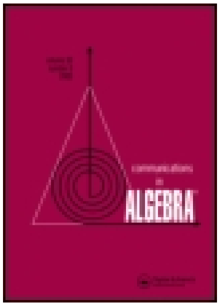

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Localization of modules and the central closure of rings

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LOCALIZATION OF MODULES AND THE CENTRAL CLOSURE OF RINGS

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Introduction

Let R be an associative ring with unity and R-MOD the category of unitary left R-modules. For any $M \in$ R-MOD we denote by $\sigma[M]$ the full subcategory of R-MOD, whose objects are the submodules of M-generated modules.

$\sigma[M]$ is a (locally finite) Grothendieck category and hence we can apply the abstract localization theory for this type of category as presented in Gabriel [8]. The techniques involved are quite similar to those used for localization in the full module category

R-MOD (Goldman [11], Golan [9])with two major differences:

(1) there is no distinguished (projective) generator in $\sigma[M]$ and for this reason torsion theory is not described by a filter of subobjects of a single object;

(2) the ring R need not to be contained in $\sigma[M]$ and hence we do not, in fact <u>can</u> not, aim at a quotient <u>ring</u> (of R) in our theory.

In case R is in $\sigma[M]$ we have $\sigma[M]$ = R-MOD and our considerations lead to the usual torsion theory in R-MOD, i.e. we also obtain a quotient ring for R.

In §1 we develop a torsion and localization theory in $\sigma[M]$ starting from a <u>torsion class</u> $\mathfrak{T}$ of modules. The main result gives us a direct limit representation of the endomorphism ring of the quotient module $Q_{\mathfrak{T}}(N)$ of an $N \in \sigma[M]$ (Theorem (1.9)). Its application to R-MOD yields known results on the quotient ring of R (Theorem (1.11)).

§§ 2 and 3 are devoted to special torsion theories in $\sigma[M]$. Call a module N <u>M-singular</u>, if there is an exact sequence in $\sigma[M]$ $0 \to K \to L \to N \to 0$ with K essential in L. The class of M-singular modules is a subclass of the singular modules in R-MOD. In §2 we study the torsion theory determined by the M-singular modules in $\sigma[M]$. Here the quotient modules are M-injective and the endomorphism ring of a quotient module is regular and self-injective. In case $R\in\sigma[M]$ we obtain the Goldie torsion theory in R-MOD and the quotient module of M is a generator for the quotient ring of R (Theorem (2.5)).

The torsion theory in $\sigma[M]$ defined by the M-injective hull $\hat{M}$ of M is developed in §3. Notice that $\hat{M}$ is a quasi-injective module in R-MOD but an injective module in $\sigma[M]$. In general quasi-injective modules do not permit localization in R-MOD (see Lambek [15]). We call a submodule $K\subset M$ rational in M, if $\mathrm{Hom}_R(M/K,\hat{M})=0$. Our main interest is in modules whose essential submodules are rational in M. In this case the quotient module of M is just the M-injective hull $\hat{M}$ of M. The condition is in fact weaker than M being non M-singular (Prop.(3.2)). The theorems obtained ((3.4)-(3.9)) have as special cases

- the Goldie theorem for associative semiprime rings;
- properties of the endomorphism ring of a non-singular torsionless module over a semiprime ring (Zelmanowitz [28]);
- a theorem on semiprime modules by Zelmanowitz [29];
- a result on critically compressible modules by Zelmanowitz [30];

- the construction and properties of the extended centroid and the central closure of (nonassociative) semiprime rings.

The last point is a consequence of an application of §3 to the following situation (§4):

For an arbitrary ring A let $\Omega(A)$ be the <u>multiplication ring</u> of A and $\sigma_\Omega[A]$ the subcategory of $\Omega(A)$-MOD subgenerated by A. It was already seen in [25] and [26] that many results of module theory over commutative associative rings can be generalized to $\sigma_\Omega[A]$ - that is $\sigma_\Omega[A]$ is a useful category of two-sided A-modules. It is clear from recent research in one-sided module and ring theory that the crucial test for any two-sided module theory is how well it can handle prime and semiprime rings. For associative rings, the first success in this direction was due to Delale [6] and van Oystaeyen-Verschoren [19], [23], [24]. The latter studied localization for (central) R-bimodules by restricting localization in $R\otimes R^\circ$-MOD to the subcategory of (central) bimodules ("relative localization"). A more general way to study (even nonassociative) semiprime rings is opened by localization in $\sigma_\Omega[A]$. The key to this is the observation that for a semiprime ring A every essential ideal is rational in A. In case A has a unit and every non-zero ideal has non-zero intersection with the centre, this is also a sufficient condition for A to be semiprime (Prop.(4.3)). Now the results of §3 apply and we obtain for a semiprime ring A with A-injective hull $\hat{A}$ in $\sigma_\Omega[A]$:

