ON UNIFORM BOUNDS OF PRIMENESS IN MATRIX RINGS

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Abstract. A subset $\mathcal{S}$ of an associative ring R is a *uniform insulator* for R provided $a\mathcal{S}b \neq 0$ for any nonzero $a, b \in R$. The ring R is called *uniformly strongly prime of bound* m if R has uniform insulators and the smallest of those has cardinality m. Here we compute these bounds for matrix rings over fields and obtain refinements of some results of van den Berg in this context.

More precisely, for a field F and a positive integer k, let m be the bound of the matrix ring $M_k(F)$, and let n be $\dim_F(\mathcal{V})$, where $\mathcal{V}$ is a subspace of $M_k(F)$ of maximal dimension with respect to not containing rank one matrices. We show that $m+n = k^2$. This implies, for example, that $n = k^2 - k$ if and only if there exists a (nonassociative) division algebra over F of dimension k.

1. Introduction

Following Handelman and Lawrence [1, p. 211], we call a subset $\mathcal{S}$ of an associative ring $\mathcal{R}$ a *uniform insulator* for $\mathcal{R}$ if $a\mathcal{S}b \neq 0$ for all $a, b \in \mathcal{R}$ with $a \neq 0 \neq b$. The ring $\mathcal{R}$ is said to be *uniformly strongly prime* if it contains a finite uniform insulator. For such a ring we set

$$m(\mathcal{R}) = \min\{|\mathcal{S}| \mid \mathcal{S} \text{ is a uniform insulator of } \mathcal{R}\},$$

and we say $\mathcal{R}$ is *uniformly strongly prime of bound* n provided $m(\mathcal{R}) = n$.

In what follows F is a field and $M_k(F)$ stands for the algebra of $k \times k$ matrices over F, where k is a positive integer. For $\mathcal{R} = M_k(F)$ we put $m_k(F) := m(\mathcal{R})$.

The systematic study of $m(\mathcal{R})$ was initiated by van den Berg in [2, 3] and we recall the following of his results ([3], Theorems 4, 7, 11).

Theorem 1.1.

(i) *Let $\mathcal{D}$ be a division ring and $\mathcal{R} = M_k(\mathcal{D})$. Then $k \leq m(\mathcal{R}) \leq 2k-1$.*

(ii) *If F is an algebraically closed field, then $m_k(F) = 2k-1$.*

(iii) *Let F be a field and assume there exists a nonassociative division F-algebra of dimension k, then $m_k(F) = k$.*

In [3], Remark 2, van den Berg asks if the converse of assertion (iii) holds. In the present paper we obtain a positive answer to this question (see 1.4(iii)). We sharpen the above results by studying connections of the

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uniform bound of $M_k(F)$ with (maximal) dimension of certain subspaces of $M_k(F)$ and $M_{k^2}(F)$. We also pose some open questions.

Before stating our results we fix some notation. Given positive integers k, ℓ we denote by $M_{k,\ell}(F)$ the $k \times \ell$-matrices over the field F.

For $A = (a_{ij})_{1\le i\le k, 1\le j\le \ell} \in M_{k,\ell}(F)$ and $B \in M_{\ell,k}(F)$, we define

$$A \bullet B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1\ell}B \\ a_{21}B & a_{22}B & \dots & a_{2\ell}B \\ \dots & & & \\ a_{k1}B & a_{k2}B & \dots & a_{k\ell}B \end{pmatrix} \in M_{k\ell}(F).$$

If $\ell = 1$, then $A \bullet B = AB$, and it is known that a matrix $C \in M_k(F)$ has rank one if and only if there exist nonzero matrices $A \in M_{k,1}(F)$ and $B \in M_{1,k}(F)$ such that $C = AB = A \bullet B$.

If $\ell = k$, it is well-known that $\phi : M_k(F) \otimes_F M_k(F) \to M_{k^2}(F)$, the linear extension of the map $A \otimes B \mapsto A \bullet B$, is an algebra isomorphism.

