepsilon(\beta) = \mathcal{E}(m) = m.
\end{aligned}$$

On the other hand, for any $m \in M^{\overline{coH}}$,

$$\begin{aligned}
\overline{\mathcal{E}}(\mathcal{E}(m)) &= \overline{\mathcal{E}}(\sum \tilde{X}^1_\rho\,m_0\,\beta S(\tilde{X}^2_\rho m_1)\alpha\tilde{X}^3_\rho) = \overline{\mathcal{E}}(\sum \tilde{X}^1_\rho\,m_0\,\beta S(m_1)S(\tilde{X}^2_\rho)\alpha\tilde{X}^3_\rho)\\
&= \overline{\mathcal{E}}(\sum \tilde{X}^1_\rho\,\overline{\mathcal{E}}(m)\,S(\tilde{X}^2_\rho)\alpha\tilde{X}^3_\rho)\\
{\scriptstyle m\in M^{\overline{coH}}} &= \overline{\mathcal{E}}(\sum \tilde{X}^1_\rho\,m\,S(\tilde{X}^2_\rho)\alpha\tilde{X}^3_\rho)\\
&= \sum(\tilde{X}^1_\rho\,m\,S(\tilde{X}^2_\rho)\alpha\tilde{X}^3_\rho)_0\,\beta S([\tilde{X}^1_\rho\,m\,S(\tilde{X}^2_\rho)\alpha\tilde{X}^3_\rho]_1)\\
&= \sum(\tilde{X}^1_\rho\,m)_0\,S(\tilde{X}^2_\rho)_1(\alpha\tilde{X}^3_\rho)_1\beta S((\tilde{X}^1_\rho\,m)_1 S(\tilde{X}^2_\rho)_2(\alpha\tilde{X}^3_\rho)_2)\\
&= \sum \varepsilon(\tilde{X}^2_\rho)\varepsilon(\alpha\tilde{X}^3_\rho)(\tilde{X}^1_\rho\,m)_0\,\beta S((\tilde{X}^1_\rho\,m)_1)\\
&= \sum \varepsilon(\tilde{X}^2_\rho)\varepsilon(\alpha\tilde{X}^3_\rho)\overline{\mathcal{E}}(\tilde{X}^1_\rho\,m) = \overline{\mathcal{E}}(m) = m.
\end{aligned}$$

For left $\mathcal{A}$-linearity of $\mathcal{E}$ we compute

$$\begin{aligned}
\mathcal{E}(a\triangleright m) &= \sum \mathcal{E}(a_{(0)}\,m_0\,\beta S(a_{(1)}m_1))\\
&= \sum \tilde{X}^1_\rho a_{(0)_{(0)}}\,m_{00}\,\beta_1 S(a_{(1)}m_1)_1\beta S(\tilde{X}^2_\rho a_{(0)_{(1)}}m_{01}\beta_2 S(a_{(1)}m_1)_2)\alpha\tilde{X}^3_\rho\\
&= \sum \tilde{X}^1_\rho a_{(0)_{(0)}}\,m_{00}\,\beta_1 S(a_{(1)}m_1)_1\beta S(\beta_2 S(a_{(1)}m_1)_2)S(\tilde{X}^2_\rho a_{(0)_{(1)}}m_{01})\alpha\tilde{X}^3_\rho\\
&= \sum \tilde{X}^1_\rho a_{(0)_{(0)}}\,m_{00}\,\varepsilon(\beta)\varepsilon(S(a_{(1)}m_1))\beta S(\tilde{X}^2_\rho a_{(0)_{(1)}}m_{01})\alpha\tilde{X}^3_\rho\\
&= \sum \varepsilon(a_{(1)}m_1)\tilde{X}^1_\rho a_{(0)_{(0)}}\,m_{00}\,\beta S(a_{(0)_{(1)}}m_{01})S(\tilde{X}^2_\rho)\alpha\tilde{X}^3_\rho\\
&= \sum \varepsilon(a_{(1)}m_1)\tilde{X}^1_\rho\,\overline{\mathcal{E}}(a_{(0)}\,m_0)\,S(\tilde{X}^2_\rho)\alpha\tilde{X}^3_\rho = \mathcal{E}(a\,m) = a\blacktriangleright\mathcal{E}(m).
\end{aligned}$$

□

With similar arguments as in [5, Lemma 3.6], we show

5.8. Characterisation of BC-type coinvariants in ${}_{\mathcal{A}}\mathbb{M}^H_H$.

$$M^{\overline{coH}} = \{m\in M \mid \varrho^M(m) = \sum \tilde{x}^1_\rho\,m\,S((\tilde{x}^3_\rho)_2X^3)f^1\otimes\tilde{x}^2_\rho X^1\beta S((\tilde{x}^3_\rho)_1X^2)f^2\}. \tag{5.3}$$