- $T=End_\Omega(\hat{A})$ is a self-injective, commutative and regular ring and equal to the <u>extended centroid</u> of A ([7], [4]);
- $\hat{A}=A\cdot T$ can be endowed with a ring structure to become the <u>central closure</u> of A.

In further investigations we observe that the associativity of A is of no special advantage. Two other properties which hold for associative commutative rings turn out to be of importance:

(i) every proper ideal of A intersects the centre non-trivially;

(ii) $\Omega(A)\in\sigma_\Omega(A)$, i.e. A is a subgenerator for $\Omega(A)$-MOD.

If A is a semiprime ring A with unit satisfying property (i) and having finite Goldie dimension over $\Omega(A)$, then $\hat{A}$ is a direct sum of simple rings and T is the classical ring of quotients of the centre of A (Theorem (4.5)). If A is semiprime with property (ii), the maximal ring of quotients of $\Omega(A)$ is equal to $End_T(\hat{A})$ (Theorem (4.7)). For semiprime rings (ii) is actually equivalent to $\hat{A}_T$ being a finitely generated

T-module. Finally, a semiprime ring with (i) and (ii) and finite Goldie dimension over $\Omega(A)$ is an Azumaya algebra (Theorem (4.9)). Examples of the situations considered are given at the end of §4.

It is one aim of the present paper to demonstrate to what extent theorems on rings can be deduced from one-sided associative module theory. The approach suggested in §4 provides straightforward proofs for properties of the central closure. For example, the direct limit construction of the extended centroid can be avoided.

1. General torsion theory in $\sigma[M]$

Let R be an associative ring with unit and R-MOD the category of unitary left R-modules. Homomorphisms are written on the right side. An R-module N is generated by an R-module M (M-generated), if it is a homomorphic image of a direct sum of copies of M. N is subgenerated by M (M-subgenerated), if it is a submodule of an M-generated module.

$\sigma_R[M]$, or $\sigma[M]$, denotes the full subcategory of R-MOD whose objects are all M-subgenerated modules (Wisbauer [26]) . $\sigma[M]$ is a locally finite Grothendieck category. There are enough injectives in $\sigma[M]$

and any injective object in $\sigma[M]$ is M-generated. For $N \in \sigma[M]$ the injective hull $\hat{N}$ in $\sigma[M]$ is called the <u>M-injective hull</u> of N.

In Gabriel [8] an abstract theory of localization in Grothendieck categories is given. We begin with a short outline of this theory applied to $\sigma[M]$. In general there is no distinguished (projective) generator in $\sigma[M]$ and therefore we cannot transfer the full localization theory in R-MOD as presented in Goldman [11] or Golan [9] to our situation. However, many of the fundamental proofs remain valid and we shall not repeat them here.

One way to define a torsion theory is to designate a distinguished class of modules as torsion modules:

(1.1) A class $\mathfrak{T}$ of modules in $\sigma[M]$ is called a (hereditary) <u>torsion class</u>, if it is closed under taking

submodules, homomorphic images,

extensions and direct sums.

$\mathfrak{T}$ will always denote a torsion class in $\sigma[M]$.

(1.2) For any $N \in \sigma[M]$ the submodule

$$\mathfrak{T}(N) = Tr(\mathfrak{T}, N) = \text{trace of } \mathfrak{T} \text{ in } N$$

is called <u>$\mathfrak{T}$-torsion submodule</u> of N.

By definition we have $N = \mathfrak{T}(N)$ if and only if $N \in \mathfrak{T}$.

From the properties of $\mathfrak{T}$ and the trace we deduce:

- $\mathfrak{T}(N) \in \mathfrak{T}$;
- $\mathfrak{T}(N/\mathfrak{T}(N)) = 0$;
- if $L \in \sigma[M]$ and $f \in Hom_R(N,L)$ then $\mathfrak{T}(N)f \subset \mathfrak{T}(L)$;
- if $K \subset N$ then $\mathfrak{T}(K) = K \cap \mathfrak{T}(N)$.