With this in mind we introduce the following entities which will be helpful for our purposes:

$$\begin{aligned} n_k(F) &= \max\{\dim_F(\mathcal{V}) \mid \ \mathcal{V} \text{ is a subspace of } M_k(F) \\ &\qquad \text{and } \mathcal{V} \cap \{M_{k,1}(F) \bullet M_{1,k}(F)\} = 0\}, \\ \ell_k(F) &= \max\{\dim_F(\mathcal{K}) \mid \ \mathcal{K} \subseteq M_{k^2}(F) \text{ is a left ideal} \\ &\qquad \text{and } \mathcal{K} \cap \{M_k(F) \bullet M_k(F)\} = 0\}. \end{aligned}$$

We are now in a position to state the main results of the present paper.

Theorem 1.2. *Given a field F and positive integer k, we have :*

(i) $m_k(F) = 2k - 1$*, for all k, if and only if F is algebraically closed.*
(ii) $m_k(F) = k$ *if and only if there exists a nonassociative division F-algebra of dimension k.*

The above result sharpens (ii) and (iii) in Theorem 1.1. We note that the theorem is essentially a corollary to van den Berg's results. The next observations provide relationships between the dimensions under consideration.

Theorem 1.3. *Given a field F and positive integer k, we have*

$$m_k(F) + n_k(F) = k^2 \quad \text{and} \quad \ell_k(F) = k^2 \cdot n_k(F).$$

We list some immediate implications.

Corollary 1.4. *Let $\mathcal{V}$ be a k dimensional vector space over a field F and let $\overline{F}$ be the algebraic closure of F. Then:*

(i) $k^2 - 2k + 1 \le n_k(F) \le k^2 - k$.
(ii) $n_k(F) = k^2 - 2k + 1$*, for all k, if and only if F is algebraically closed.*
(iii) $n_k(F) = k^2 - k$ *if and only if there exists a nonassociative division F-algebra of dimension k.*
(iv) *A subspace $\mathcal{W} \subset M_k(F)$ contains a rank one matrix, provided*

$$\dim_F(\mathcal{W}) > k^2 - k, \ \text{or } F = \overline{F} \text{ and } \dim_F(\mathcal{W}) > k^2 - 2k + 1.$$

(v) *A subspace* $\mathcal{W} \subset \mathcal{V} \otimes_F \mathcal{V}$ *contains a non-zero element of the form* $A \otimes B$ *for some* $A, B \in \mathcal{V}$*, provided*

$$\dim(\mathcal{W}) > k^2 - k, \text{ or } F = \overline{F} \text{ and } \dim(\mathcal{W}) > k^2 - 2k + 1.$$

Proof. (i) follows at once from Theorem 1.1 and Theorem 1.3. (ii) and (iii) are immediate consequences of Theorem 1.2(ii) together with Theorem 1.3. (iv) follows from (i) and (ii). Clearly $\mathcal{V} \cong M_{k1}(F)$ and $\mathcal{V} \cong M_{1k}(F)$ as vector spaces. Next, the linear extension of the map $A \otimes B \mapsto AB$, $A \in M_{k1}(F)$, $B \in M_{1k}(F)$, is an isomorphism of vector spaces $M_{k1}(F) \otimes_F M_{1k}(F) \to M_k(F)$. Therefore there exists an isomorphism $\mathcal{V} \otimes_F \mathcal{V} \to M_k(F)$ of vector spaces sending vectors of the form $v \otimes u$ to matrices of rank 1. The result now follows from (iv). □

2. **Proof of the Main Theorems**

We need the following result.