Proof. Let $m \in M^{\overline{coH}}$. Then

$$\begin{array}{rcl}
\varrho^M(m) & = & \varrho^M(\overline{\mathcal{E}}(m)) = \sum m_{00}\,\beta_1 S(m_1)_1 \otimes m_{01}\beta_2 S(m_1)_2 \\
\text{by (2.10)} & = & \sum m_{00}\,\delta^1 f^1 S(m_1)_1 \otimes m_{01}\delta^2 f^2 S(m_1)_2 \\
\text{by (2.9)} & = & \sum m_{00}\,\delta^1 S(m_{12}) f^1 \otimes m_{01}\delta^2 S(m_{11}) f^2 \\
\text{by (2.8)} & = & \sum m_{00}\,x^1 Y^1 \beta S((m_{12}x_2^3 X^3 Y^3) f^1 \otimes m_{01} x^2 X^1 Y_1^2 \beta S((m_{11}x_1^3 X^2 Y_2^2) f^2 \\
\text{by (2.6)} & = & \sum m_{00}\,x^1 Y^1 \beta S((m_{12}x_2^3 X^3 Y^3) f^1 \otimes m_{01} x^2 X^1 \varepsilon(Y^2) \beta S((m_{11}x_1^3 X^2) f^2 \\
\text{by (2.4)} & = & \sum m_{00}\,x^1 \beta S((m_1 x^3)_2 X^3) f^1 \otimes m_{01} x^2 X^1 \beta S((m_1 x^3)_1 X^2) f^2 \\
\text{by (4.3)} & = & \sum \tilde{x}_\rho^1\, m_0\, \beta S((\tilde{x}_\rho^3 m_{12})_2 X^3) f^1 \otimes \tilde{x}_\rho^2 m_{11} X^1 \beta S((\tilde{x}_\rho^3 m_{12})_1 X^2) f^2 \\
\text{by (2.2)} & = & \sum \tilde{x}_\rho^1\, m_0\, \beta S((\tilde{x}_\rho^3)_2 X^3 m_{12})) f^1 \otimes \tilde{x}_\rho^2 X^1 m_{111} \beta S((\tilde{x}_\rho^3)_1 X^2 m_{112})) f^2 \\
\text{by (2.6)} & = & \sum \tilde{x}_\rho^1\, m_0\, \beta S((\tilde{x}_\rho^3)_2 X^3 m_{12})) f^1 \otimes \tilde{x}_\rho^2 X^1 \varepsilon(m_{11}) \beta S((\tilde{x}_\rho^3)_1 X^2) f^2 \\
 & = & \sum \tilde{x}_\rho^1\, m_0\, \beta S((\tilde{x}_\rho^3)_2 X^3 m_1) f^1 \otimes \tilde{x}_\rho^2 X^1 \beta S((\tilde{x}_\rho^3)_1 X^2) f^2 \\
 & = & \sum \tilde{x}_\rho^1\, \overline{\mathcal{E}}(m)\, S((\tilde{x}_\rho^3)_2 X^3) f^1 \otimes \tilde{x}_\rho^2 X^1 \beta S((\tilde{x}_\rho^3)_1 X^2) f^2 \\
(m \in M^{\overline{coH}}) & = & \sum \tilde{x}_\rho^1\, m\, S((\tilde{x}_\rho^3)_2 X^3) f^1 \otimes \tilde{x}_\rho^2 X^1 \beta S((\tilde{x}_\rho^3)_1 X^2) f^2.
\end{array}$$

Conversely, if $\varrho^M(m) = \sum \tilde{x}_\rho^1\, m\, S((\tilde{x}_\rho^3)_2 X^3) f^1 \otimes \tilde{x}^2 X^1 \beta S((\tilde{x}_\rho^3)_1 X^2) f^2$, then