Thus $\mathfrak{T}(-)$ is an idempotent kernel functor in the sense of Goldman [11].

(1.3) A module $N \in \sigma[M]$ with $\mathfrak{T}(N) = 0$ is called <u>$\mathfrak{T}$-torsion-free.</u> This is the case if and only if $Hom_R(T,N) = 0$ for all $T \in \mathfrak{T}$. The class $\mathfrak{F}$ of $\mathfrak{T}$-torsion-free modules in $\sigma[M]$ is closed under

submodules, isomorphic images,
injective hulls and direct products in $\sigma[M]$.

There is an M-injective module in $\sigma[M]$ which cogenerates all $\mathfrak{T}$-torsion-free modules in $\sigma[M]$. On the other hand, any M-injective module E in $\sigma[M]$ defines a torsion class

$$\mathfrak{T}_E = \{K \in \sigma[M] \mid Hom_R(K,E) = 0\}.$$

(1.4) A submodule $K \subset N \in \sigma[M]$ is called <u>$\mathfrak{T}$-dense</u> in N, if $N/K \in \mathfrak{T}$. The set of $\mathfrak{T}$-dense submodules of N,

$$\mathfrak{L} = \mathfrak{L}(N,\mathfrak{T}) = \{K \subset N \mid N/K \in \mathfrak{T}\}$$

has the following properties:

(1) if $K \in \mathfrak{L}$, $K \subset L \subset N$, then $L \in \mathfrak{L}$;

(2) if $K,L \in \mathfrak{L}$ then $K \cap L \in \mathfrak{L}$;

(3) if $K \in \mathfrak{L}$ and $f \in End_R(N)$, then $Kf^{-1} \in \mathfrak{L}$;

(4) if $K \subset L \subset N$, $L \in \mathfrak{L}$ and $L/K \in \mathfrak{T}$, then $K \in \mathfrak{L}$.

In a $\mathfrak{T}$-torsion-free module N any $\mathfrak{T}$-dense submodule is essential in N. If N is a generator in $\sigma[M]$ the set of $\mathfrak{T}$-dense submodules of N uniquely determines the torsion class $\mathfrak{T}$.

(1.5) A module $N\in\sigma[M]$ is <u>$(M,\mathfrak{T})$-injective</u>, if N is injective with respect to any exact sequence $0\to K\to L\to L/K\to 0$ in $\sigma[M]$ with $L/K\in\mathfrak{T}$.

Any $N\in\sigma[M]$ possesses an <u>$(M,\mathfrak{T})$-injective hull</u> $E_{\mathfrak{T}}(N)$, that is a module $E_{\mathfrak{T}}(N)$ with the properties

- N is essential in $E_{\mathfrak{T}}(N)$;
- $E_{\mathfrak{T}}(N)/N\in\mathfrak{T}$;
- $E_{\mathfrak{T}}(N)$ is $(M,\mathfrak{T})$-injective.

We may identify $E_{\mathfrak{T}}(N)$ with the submodule E of the M-injective hull $\hat{N}$ of N for which $E/N = \mathfrak{T}(\hat{N}/N)$. If N is $\mathfrak{T}$-torsion-free, the same is true for $\hat{N}$ and $E_{\mathfrak{T}}(N)$ (see (1.3)). In case every essential submodule of M is $\mathfrak{T}$-dense in M, "$(M,\mathfrak{T})$-injective" is equivalent to "M-injective" and the $(M,\mathfrak{T})$-injective hull of any $N\in\sigma[M]$ is equal to the M-injective hull N.

(1.6) We call a module $N\in\sigma[M]$ <u>faithfully $(M,\mathfrak{T})$-injective</u>, if any diagram in $\sigma[M]$.

$$\begin{array}{ccccccccc} 0 & \to & K & \to & L & \to & L/K & \to & 0 \\ & & \downarrow & & & & & & \\ & & N & & & & & & \end{array}$$

with exact row and $L/K\in\mathfrak{T}$ can be completed to a commutative diagram by a <u>uniquely determined</u> $L\to N$. N is faithfully $(M,\mathfrak{T})$-injective if and only if N is $(M,\mathfrak{T})$-injective and $\mathfrak{T}$-torsion-free.