Corollary 2.1 ([3, Corollary 5])**.** *The following assertions are equivalent for a division ring* $\mathcal{D}$ *and a positive integer* k*:*

(i) $M_k(\mathcal{D})$ *is uniformly strongly prime of bound* k*;*
(ii) $GL(k;\mathcal{D}) \cup \{0\}$ *contains a* k*-dimensional* $\mathcal{D}$*-subspace of* $M_k(\mathcal{D})$*.*

Recall that a nonassociative F-algebra $\mathcal{D}$ is said to be a *division algebra* provided for any $a, b \in \mathcal{D}$ with $a \neq 0$ both equations $ax = b$ and $ya = b$ have unique solutions in $\mathcal{D}$. We are now in a position to prove Theorem 1.2.

Proof. (i) If F is algebraically closed, then $m_k(F) = 2k - 1$ by Theorem 1.1. Conversely, if F is not algebraically closed, then it has a finite extension $\mathcal{E}$ of dimension $k > 1$. Therefore $m_k(F) = k < 2k - 1$ by Theorem 1.1(iii).

(ii) If there exists a nonassociative division F-algebra of dimension k, then $m_k(F) = k$ by Theorem 1.1(iii). Conversely, assume that $m_k(F) = k$. Then Corollary 2.1 yields that $GL(k;F) \cup \{0\}$ contains a k-dimensional F-subspace $\mathcal{V}$ of $M_k(F)$. Considering $M_k(F)$ as the endomorphism algebra of the vector space $\mathcal{V}$, we define a product $\cdot : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ by the rule $AB = A(B)$ for all $A, B \in \mathcal{V}$. We claim that $(\mathcal{V}, \cdot)$ is a nonassociative division algebra over F of dimension k. Indeed, let $A, B \in \mathcal{V}$ with $A \neq 0$. Consider the map $\phi : \mathcal{V} \to \mathcal{V}$ given by $\phi(X) = XA = X(A)$. Clearly ϕ is an endomorphism of the vector space $\mathcal{V}$. Since $\mathcal{V} \setminus \{0\} \subseteq GL(k;F)$ and $A \neq 0$, $X(A) \neq 0$ for all $X \in \mathcal{V}$ with $X \neq 0$. That is $\ker(\phi) = 0$ and so ϕ is an automorphism of $\mathcal{V}$. In particular, there exists a unique $Y \in \mathcal{V}$ such that $YA = B$. Finally, since $A \in GL(k;F)$, there exists a unique $X \in \mathcal{V}$ with $AX = A(X) = B$. Thus $(\mathcal{V}, \cdot)$ is a nonassociative division algebra. □

Let $\mathrm{tr}_k : M_k(F) \to F$ be the trace map. Given a subspace $\mathcal{W} \subseteq M_k(F)$, we set

$$\mathcal{W}^{\perp} = \{A \in M_k(F) \mid \mathrm{tr}_k(A\mathcal{W}) = 0\}.$$

Given $A \in M_{k,\ell}(F)$ and $B \in M_{\ell,k}(F)$, one can easily check that

$$\mathrm{tr_k}(AB) = \mathrm{tr}_\ell(BA). \tag{1}$$

Lemma 2.2. *Let $\mathcal{W} \subseteq M_k(F)$ be a subspace containing no rank one matrices. Then any basis of $\mathcal{W}^\perp$ is a uniform insulator for $M_k(F)$. Conversely, let $\mathcal{S}$ be a uniform insulator for $M_k(F)$ and let $\mathcal{V} = \sum_{A\in\mathcal{S}} FA$. Then $\mathcal{V}^\perp$ contains no rank one matrices.*

Proof. It is well-known that the map $(A,B) \mapsto \mathrm{tr}_k(AB)$, $A, B \in M_k(F)$, is a nondegenerate symmetric bilinear form. Therefore

$$\dim_F(\mathcal{U}) + \dim_F(\mathcal{U}^\perp) = k^2 \quad \text{and} \quad \{\mathcal{U}^\perp\}^\perp = \mathcal{U} \tag{2}$$

for any subspace $\mathcal{U} \subseteq M_k(F)$.