$$\begin{array}{rcl}
\overline{\mathcal{E}}(m) & = & \sum m_0\, \beta S(m_1) \\
 & = & \sum \tilde{x}_\rho^1\, m\, S((\tilde{x}_\rho^3)_2 X^3) f^1 \beta S(\tilde{x}_\rho^2 X^1 \beta S((\tilde{x}_\rho^3)_1 X^2) f^2) \\
 & = & \sum \tilde{x}_\rho^1\, m\, S((\tilde{x}_\rho^3)_2 X^3) f^1 \beta S(f^2) S(\tilde{x}_\rho^2 X^1 \beta S((\tilde{x}_\rho^3)_1 X^2)) \\
\text{by (2.11)} & = & \sum \tilde{x}_\rho^1\, m\, S((\tilde{x}_\rho^3)_2 X^3) S(\alpha) S(\tilde{x}_\rho^2 X^1 \beta S((\tilde{x}_\rho^3)_1 X^2)) \\
 & = & \sum \tilde{x}_\rho^1\, m\, S(\tilde{x}_\rho^2 X^1 \beta S((\tilde{x}_\rho^3)_1 X^2) \alpha (\tilde{x}_\rho^3)_2 X^3) \\
 & = & \sum \tilde{x}_\rho^1\, m\, S(\tilde{x}_\rho^2 X^1 \beta S(X^2) S((\tilde{x}_\rho^3)_1) \alpha (\tilde{x}_\rho^3)_2 X^3) \\
 & = & \sum \tilde{x}_\rho^1\, m\, S(\tilde{x}_\rho^2 X^1 \beta S(X^2) \varepsilon(\tilde{x}_\rho^3) \alpha X^3) \\
 & = & \sum m\, S(X^1 \beta S(X^2) \alpha X^3) = m.
\end{array}$$

□

The above characterisation generalises the BC-coinvariants in (3.8). It can be also be written as

$$M^{\overline{coH}} \;=\; Ke(\varrho^M - \{\sum (\tilde{x}_\rho^1 \otimes \tilde{x}_\rho^2)\,(-\otimes 1_H)\,[S((\tilde{x}_\rho^3)_2 X^3) f^1 \otimes X^1 \beta S((\tilde{x}_\rho^3)_1 X^2) f^2]\})$$

where $f = \sum f^1 \otimes f^2$ is the Drinfeld gauge element defined in equation (2.9).

A new left $\mathcal{A}$-module structure on $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$ can be defined by

$$a \triangleright m := \sum a_{(0)}\, m\, S(a_{(1)}),$$

for $a \in \mathcal{A}$, and $m \in M$, where $\rho(a) = \sum a_{(0)} \otimes a_{(1)}$. With this left $\mathcal{A}$-action, $M^{\overline{coH}}$ can be considered as a left $\mathcal{A}$-submodule of M. It is straightforward to see that for any morphism $g : M \to L$ in ${}_{\mathcal{A}}\mathbb{M}_H^H$, we have $g(M^{\overline{coH}}) \subseteq L^{\overline{coH}}$. This leads to an alternative *coinvariants functor*

$$(-)^{\overline{coH}} : {}_{\mathcal{A}}\mathbb{M}_H^H \to {}_{\mathcal{A}}\mathbb{M},$$

which we will show to be right adjoint to the comparison functor $- \otimes_k H$ (from 4.4).

5.9. The adjoint pair $(- \otimes_k H, (-)^{\overline{coH}})$ for BC-type coinvariants. *Let $\mathcal{A}$ be a right H-comodule algebra, $N \in {}_{\mathcal{A}}\mathbb{M}$ and $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$.*

(1) *There is a functorial isomorphism*

$$\psi_{N,M} : {}_{\mathcal{A}}\mathrm{Hom}_H^H(N \otimes H, M) \longrightarrow {}_{\mathcal{A}}\mathrm{Hom}(N, M^{\overline{coH}}), \quad f \mapsto [n \mapsto f(\tilde{x}_\rho^1 n \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3))],$$

with inverse map $\psi'_{N,M}$ *given by* $g \mapsto [n \otimes h \mapsto \sum \tilde{X}_\rho^1\, g(n)\, S(\tilde{X}_\rho^2)\alpha \tilde{X}_\rho^3 h)]$.

(2) *The functors* $(- \otimes_k H, (-)^{\overline{coH}})$ *form an adjoint pair with unit and counit*

$$\bar{\eta}_N : N \to (N \otimes H)^{\overline{coH}}, \quad n \mapsto \sum \tilde{x}_\rho^1 n \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3),$$
$$\bar{\varepsilon}_M : M^{\overline{coH}} \otimes_k H \to M, \quad m \otimes h \mapsto \sum \tilde{X}_\rho^1\, m\, S(\tilde{X}_\rho^2)\alpha \tilde{X}_\rho^3 h.$$

(3) *the unit map* $\bar{\eta}_N$ *is an isomorphism, in particular*

$$(N \otimes H)^{\overline{coH}} = \{\sum \tilde{x}_\rho^1 n \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3) \mid n \in N\}.$$