(1.7) The <u>$\mathfrak{T}$-quotient</u> of a module $N\in\sigma[M]$ is defined to be the $(M,\mathfrak{T})$-injective hull of the factor module $N/\mathfrak{T}(N)$:

$$Q_{\mathfrak{T}}(N)=E_{\mathfrak{T}}(N/\mathfrak{T}(N)).$$

By construction, $Q_{\mathfrak{T}}(N)$ is $\mathfrak{T}$-torsion-free and $(M,\mathfrak{T})$-injective, which means $Q_{\mathfrak{T}}(N)$ is faithfully $(M,\mathfrak{T})$-injective (see (1.6)).

For $N,L\in\sigma[M]$ and $f\in Hom_R(N,L)$ we obtain in a canonical way a homomorphism $\bar{f}:N/\mathfrak{T}(N) \to L/\mathfrak{T}(L)$. Since $Q_{\mathfrak{T}}(L)$ is faithfully $(M,\mathfrak{T})$-injective, the diagram

$$\begin{array}{ccccc} 0 \to & N/\mathfrak{T}(N) & \to & Q_{\mathfrak{T}}(N) \\ & \bar{f}\downarrow & & \\ 0 \to & L/\mathfrak{T}(L) & \to & Q_{\mathfrak{T}}(L) \end{array}$$

can be completed to a commutative diagram by exactly one

$$Q_{\mathfrak{T}}(f) : Q_{\mathfrak{T}}(N) \to Q_{\mathfrak{T}}(L).$$

Thus building the quotient modules defines a (left exact) functor

$$Q_{\mathfrak{T}}(\ ) : \sigma[M] \to \sigma[M].$$

For a monomorphism $f : N \to L$ the quotient map $Q_{\mathfrak{T}}(f)$ is an isomorphism if and only if $L/Nf\in\mathfrak{T}$.

(1.8) For any module $N\in\sigma[M]$ the set of $\mathfrak{T}$-dense submodules $\mathfrak{L}(N,\mathfrak{T})$ is left directed with respect to inclusion (see (1.4)).

Take $U\in\sigma[M]$ and $K,L\in\mathfrak{L}(N,\mathfrak{T})$ with $K\subset L$. We have a canonical $\mathbb{Z}$-homomorphism

$$\lambda_{L,K} : \mathrm{Hom}_R(L,U) \to \mathrm{Hom}_R(K,U)$$

and this gives us an inductive system of $\mathbb{Z}$-modules

$$(\mathrm{Hom}_R(K,U),\ \lambda_{L,K},\ \Omega(N,\mathfrak{T})).$$

(1.9) <u>THEOREM</u> Let $\mathfrak{T}$ be a torsion class in $\sigma[M]$, $N,U\in\sigma[M]$ and $\mathfrak{T}(U) = 0$. With the notations above we have:

(1) $\mathrm{Hom}_R(N,Q_{\mathfrak{T}}(U)) \cong \varinjlim \mathrm{Hom}_R(K,U)$, $K\in\Omega(N,\mathfrak{T})$;

(2) $\mathrm{Hom}_R(N,Q_{\mathfrak{T}}(U)) = \mathrm{Hom}_R(Q_{\mathfrak{T}}(N),\ Q_{\mathfrak{T}}(U))$;

(3) if N generates $Q_{\mathfrak{T}}(U)$, then $Q_{\mathfrak{T}}(N)$ also generates $Q_{\mathfrak{T}}(U)$;

(4) $\mathrm{End}_R(Q_{\mathfrak{T}}(N)) \cong \varinjlim \mathrm{Hom}_R(K,N/\mathfrak{T}(N))$, $K\in\Omega(N,\mathfrak{T})$.

Proof: (1) For $K\in\Omega(N,\mathfrak{T})$, any $f\in\mathrm{Hom}_R(K,U)$ can be extended to a unique $\tilde{f}\in\mathrm{Hom}_R(N,Q_{\mathfrak{T}}(U))$. This yields an inverse system of $\mathbb{Z}$-monomorphisms

$$\Phi_K : \mathrm{Hom}_R(K,U) \to \mathrm{Hom}_R(N,Q_{\mathfrak{T}}(U)).$$

By the universal property of direct limits we obtain a monomorphism

$$\Phi : \varinjlim \mathrm{Hom}_R(K,U) \to \mathrm{Hom}_R(N,Q_{\mathfrak{T}}(U)).$$

Φ is epimorphic since, for $h\in\mathrm{Hom}_R(N,Q_{\mathfrak{T}}(U))$, we have $V = Uh^{-1}\in\Omega(N,\mathfrak{T})$. By restriction we get $\bar{h}=h|_V\in\mathrm{Hom}_R(V,U)$ and $\Phi_V(\bar{h})=h$.