Let $\mathcal{W}$ be as in the lemma and let $\mathcal{S}$ be a basis of $\mathcal{W}^\perp$. Given $0 \neq A \in M_{k,1}(F)$ and $0 \neq B \in M_{1,k}(F)$, $AB \in M_k(F)$ has rank one and so $AB \notin \mathcal{W} = \{\mathcal{W}^\perp\}^\perp$ forcing $0 \neq \mathrm{tr}_k(ABX)$ for some $X \in \mathcal{S}$. Making use of (1), we conclude that $BXA = \mathrm{tr}_1(BXA) \neq 0$. We see that $B\mathcal{S}A \neq 0$ for all $0 \neq A \in M_{k,1}(F)$ and $0 \neq B \in M_{1,k}(F)$. Now let $P, Q \in M_k(F)$ be nonzero. Write

$$P = \begin{pmatrix} P_1 \\ P_2 \\ \dots \\ P_k \end{pmatrix} \quad \text{and} \quad Q = (Q^1, Q^2, \dots, Q^k)$$

where $P_i \in M_{1,k}(F)$ and $Q^j \in M_{k,1}(F)$. Then $PXQ = (P_iXQ^j)_{i,j=1}^k$ for all $X \in \mathcal{S}$ and so $P\mathcal{S}Q \neq 0$. Therefore $\mathcal{S}$ is a uniform insulator for $M_k(F)$.

Now let $\mathcal{S}$ and $\mathcal{V}$ be as in the lemma. Assume to the contrary that $\mathcal{V}^\perp$ contains a matrix C of rank one. Write $C = AB$ where $A \in M_{k,1}(F)$ and $B \in M_{1,k}(F)$. Clearly $A \neq 0$ and $B \neq 0$ (otherwise $C = 0$ would be of rank 0). Since $AB = C \in \mathcal{V}^\perp$, $BXA = \mathrm{tr}_1(BXA) = \mathrm{tr}_k(ABX) = 0$ for all $X \in \mathcal{S}$. Let $P, Q \in M_k(F)$ be matrices such that the first row of P is equal to B and all the other ones are equal to 0, the first column of Q is equal to A and all the other ones are equal to 0. Clearly $P \neq 0 \neq Q$ and $P\mathcal{S}Q = 0$, a contradiction. □

We denote by $A \mapsto {}^tA$, $A \in M_k(F)$, the transpose map of $M_k(F)$. Define an action of $M_k(F) \otimes_F M_k(F)$ on $M_k(F)$ by the rule

$$UX = \left(\sum_{i=1}^n A_i \otimes B_i\right) X = \sum_{i=1}^n A_i X\, {}^tB_i$$

whenever $U = \sum_{i=1}^n A_i \otimes B_i$. It is well-known that $M_k(F)$ is a simple faithful left module over the algebra $M_k(F) \otimes_F M_k(F)$ under this action and $M_k(F) \otimes_F M_k(F)$ is the algebra of all linear transformations of the vector space $M_k(F)$.

Lemma 2.3. *With the above notation we have:*

(i) *If $\mathcal{S}$ is a finite uniform insulator for $M_k(F)$ such that the set $\mathcal{S}$ is linearly independent over F, then*

$$\mathcal{K} = \{U \in M_k(F) \otimes_F M_k(F) \mid U\mathcal{S} = 0\}$$

is a left ideal in $M_k(F) \otimes_F M_k(F)$ containing no nonzero elements of the form $A \otimes B$, $A, B \in M_k(F)$, and $\dim_F(\mathcal{K}) = k^2(k^2 - |\mathcal{S}|)$.