Proof. (1) We show that ψ and ψ' are inverse to each other. For $n \in N$, $h \in H$ and $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(N \otimes H, M)$,

$$\begin{aligned}
[(\psi' \circ \psi)(f)](n \otimes h) &= \sum \tilde{X}_\rho^1\, \psi(f)(n)\, S(\tilde{X}_\rho^2)\alpha \tilde{X}_\rho^3 h \\
&= \sum \tilde{X}_\rho^1\, f(\tilde{x}_\rho^1 n \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3)\, S(\tilde{X}_\rho^2))\alpha \tilde{X}_\rho^3 h \\
\text{f is $(\mathcal{A},H)$-bilinear} \quad &= \sum f(\rho(\tilde{X}_\rho^1)[\tilde{x}_\rho^1 n \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3)\, S(\tilde{X}_\rho^2)\alpha \tilde{X}_\rho^3 h]) \\
&= \sum f((\tilde{X}_\rho^1)_{(0)} \tilde{x}_\rho^1 n \otimes (\tilde{X}_\rho^1)_{(1)} \tilde{x}_\rho^2 \beta S(\tilde{X}_\rho^2 \tilde{x}_\rho^3)\alpha \tilde{X}_\rho^3 h) \underset{(5.1)}{=} f(n \otimes h).
\end{aligned}$$

Conversely, for $n \in N$ and $g \in {}_{\mathcal{A}}\mathrm{Hom}(N, M^{\overline{coH}})$,

$$\begin{aligned}
[(\psi \circ \psi')(g)](n) &= \psi'(g)(\tilde{x}_\rho^1 n \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3)) \\
&= \sum \tilde{X}_\rho^1\, g(\tilde{x}_\rho^1 n)\, S(\tilde{X}_\rho^2)\alpha \tilde{X}^3 \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3) \\
\text{g is left $\mathcal{A}$-linear} \quad &= \sum \tilde{X}_\rho^1\, (\tilde{x}_\rho^1 \triangleright g(n))\, S(\tilde{X}_\rho^2)\alpha \tilde{X}_\rho^3 \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3) \\
&= \sum \tilde{X}_\rho^1 (\tilde{x}_\rho^1)_{(0)}\, g(n)\, S((\tilde{x}_\rho^1)_{(1)}) S(\tilde{X}_\rho^2)\alpha \tilde{X}_\rho^3 \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3) \\
&= \sum \tilde{X}_\rho^1 (\tilde{x}_\rho^1)_{(0)} \cdot g(n) \cdot S(\tilde{X}_\rho^2 (\tilde{x}_\rho^1)_{(1)})\alpha \tilde{X}_\rho^3 \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3) \underset{(5.2)}{=} g(n).
\end{aligned}$$

(2) is a consequence of (1).

(3) For $n \otimes h \in (N \otimes H)^{\overline{coH}}$,

$$\begin{aligned}
\varrho^{N \otimes H}(n \otimes h) &= \sum \tilde{x}_\rho^1 \cdot (n \otimes h) \cdot S((\tilde{x}_\rho^3)_2 X^3) f^1 \otimes \tilde{x}_\rho^2 X^1 \beta S((\tilde{x}_\rho^3)_1 X^2) f^2 \\
&= \sum (\tilde{x}_\rho^1)_{(0)}\, n \otimes (\tilde{x}_\rho^1)_{(1)} h S((\tilde{x}_\rho^3)_2 X^3) f^1 \otimes \tilde{x}_\rho^2 X^1 \beta S((\tilde{x}_\rho^3)_1 X^2) f^2.
\end{aligned}$$

On the other hand, $\varrho^{N \otimes H}(n \otimes h) = \sum \tilde{x}_\rho^1 n \otimes \tilde{x}_\rho^2 h_1 \otimes \tilde{x}_\rho^3 h_2$.

Comparing this two values for $\varrho^{N \otimes H}(n \otimes h)$ and applying $id \otimes \varepsilon \otimes id$ on both sides, we obtain

$$n \otimes h = \sum \varepsilon(h)(\tilde{x}_\rho^1 n \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3)).$$

This shows that the unit map $\bar{\eta}_N$ is an isomorphism with inverse map $n \otimes h \mapsto n\varepsilon(h)$. This shows again that the comparison functor is fully faithful. □

5.10. Fundamental Theorem for ${}_{\mathcal{A}}\mathbb{M}_H^H$ with BC-type coinvariants. *Let* $(\mathcal{A}, \rho, \phi_\rho)$ *be a right comodule algebra and* $M \in {}_{\mathcal{A}}\mathbb{M}_H^H$. *Consider* $M^{\overline{coH}} \otimes H$ *as an object in* ${}_{\mathcal{A}}\mathbb{M}_H^H$ *with the structures*

$$a \cdot (n \otimes h) \cdot h' = \sum a_1 \triangleright n \otimes a_2 h h', \quad \varrho'(n \otimes h) = \sum \tilde{x}_\rho^1 \triangleright n \otimes \tilde{x}_\rho^2 h_1 \otimes \tilde{x}_\rho^3 h_2,$$

for $h, h' \in H$, $a \in \mathcal{A}$ *and* $n \in M^{\overline{coH}}$. *Then the map*

$$\bar{\varepsilon}_M : M^{\overline{coH}} \otimes H \to M, \quad n \otimes h \mapsto \sum \tilde{X}_\rho^1\, n\, S(\tilde{X}_\rho^2)\alpha \tilde{X}_\rho^3 h$$

is an isomorphism in ${}_{\mathcal{A}}\mathbb{M}_H^H$ *with inverse map* $\bar{\varepsilon}'_M$ *given by* $m \mapsto \sum \overline{E}(m_0) \otimes m_1$.