(2) Any $f\in\mathrm{Hom}_R(N,Q_{\mathfrak{T}}(U))$ defines a unique $f'\in\mathrm{Hom}_R(Q_{\mathfrak{T}(N)},$ $Q_{\mathfrak{T}}(U))$. On the other hand every $g\in\mathrm{Hom}_R(Q_{\mathfrak{T}}(N),\ Q_{\mathfrak{T}}(U))$ is uniquely determined by its restriction to $N/\mathfrak{T}(N)$.

(3) From (2) we deduce that $Q_{\mathfrak{T}}(U) = N\cdot Hom_R(N, Q_{\mathfrak{T}}(U)) =$ $= N\cdot Hom_R(Q_{\mathfrak{T}}(N), Q_{\mathfrak{T}}(U)) = Q_{\mathfrak{T}}(N)Hom_R(Q_{\mathfrak{T}}(N), Q_{\mathfrak{T}}(U))$.

(4) This follows from (1) and (2).

(1.10) We call M a <u>subgenerator</u> in R-MOD, if M has one of the three equivalent properties

(i) $R\epsilon\sigma[M]$;

(ii) $R\subset M^k$ for a $k\in\mathbb{N}$;

(iii) $R\text{-MOD} = \sigma[M]$.

In this case the torsion theory considered above is just a torsion theory in R-MOD and is determined by the set of all $\mathfrak{T}$-dense left ideals of R (Gabriel filter),

$$\mathfrak{L}(R,\mathfrak{T}) = \{K\subset_R R \mid R/K\epsilon\mathfrak{T}\}.$$

From Theorem (1.9) we obtain the following well-known properties of localization in R-Mod:

(1.11) <u>THEOREM</u> Let $\mathfrak{T}$ be a torsion class in R-Mod and $N, L\epsilon$ R-Mod. Then

(1) $Q_{\mathfrak{T}}(N) = \varinjlim Hom_R(K, N/\mathfrak{T}(N))$, $K\epsilon\mathfrak{L}(R,\mathfrak{T})$;

(2) $Q_{\mathfrak{T}}(R) \cong End_R(Q_{\mathfrak{T}}(R))$ as $\mathbb{Z}$-modules; by this isomorphism a ring structure is defined on $Q_{\mathfrak{T}}(R)$;

(3) $Q_{\mathfrak{T}}(N) \cong Hom_R(R, Q_{\mathfrak{T}}(N)) \cong Hom_R(Q_{\mathfrak{T}}(R), Q(N))$; this isomorphism allows the structure of a left $Q_{\mathfrak{T}}(R)$-module for $Q_{\mathfrak{T}}(N)$ which extends the R-module structure;

(4) $Hom_R(Q_{\mathfrak{T}}(N), Q_{\mathfrak{T}}(L)) \cong Hom_{Q_{\mathfrak{T}}(R)}(Q_{\mathfrak{T}}(N), Q_{\mathfrak{T}}(L))$.

$End_R(Q_{\mathfrak{T}}(M))$ is just the maximal ring of quotients of $End_R(M)$ in case M satisfies the following condition :

(*) For any $\mathfrak{T}$-dense submodule $K \subset M$ there is a monomorphism $g: M \to K$ with $Im\, g$ $\mathfrak{T}$-dense in K (hence in M).

(1.12) <u>THEOREM</u> Let $\mathfrak{T}$ be a torsion class in $\sigma[M]$. If M is $\mathfrak{T}$-torsion-free and M satisfies condition (*), then $End_R(Q_{\mathfrak{T}}(M)) = Q_{cl}(S)$, the classical left ring of quotients of $S = End_R(M)$.