(ii) *If $\mathcal{K}'$ is a left ideal of $M_k(F) \otimes_F M_k(F)$ containing no nonzero elements of the form $A \otimes B$ and $\mathcal{S}'$ is a basis of the vector space $\{X \in M_k(F) \mid \mathcal{K}X = 0\}$, then $\mathcal{S}'$ is a uniform insulator for $M_k(F)$ and $\dim_F(\mathcal{K}') = k^2(k^2 - |\mathcal{S}'|)$.*

Proof. Let $\mathcal{S}$ and $\mathcal{K}$ be as in the lemma. Clearly $\mathcal{K}$ is a left ideal of the algebra $M_k(F) \otimes_F M_k(F)$. Since $\mathcal{S}$ is a uniform insulator for $M_k(F)$, $(A \otimes B)\mathcal{S} \neq 0$ for all nonzero $A, B \in M_k(F)$ and so $\mathcal{K}$ contains no nonzero elements of the form $A \otimes B$. Write $\mathcal{S} = \{X_1, X_2, \dots, X_m\}$ where $m = |\mathcal{S}|$. Define a linear map

$$\psi_{\mathcal{S}} : M_k(F) \otimes_F M_k(F) \to M_k(F)^m, \quad \psi_{\mathcal{S}}(U) = (UX_1, UX_2, \dots, UX_m)$$

for all $U \in M_k(F) \otimes_F M_k(F)$. Clearly $\psi_{\mathcal{S}}$ is a left $M_k(F) \otimes_F M_k(F)$-module map and $\mathcal{K} = \ker(\psi_{\mathcal{S}})$. Since $\{X_1, X_2, \dots, X_m\}$ is linearly independent over F and $M_k(F) \otimes_F M_k(F)$ is the algebra of all linear transformations of the vector space $M_k(F)$, we conclude that $\psi_{\mathcal{S}}$ is an epimorphism. Therefore

$$\dim_F(\mathcal{K}) = \dim_F(\ker(\psi_{\mathcal{S}})) = k^4 - \dim_F(\mathrm{Im}(\psi_{\mathcal{S}})) = k^4 - k^2|\mathcal{S}| = k^2(k^2 - |\mathcal{S}|).$$

Further let $\mathcal{K}'$ and $\mathcal{S}'$ be as in the lemma. Since $\mathcal{K}'$ is a proper left ideal of $M_k(F) \otimes_F M_k(F) \cong M_{k^2}(F)$, there exists an idempotent $E \in M_k(F) \otimes_F M_k(F)$ such that $\mathcal{K}' = (M_k(F) \otimes_F M_k(F))E$ and $E \neq 1$ where 1 is the identity of the algebra $M_k(F) \otimes_F M_k(F)$. Clearly

$$(1 - E)M_k(F) = \{X \in M_k(F) \mid \mathcal{K}'X = 0\}$$

and so $\mathcal{S}'$ is a basis of the vector space $(1-E)M_k(F)$. Write $\mathcal{S}' = \{Y_1, \dots, Y_r\}$ where $r = |\mathcal{S}'|$. Consider the linear map

$$\psi_{\mathcal{S}'} : M_k(F) \otimes_F M_k(F) \to M_k(F)^r, \quad U \mapsto (UY_1, UY_2, \dots, UY_r).$$

We claim that $\ker(\psi_{\mathcal{S}'}) = (M_k(F) \otimes_F M_k(F))E = \mathcal{K}'$. Indeed, the inclusion $\ker(\psi_{\mathcal{S}'}) \supseteq \mathcal{K}'$ follows from the definition of $\psi_{\mathcal{S}'}$. Next, let $U \in \ker(\psi_{\mathcal{S}'})$. Then $UY_i = 0$ for all $i = 1, 2, \dots, r$. Since $\{Y_1, Y_2, \dots, Y_r\}$ is a basis of $(1 - E)M_k(F)$, we conclude that $[U(1 - E)]M_k(F) = 0$. Recalling that $M_k(F)$ is a faithful left $M_k(F) \otimes_F M_k(F)$-module, we get that $U(1-E) = 0$ forcing $U = UE$. That is $U \in \mathcal{K}'$ and our claim is proved.