Proof. By 5.7, we have the isomorphism $\mathcal{E} : M^{\overline{coH}} \to M^{coH}$ in ${}_{\mathcal{A}}\mathbb{M}$ and tensoring it with H, we obtain

$$\mathcal{E} \otimes id_H : M^{\overline{coH}} \otimes H \to M^{coH} \otimes H,$$

as an isomorphism in ${}_{\mathcal{A}}\mathbb{M}^H_H$. By the Hausser-Nill version of the Fundamental Theorem for ${}_{\mathcal{A}}\mathbb{M}^H_H$ 3.8, there is an isomorphism $\varepsilon_M : M^{coH} \otimes H \to M$, $m \otimes h \mapsto m\,h$ in ${}_{\mathcal{A}}\mathbb{M}^H_H$. Combining these two isomorphisms, we have the isomorphism

$$\bar{\varepsilon}_M = \varepsilon_M \circ (\mathcal{E} \otimes id) : M^{\overline{coH}} \otimes H \to M^{coH} \otimes H \to M,$$

$$\begin{aligned} m \otimes h \mapsto \mathcal{E}(m) \otimes h \quad &\mapsto \quad \mathcal{E}(m)\,h = \sum \tilde{X}^1_\rho\, m_0\, \beta S(\tilde{X}^2_\rho m_1)\alpha \tilde{X}^3_\rho h \\ &= \quad \sum \tilde{X}^1_\rho\, m_0\, \beta S(m_1) S(\tilde{X}^2_\rho)\alpha \tilde{X}^3_\rho h \;=\; \sum \tilde{X}^1_\rho\, \overline{\mathcal{E}}(m)\, S(\tilde{X}^2_\rho)\alpha \tilde{X}^3_\rho h \\ {\scriptstyle m\in M^{\overline{coH}}} \quad &= \quad \sum \tilde{X}^1_\rho\, m\, S(\tilde{X}^2_\rho)\alpha \tilde{X}^3_\rho h. \end{aligned}$$

The inverse map $\bar{\varepsilon}'_M$ can also be computed directly as

$$\begin{aligned} \bar{\varepsilon}'_M(m) \quad &= \quad (\overline{\mathcal{E}} \otimes id)(\sum \mathcal{E}(m_0) \otimes m_1) \;=\; \sum \overline{\mathcal{E}}(\mathcal{E}(m_0)) \otimes m_1 \\ &= \quad \sum \overline{\mathcal{E}}(\tilde{X}^1_\rho\, m_{00}\, \beta S(\tilde{X}^2_\rho m_{01})\alpha \tilde{X}^3_\rho) \otimes m_1 \\ &= \quad \sum \overline{\mathcal{E}}(\tilde{X}^1_\rho\, m_{00})\varepsilon(\beta)\varepsilon(\tilde{X}^2_\rho m_{01})\varepsilon(\alpha \tilde{X}^3_\rho) \otimes m_1 = \sum \overline{\mathcal{E}}(m_0) \otimes m_1. \end{aligned}$$

□

As shown in the proceding sections, for any comodule algebra $\mathcal{A}$ over H, the right adjoint of the comparison functor $- \otimes_k H$ (from 4.4) can be written in three different forms, namely

$${}_{\mathcal{A}}\mathrm{Hom}^H_H(H \otimes H, -), \quad (-)^{coH} \text{ and } (-)^{\overline{coH}} : {}_{\mathcal{A}}\mathbb{M}^H_H \to {}_{\mathcal{A}}\mathbb{M}.$$

These have to be isomorphic and we describe the isomorphisms explicitly.

5.11. Coinvariants for ${}_{\mathcal{A}}\mathbb{M}^H_H$ as Hom-functor. *Let H be a quasi-Hopf algebra, $(\mathcal{A}, \rho, \phi_\rho)$ a right H-comodule algebra, and $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$.*

(1) *There is a functorial isomorphism in ${}_{\mathcal{A}}\mathbb{M}$,*

$$\bar{\psi}_M : {}_{\mathcal{A}}\mathrm{Hom}^H_H(\mathcal{A} \otimes_k H, M) \to M^{coH}, \quad f \mapsto f(1_{\mathcal{A}} \otimes 1_H),$$

with inverse map $\bar{\psi}'_M$ given by $m \mapsto [a \otimes h \mapsto \mathcal{E}(a\,m)\,h]$.