Proof: We have to show that any $q \in T = End_R(Q_{\mathfrak{T}}(M))$ can be written as $q = s^{-1}t$ for $s, t \in S$:
Take $N = Mq^{-1} \cap M$ and chose a monomorphism $s: M \to N$ with Ms $\mathfrak{T}$-dense in M (condition (*)). Then $0 \neq sq = t \in S$. Since s is invertible in T (see end of (1.7)) we get $q = s^{-1}t$.
A special case of (1.12) will appear in Corollary (3.2).

2. The singular torsion theory

As in torsion theories in R-MOD, the essential submodules play a special part here too. A first observation is:

(2.1) <u>PROPOSITION</u> Let $\mathfrak{T}$ be a torsion class in $\sigma[M]$. If every essential submodule $L \subset M$ is $\mathfrak{T}$-dense in M (i.e. $M/L \in \mathfrak{T}$), then

(1) $T = End_R(Q_{\mathfrak{T}}(M))$ is (von Neumann) regular and ${}_T T$ is injective;

(2) if $M/\mathfrak{T}(M)$ has finite (Goldie) dimension, then T is semisimple artinian.

Proof: (1) Under the given assumption, $Q=Q_{\mathfrak{T}}(M)$ is just the M-injective hull of $M/\mathfrak{T}(M)$ (see (1.5)). Hence ${}_RQ$ is quasi-injective and $T/Jac(T)$ is self-injective and regular. It remains to show that $Jac(T)=0$. For $f\epsilon Jac(T)$ we have $K=Ker\ f$ is essential in Q. Then, for any $\alpha\epsilon Hom_R(M,Q)$, $K\alpha^{-1}$ is essential -hence $\mathfrak{T}$-dense- in M. Since M generates the M-injective module Q, this means $Q/K\epsilon\mathfrak{T}$ and $Q/K\subset\mathfrak{T}(Q)=0$, i.e., $f=0$.

(3) If $M/\mathfrak{T}(M)$ is finite dimensional, the same is true for Q. In this case the regular ring T has no infinite set of orthogonal idempotents and hence it is semisimple artinian.

(2.2) COROLLARY Let $\mathfrak{T}$ be a torsion class in $\sigma[M]$. If M is $\mathfrak{T}$-torsion-free and every non-zero submodule is $\mathfrak{T}$-dense in M, then $T=End_R(Q_{\mathfrak{T}}(M))$ is a skew field. If, in addition, $Hom_R(M,N)\neq 0$ for all submodules $N\subset M$, then T is the quotient skew field of $S=End_R(M)$.

Proof: Under the given condition, $Q_{\mathfrak{T}}(M)=\hat{M}$, the M-injective hull of M. Every $t\epsilon T$ is a monomorphism. Thus T is a regular ring (by (2.1)) without zero divisors, i.e. T is a skew field. The second assertion follows from Theorem (1.12).

(2.3) A module $N \in \sigma[M]$ is called <u>singular</u> in $\sigma[M]$, or <u>M-singular</u>, if there is an exact sequence in $\sigma[M]$ $0 \to K \to L \to N \to 0$, with K essential in L. Observe that a projective module N in $\sigma[M]$ is never M-singular, since the above sequence splits. If $\sigma[M]$=R-MOD, the notion "M-singular" is identical to the usual "singular" for modules. Of course, every M-singular module is singular in R-MOD. However, a simple module M can be singular in R-MOD but never is M-singular, because it is projective in $\sigma[M]$.

The class $\mathfrak{S}'$ of all M-singular modules in $\sigma[M]$ is closed by taking submodules, factor modules and direct sums (e.g.Prop.(1.1) in Goodearl [12]). Therefore any $L \in \sigma[M]$ has a largest M-singular submodule $Tr(\mathfrak{S}', L)$. L is M-singular if and only if $Tr(\mathfrak{S}', L) = L$; L is called <u>non M-singular</u>, if $Tr(\mathfrak{S}', L) = 0$. For example, a semihereditary module P in $\sigma[M]$ (Wisbauer [27]) is non M-singular: Since every finitely generated submodule of P is M-projective, P cannot contain an M-singular submodule.