Since $\ker(\psi_{\mathcal{S}'}) = \mathcal{K}'$, it follows from our assumption on K' that $\ker(\psi_{\mathcal{S}'})$ contains no nonzero matrices of the form $A \otimes B$, $A, B \in M_k(F)$. That is to say, $\mathcal{S}'$ is a uniform insulator for $M_k(F)$. As above we get

$$\dim_F(\mathcal{K}') = \dim_F(\psi_{\mathcal{S}'}) = k^4 - k^2|\mathcal{S}'| = k^2(k^2 - |\mathcal{S}'|).$$

The proof is thereby complete. □

We are now ready to prove Theorem 1.3.

Proof. Let $\mathcal{S}$ be a uniform insulator for $M_k(F)$ with $|\mathcal{S}| = m_k(F)$ and let $\mathcal{V} = \sum_{A\in\mathcal{S}} FA$. According to Lemma 2.2, $\mathcal{V}^\perp$ contains no rank one matrices and so (2) yields

$$n_k(F) \geq \dim_F(\mathcal{V}^\perp) = k^2 - \dim_F(\mathcal{V}) = k^2 - m_k(F).$$

That is to say $m_k(F) + n_k(F) \geq k^2$. On the other hand, if $\mathcal{W}$ is a subspace of $M_k(F)$ containing no rank one matrices and $\mathcal{T}$ is a basis of $\mathcal{W}^\perp$, then $\mathcal{T}$ is a uniform insulator for $M_k(F)$ by Lemma 2.2 and so

$$m_k(F) \leq |\mathcal{T}| = \dim_F(\mathcal{W}^\perp) = k^2 - \dim_F(\mathcal{W}) \leq k^2 - n_k(F)$$

forcing $m_k(F) + n_k(F) \leq k^2$. Therefore $m_k(F) + n_k(F) = k^2$.

Let $\mathcal{K}'$ be any left ideal of $M_k(F) \otimes_F M_k(F)$ containing no nonzero elements of the form $A \otimes B$, $A, B \in M_k(F)$. We claim that

$$\dim_F(\mathcal{K}') \leq k^2 \cdot n_k(F). \tag{3}$$

Indeed, let $\mathcal{S}'$ be a basis of the vector space $\{X \in M_k(F) \mid \mathcal{K}'X = 0\}$. According to Lemma 2.3, $\mathcal{S}'$ is a uniform insulator for $M_k(F)$ and since $|\mathcal{S}'| \geq m_k(F)$,

$$\dim_F(\mathcal{K}') = k^2(k^2 - |\mathcal{S}'|) \leq k^2(k^2 - m_k(F)) = k^2 n_k(F).$$

Now let $\mathcal{S}$ be a uniform insulator for $M_k(F)$ with $|\mathcal{S}| = m_k(F)$. It follows at once from the definition of $m_k(F)$ that $\mathcal{S}$ is a linearly independent subset of $M_k(F)$. Therefore Lemma 2.3 implies that $\mathcal{K} = \{U \in M_k(F) \otimes_F M_k(F) \mid U\mathcal{S} = 0\}$ is a left ideal of $M_k(F) \otimes_F M_k(F)$ containing no nonzero elements of the form $A \otimes B$ and $\dim_F(\mathcal{K}) = k^2(k^2 - m_k(F)) = k^2 n_k(F)$ by the above result. It now follows from (3) that

$$\max\{\dim_F(\mathcal{K}')\} = k^2 n_k(F), \tag{4}$$

where $\mathcal{K}'$ is a left ideal of $M_k(F) \otimes_F M_k(F)$ containing no nonzero elements of the form $A \otimes B$.

Since $M_k(F) \otimes_F M_k(F)$ is isomorphic to $M_{k^2}(F)$ under $\phi : A \otimes B \mapsto A \bullet B$ (see Section 1.), we conclude from (4) that $\ell_k(F) = k^2 \cdot n_k(F)$. □

Remark 2.4. We conclude our discussion of the uniform bounds of primeness by considering the following implications for a field F and a positive integer k.