(2) *There is a functorial isomorphism in ${}_{\mathcal{A}}\mathbb{M}$,*

$$\bar{\theta}_M : {}_{\mathcal{A}}\mathrm{Hom}^H_H(\mathcal{A} \otimes_k H, M) \to M^{\overline{coH}}, \quad f \mapsto \sum f(\tilde{x}^1_\rho \otimes \tilde{x}^2_\rho \beta S(\tilde{x}^3_\rho)),$$

with inverse map $\bar{\theta}'_M$ given by $m \mapsto [a \otimes h \mapsto \mathcal{E}(a\,m)\,h]$.

Proof. (1) Substituting $N = \mathcal{A}$ in the isomorphism in 5.3, we obtain for $M \in {}_{\mathcal{A}}\mathbb{M}^H_H$ the isomorphisms

$$\bar{\psi}_M : {}_{\mathcal{A}}\mathrm{Hom}^H_H(\mathcal{A} \otimes_k H, M) \xrightarrow{\psi_{\mathcal{A},M}} {}_{\mathcal{A}}\mathrm{Hom}(\mathcal{A}, M^{coH}) \cong M^{coH},$$

$$f \mapsto [a \mapsto f(a \otimes 1_H)] \mapsto f(1_{\mathcal{A}} \otimes 1_H).$$

The inverse map $\bar{\psi}'_M$ is obtained as the composition

$$M^{coH} \cong {}_{\mathcal{A}}\mathrm{Hom}(\mathcal{A}, M^{coH}) \xrightarrow{\psi'_{\mathcal{A},M}} {}_{\mathcal{A}}\mathrm{Hom}^H_H(\mathcal{A} \otimes_k H, M),$$

$$m \mapsto [a \mapsto a \blacktriangleright m = \mathcal{E}(a\,m)] \mapsto [a \otimes h \mapsto \mathcal{E}(a\,m)\,h].$$

Here, $\psi_{\mathcal{A},M}$ is the isomorphism given in 5.3 and $\psi'_{\mathcal{A},M}$ is its inverse.

It remains to show that $\bar{\psi}_M$ is left $\mathcal{A}$-linear: For $a \in \mathcal{A}$ and $f \in {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A}\otimes H, M)$,

$$\begin{array}{rcl} a \blacktriangleright \bar{\psi}_M(f) & = & \mathcal{E}(a\, f(1_{\mathcal{A}}\otimes 1_H)) = \sum \mathcal{E}(f(a_{(0)}\otimes a_{(1)})) \\ & = & \sum \tilde{X}_\rho^1\, f(a_{(0)}\otimes a_{(1)})_0\, \beta S(\tilde{X}_\rho^2 f(a_{(0)}\otimes a_{(1)})_1)\alpha \tilde{X}_\rho^3 \\ f \text{ is } H\text{-colinear} & = & \sum \tilde{X}_\rho^1\, f(\tilde{x}_\rho^1 a_{(0)}\otimes \tilde{x}_\rho^2 a_{(1)_1}\beta S(\tilde{X}_\rho^2\tilde{x}_\rho^3 a_{(1)_2})\alpha\tilde{X}_\rho^3) \\ f \text{ is } \mathcal{A}\text{-linear} & = & \sum f(\rho(\tilde{X}_\rho^1)\,(\tilde{x}_\rho^1 a_{(0)}\otimes \tilde{x}_\rho^2 a_{(1)_1})\beta S(a_{(1)_2})S(\tilde{x}_\rho^3)S(\tilde{X}_\rho^2)\alpha\tilde{X}_\rho^3) \\ \text{by (2.6)} & = & \sum f(\rho(\tilde{X}_\rho^1)\,(\tilde{x}_\rho^1 a\otimes \tilde{x}_\rho^2\beta S(\tilde{x}_\rho^3)S(\tilde{X}_\rho^2)\alpha\tilde{X}_\rho^3) \\ & = & \sum f([(\tilde{X}_\rho^1)_{(0)}\,\tilde{x}_\rho^1\otimes (\tilde{X}_\rho^1)_{(1)}\tilde{x}_\rho^2\beta S(\tilde{X}_\rho^2\tilde{x}_\rho^3)\alpha\tilde{X}_\rho^3](a\otimes 1)) \\ \text{by (5.1)} & = & f(a\otimes 1_H) = (a\cdot f)(1_{\mathcal{A}}\otimes 1_H) = \bar{\psi}_M(a\cdot f). \end{array}$$