(2.4) By $\mathfrak{S}$ we denote the smallest torsion class in $\sigma[M]$ (as defined in (1.1)) which contains all M-singular modules, and the resulting torsion theory is called the <u>singular torsion theory</u> in $\sigma[M]$. Applying standard arguments we obtain ($L \in \sigma[M]$):

(1) If L is non M-singular, then L is $\mathfrak{S}$-torsion-free and every $\mathfrak{S}$-dense submodule is essential in L (see (1.4));

(2) Any $(M,\mathfrak{S})$-injective module is M-injective (see (1.5));

(3) $L\in\mathfrak{S}$ if and only if its M-injective hull $\hat{L}$ belongs to $\mathfrak{S}$;

(4) If M is projective in $\sigma[M]$, then $\hat{L}\in\mathfrak{S}$ iff, for every $\delta\in Hom_R(M,\hat{L})$, $Ker\delta$ is $\mathfrak{S}$-dense in M. In this case the class $\mathfrak{S}$ is uniquely determined by the set of all $\mathfrak{S}$-dense submodules of M.

(5) In case M is non M-singular and projective in $\sigma[M]$, $\mathfrak{S}$ is determined by the set of all essential submodules of M (this follows from (1) and (4)).

(6) $End_R(Q_{\mathfrak{S}}(L))$ is a regular, left self-injective ring (Prop.(2.1)).

If M is a subgenerator in R-MOD (see (1.10)), the M-singular modules are just the singular modules and the singular torsion theory is the Goldie torsion theory in R-MOD. We know from Theorem (1.11) that in this case the quotient module $Q_{\mathfrak{S}}(R)$ allows a ring structure.

(2.5) THEOREM Let M be a subgenerator in R-MOD and $\mathfrak{S}$ the smallest torsion class containing the (M-)singular modules. Then

(1) $Q_{\mathfrak{C}}(M)$ is a generator in $Q_{\mathfrak{C}}(R)$-MOD;

(2) $T=End_R(Q_{\mathfrak{C}}(M))=End_{Q_{\mathfrak{C}}(R)}(Q_{\mathfrak{C}}(M))$ is a regular, left selfinjective ring;

(3) $Q_{\mathfrak{C}}(M)_T$ is a finitely generated, projective T-module;

(4) $Q_{\mathfrak{C}}(R) \cong End_T(Q_{\mathfrak{C}}(M))$.

Proof: (1) By part (2) of (2.4), $Q_{\mathfrak{C}}(R)$ is M-injective (=R-injective), hence generated by M, and the assertion follows from part (3) of Theorem (1.9). (2) This is a special case of Prop. (2.1) and Theorem(1.11), (4). (3) and (4) are consequences of (1).

Results on non M-singular modules M will be special cases of the more general theorems of the next paragraph.

3. The torsion theory determined by the M-injective hull of M

(3.1) As mentioned in (1.3) any M-injective module in $\sigma[M]$ defines a torsion theory in $\sigma[M]$. Here we are concerned with the torsion theory determined by the M-injective hull $\hat{M}$ of M. Thus we are considering the torsion class

$$\mathfrak{T}_M=\{K\in\sigma[M] \mid Hom_R(K,\hat{M})=0\}.$$

It is immediate that M is $\mathfrak{T}$-torsion-free, and $\mathfrak{T}_M$ is, in fact, the largest torsion class in $\sigma[M]$ for which M is torsion-free. $Q_{\mathfrak{T}}(M)$ is just the $(M,\mathfrak{T}_M)$-injective hull of M.

We call a $\mathfrak{T}_M$-dense submodule $U \subset M$ _rational in M_ and M a _rational extension_ of U. The following assertions are equivalent for a submodule $U \subset M$:

(a) U is rational in M;

(b) $Hom_R(M/U, \hat{M}) = 0$;

(c) for all $U \subset V \subset M$, $Hom_R(V/U, M) = 0$.

Since M is $\mathfrak{T}_M$-torsion-free, every rational submodule is essential in M (see (1.4)). We will be mainly interested in modules M, whose _essential submodules are rational_ in M. These modules are related to non M-singular modules in the following way:

(3.2) _PROPOSITION_ Let M be an R-module.

(1) If M is non M-singular, then every essential submodule is rational in M.

(2) If M is projective in $\sigma[M]$ and every essential submodule is rational in M, then $Hom_R(M, Tr(\mathfrak{S}', M)) = 0$.

(3) If M is projective in $\sigma[M]$ and $Hom_R(M, K) \neq 0$ for all $K \subset M$, then M is non M-singular iff every essential submodule is rational in M.