(i) *If $\mathcal{S}$ is a uniform insulator for $M_k(F)$ and $\mathcal{V} = \sum_{A\in\mathcal{S}} FA$, then $\mathcal{V}$ contains a uniform insulator $\mathcal{S}'$ for $M_k(F)$ with $|\mathcal{S}'| = m_k(F)$.*

(ii) *If $\mathcal{W}$ is a subspace of $M_k(F)$ maximal with respect to the property $\mathcal{W} \cap \{M_{k,1}(F) \bullet M_{1,k}(F)\} = 0$, then $\dim_F(\mathcal{W}) = n_k(F)$.*

(iii) *If $\mathcal{K}$ is a left ideal of $M_{k^2}(F)$ maximal with respect to the property $\mathcal{K} \cap \{M_k(F) \bullet M_k(F)\} = 0$, then $\dim_F(\mathcal{K}) = \ell_k(F)$.*

We cannot prove any of these but we show that they are equivalent:

Proof. Suppose that (i) is satisfied. We prove (ii). Let $\mathcal{W}$ be as in (ii). According to Lemma 2.2 any basis of $\mathcal{W}^\perp$ is a uniform insulator for $M_k(F)$. It now follows from our assumption that $\mathcal{W}^\perp$ contains a uniform insulator $\mathcal{S}'$ for $M_k(F)$ with $\mathcal{S}' = m_k(F)$. Set $\mathcal{V} = \sum_{A\in\mathcal{S}'} FA$ and note that $\dim_F(\mathcal{V}) = m_k(F)$ because the set $\mathcal{S}'$ is linearly independent. Next, the inclusion $\mathcal{V} \subseteq \mathcal{W}$ together with (2) yield that $\mathcal{V}^\perp \supseteq (\mathcal{W}^\perp)^\perp = \mathcal{W}$. By Lemma 2.2 $\mathcal{V}^\perp$ contains no rank 1 matrices and so the maximality of $\mathcal{W}$ implies that $\mathcal{V}^\perp = \mathcal{W}$. Therefore $\mathcal{V} = (\mathcal{V}^\perp)^\perp = \mathcal{W}^\perp$ and so $\dim_F(\mathcal{W}^\perp) = \dim_F(\mathcal{V}) = m_k(F)$. Recalling that $\dim_F(\mathcal{W}) = k^2 - \dim_F(\mathcal{W}^\perp) = k^2 - m_k(F)$, we conclude that $\dim_F(\mathcal{W}) = n_k(F)$ by Theorem 1.3.

Now assume that (ii) is fulfilled and show that (i) is true. Let $\mathcal{S}$ and $\mathcal{V}$ be as in (i). Then $\mathcal{V}^\perp$ contains no rank 1 matrices by Lemma 2.2. Let $\mathcal{W}$ be a subspace of $M_k(F)$ containing $\mathcal{V}^\perp$ and maximal with respect to the property $\mathcal{W} \cap \{M_{1k}(F) \bullet M_{1k}(F)\} = 0$. By our assumption $\dim_F(\mathcal{W}) = n_k(F)$ and so (2) together with Theorem 1.3 imply that $\mathcal{V} = (\mathcal{V}^\perp)^\perp \supseteq \mathcal{W}^\perp$ and $\dim_F(\mathcal{W}^\perp) = k^2 - n_k(F) = m_k(F)$. Let $\mathcal{S}'$ be a basis of $\mathcal{W}^\perp$. Then $\mathcal{S}'$ is a uniform insulator for $M_k(F)$ by Lemma 2.2. Clearly $|\mathcal{S}'| = m_k(F)$ and $\mathcal{S}' \subseteq \mathcal{V}$.

Finally, making use of Lemma 2.3 the proof of the equivalence of statements (i) and (iii) is similar to that of (i) and (ii). □

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