(2) Setting $N=\mathcal{A}$ in the isomorphism given in 5.9, we obtain the isomorphisms

$$\bar{\theta}_M : {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A}\otimes_k H, M) \overset{\psi_{\mathcal{A},M}}{\longrightarrow} {}_{\mathcal{A}}\mathrm{Hom}(H, M^{\overline{coH}}) \cong M^{\overline{coH}},$$

$$f \mapsto [a\mapsto \overline{\mathcal{E}}(f(a\otimes 1)) = \sum f(\tilde{x}_\rho^1 a\otimes \tilde{x}_\rho^2\beta S(\tilde{x}_\rho^3))] \mapsto \overline{\mathcal{E}}(f(1_{\mathcal{A}}\otimes 1_H)) = \sum f(\tilde{x}_\rho^1\otimes\tilde{x}_\rho^2\beta S(\tilde{x}_\rho^3)).$$

The inverse map $\bar{\theta}'_M$ is obtained as the composition

$$\bar{\theta}'_M : M^{\overline{coH}} \cong {}_{\mathcal{A}}\mathrm{Hom}(\mathcal{A}, M^{\overline{coH}}) \overset{\psi'_{\mathcal{A},M}}{\longrightarrow} {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A}\otimes_k H, M),$$

$$m \mapsto [a\mapsto a\triangleright m = \overline{E}(a\,m)] \mapsto \{a\otimes b\mapsto \sum \tilde{q}_\rho^1\,\overline{\mathcal{E}}(a\,m)\,S(\tilde{q}_\rho^2)h = \mathcal{E}(a\,m)\,h\}.$$

Here, $\psi_{\mathcal{A},M}$ is the isomorphism given in 5.9 and $\psi'_{\mathcal{A},M}$ is its inverse.

Similar to part (1), considering the left $\mathcal{A}$-action $\triangleright$ on $M^{\overline{coH}}$, we must show that $\bar{\theta}_M$ is left $\mathcal{A}$-linear: for $a\in\mathcal{A}$ and $f\in {}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A}\otimes H, M)$,

$$\begin{array}{rcl} a\triangleright\bar{\theta}_M(f) & = & \overline{\mathcal{E}}(a\,f(1_{\mathcal{A}}\otimes 1_H)) = \sum\overline{\mathcal{E}}(f(a_{(0)}\otimes a_{(1)})) \\ & = & \sum f(a_{(0)}\otimes a_{(1)})_0\,\beta S(f(a_{(0)}\otimes a_{(1)})_1) \\ f \text{ is } H\text{-colinear} & = & \sum f(\tilde{x}_\rho^1 a_{(0)}\otimes\tilde{x}_\rho^2 a_{(1)_1}\beta S(\tilde{x}_\rho^3 a_{(1)_2})) \\ & = & \sum f(\tilde{x}_\rho^1\,a\otimes\tilde{x}_\rho^2\beta S(\tilde{x}_\rho^3)) \;=\; (a\cdot f)(\tilde{x}_\rho^1\otimes\tilde{x}_\rho^2\beta S(\tilde{x}_\rho^3)) \;=\; \bar{\theta}_M(a\cdot f). \end{array}$$

□

Remark. Part (2) can also be proved by composing the isomorphism $\bar{\psi}_M$ from part (1) with the isomorphism $\overline{\mathcal{E}} : M^{coH}\to M^{\overline{coH}}$ leading to the isomorphism

$${}_{\mathcal{A}}\mathrm{Hom}_H^H(\mathcal{A}\otimes_k H, M) \overset{\bar{\psi}_M}{\longrightarrow} M^{coH} \overset{\overline{\mathcal{E}}}{\longrightarrow} M^{\overline{coH}},$$

given by

$$\begin{array}{rcl} f\mapsto f(1\otimes 1) & \mapsto & \overline{\mathcal{E}}(f(1\otimes 1)) \;=\; \sum f(1\otimes 1)_0\,\beta S(f(1\otimes 1)_1) \\ \text{by } H\text{-colinearity of } f & = & \sum f(\tilde{x}_\rho^1\otimes\tilde{x}_\rho^2)\,\beta S(\tilde{x}_\rho^3) \\ \text{by } H\text{-linearity of } f & = & \sum f(\tilde{x}_\rho^1\otimes\tilde{x}_\rho^2\beta S(\tilde{x}_\rho^3)). \end{array}$$

The inverse map comes out as

$$\begin{array}{rcl} m\overset{\theta'}{\mapsto}\{a\otimes h & \mapsto & \mathcal{E}(a\,\mathcal{E}(m))\,h \;=\; \sum\mathcal{E}([a_{(0)}\blacktriangleright\mathcal{E}(m)]\,a_{(1)})\,h \\ & = & [\sum\mathcal{E}(a_{(0)}\blacktriangleright\mathcal{E}(m))\varepsilon(a_{(1)})]\,h \\ & = & [\mathcal{E}(a\blacktriangleright\mathcal{E}(m))]\,h \;=\; [\mathcal{E}(\mathcal{E}(a\,m))]\,h = \mathcal{E}(a\,m)\,h\}. \end{array}$$

References

[1] Balan, A., *A Morita context and Galois extensions for quasi-Hopf Algebras*, Comm. Alg. **37(4)**, 1129-1150 (2009).
[2] Barr, M., Wells, C., *Toposes, Triples and Theories*, Springer (1985).
[3] Böhm, G., Brzeziński, T., and Wisbauer, R., *Monads and Comonads in Module Categories*, J. Algebra **322(5)**, 1719-1747 (2009).
[4] Brzeziński, T., Wisbauer, R., *Corings and Comodules*, London Math. Soc. LNS **309**, Cambridge University Press (2003).
[5] Bulacu, D., Caenepeel, S., *Integrals for (dual) Quasi-Hopf Algebras. Applications*, J. Alg. **266(2)**, 552-583 (2003).
[6] Bulacu, D., Caenepeel, S., *Two-sided two-cosided Hopf modules and Doi-Hopf Modules for Quasi-Hopf Algebras*, J. Alg. **270**, 55-95 (2003).
[7] Bulacu, D., Nauwelaerts, E., *Relative Hopf Modules for (dual) Quasi-Hopf Algebras*, J. Alg. **229** 632-659 (2000).
[8] Bulacu, D., Panaite, F., Van Oystayen, F., *Quasi-Hopf Algebra Actions and Smash Products*, Comm. Alg. **28(2)**, 631-651 (2000).
[9] Bulacu, D., Torrecillas, B., *Two-sided two-cosided Hopf modules and Yetter-Drinfeld modules for Quasi-Hopf algebras*, Appl. Categor. Str. **14**, 503-530 (2006).
[10] Drinfeld, V.G., *Quasi-Hopf algebras*, Leningrad Math. J. **1**, 1419-1457 (1990).
[11] Eilenberg, S., Moore, J.C., *Adjoint functors and Triples*, Ill. J. Math. **9**, 381-398 (1965).
[12] Hausser, F., Nill, F., *Diagonal crossed products by duals of quasi-quantum groups*, Rev. Math. Phys. **11**, 553-629 (1999).
[13] Hausser, F., Nill, F., *Doubles of quasi-quantum groups*, Comm. Math. Phys. **199**, 547-589 (1999).
[14] Hausser, F., Nill, F., *Integral theory for quasi-Hopf Algebras*, arXiv:Math. QA/9904164 (1999).
[15] Kassel, K., *Quantum Groups*, Springer, Berlin (1995).
[16] Larson, R.G., Sweedler, M.E., *An associative orthogonal bilinear form for Hopf algebras*, Am. J. Math. **91** 75-94 (1969).
[17] Mac Lane, S., *Category for the working mathematician*, Springer (1971).
[18] Mesablishvili, B., Wisbauer, R., *Bimonads and Hopf monads on categories*, J. K-Theory (2010) doi: is010001014 jkt105 (arXiv: 0710.1163 (2007)).
[19] Schauenburg, P., *Hopf modules and the Double of a Quasi-Hopf Algebra*, Trans. Amer. Math. Soc. **354**, 3349-3378 (2002).
[20] Schauenburg, P., *Quotients of finite quasi-Hopf algebras*, Hopf algebras in noncommutative geometry and physics. Proc. conf. on Hopf algebras and quantum groups, Brussels 2002, Caenepeel, Stefaan (ed.) et al., Marcel Dekker, LN Pure Appl. Math. 239, 281-290 (2005).
[21] Schauenburg, P., *Actions of monoidal categories and generalized Hopf smash products*, J. Alg. **270**, 521-563 (2003).
[22] Sweedler, M.E., *Hopf Algebras*, Benjamin, New York (1969).
[23] Wisbauer, R., *Algebras versus Coalgebras*, Appl. Categor. Struct. **16** (1-2), 255-295 (2008).
[24] Wisbauer, R., *Foundations of Module and Ring Theory*, Gordon and Breach, Reading (1991).

[1] Department of Mathematics, Malayer University, P. O. Box 65719-95863, Malayer, Iran.
E-mail address: *bagheri_saeid@yahoo.com*

[2] Mathematisches Inst., Universitaet Duesseldorf, P. O. Box 40225, Duesseldorf, Germany.
E-mail address: `wisbauer@math.uni-duesseldorf.de`