Proof:

(1) Let U be an essential submodule of M, $U \subset V \subset M$, and $f \in Hom_R(V/U, M)$. Since U is essential in V, we get $(V/U)f \subset Tr(\mathfrak{S}', M) = 0$ and $f = 0$.

(2) Take $g \in Hom_R(M, Tr(\mathfrak{S}', M))$. Then Ker g is essential and hence rational in M by assumption. This implies $g = 0$.

(3) This is a consequence of (1) and (2).

(3.3) Applying our theory to the case M=R we obtain the torsion class in R-MOD defined by the injective hull $\hat{R}$ of R:

$$\mathfrak{T}_R = \{K \in R\text{-MOD} \mid \mathrm{Hom}_R(K,\hat{R})=0\}.$$

This leads to the Lambek torsion theory in R-MOD. The quotient module $Q_{\mathfrak{T}}(R)$ may be considered as a ring extension of R (see (1.11)) and is called the maximal left ring of quotients $Q_m(R)$ of R. By (2.8), all essential left ideals are rational in R if and only if R is non singular. In this case $Q_m(R)$ coincides with the injective hull of R and is a regular, left self-injective ring (Compare (2.4), (6)). This situation is generalized in:

(3.4) THEOREM Let M be an R-module with every essential submodule rational in M. Then

(1) $T=\mathrm{End}_R(\hat{M})$ is a regular and left self-injective ring;

(2) $S=\mathrm{End}_R(M)$ is a subring of T;

(3) $\mathrm{End}_R(\hat{M})=\varinjlim \mathrm{Hom}_R(K,M)$, K essential in M;

(4) every monomorphism $f \in S$ with Im f essential in M is invertible in T;

(5) if M is (Goldie) finite dimensional, then T is semisimple artinian.

Proof: (1) and (5) follow from Proposition (2.1) since, under the given condition, $Q_{\mathfrak{X}}(M)=\hat{M}$.
(2), (4) are special cases of the general situation described in (1.7). (3) was shown in Theorem (1.9), (4).

The following theorem gives a connection between $End_R(\hat{M})$ and a quotient ring of $End_R(M)$:

(3.5) <u>THEOREM</u> Let M be an R-module with every essential submodule rational in M and $Hom_R(M,N)\neq 0$ for all submodules $N\subset M$. Then $End_R(\hat{M})=Q_m(S)$, the maximal left ring of quotients of $S=End_R(M)$.

Proof: Since $T=End_R(\hat{M})$ is a self-injective extension of S by (3.4), it remains to show that S is an essential S-submodule of T. For $0\neq\alpha\in T$ the submodule $K=M\alpha^{-1}\cap M$ is essential in M. From $Hom_R(M,N)\neq 0$ for all $N\subset M$ we deduce that $Tr(M,K)=M\cdot Hom_R(M,K)$ is essential in K and hence essential and rational in M. This implies $M\cdot Hom_R(M,K)\alpha\neq 0$ and we can find $\beta\in Hom_R(M,K)$ with $0\neq\beta\alpha\in S$. Thus $\beta\alpha\in S\cap S\alpha\neq 0$.
Notice that in (3.5) an $s\in S$ is a non zero divisor if and only if Ker s = 0 and Im s is essential in M.

Examples for the conditions in (3.5):

(i) Let R be a semiprime ring and M a non singular R-module cogenerated by R. Then $Hom_R(M,N)\neq 0$ for all $N\subset M$ and (3.5) yields part of Theorem (2.2) in Zelmanowitz [28].

(ii) Let M be a non singular module and Tr(M,R) essential in M. Then the standard Morita context (R,M,M*,S) is non degenerate and (3.5) is a special case of Theorem (3.1) in Hutchinson-Leu [14] (see also Leu [16]).

(iii) If the standard Morita context is semiprime and M is noetherian, then we also have the situation of (3.5). This is described in Zelmanowitz [29] (e.g., Corollary (3.6)).

(iv) For any M-projective module M with Rad(M)=0 we have $Hom_R(M,N)\neq 0$ for all $N\subset M$:
Since N is not small in M there is a $K\subset M$, $K\neq M$, with N+K=M. Then the composition $N\to M\to M/K$ is an epimorphism and the diagram

$$\begin{array}{ccc} & M & \\ & \downarrow & \\ N \to & M/K & \to 0 \end{array}$$

can be completed to a commutative diagram by an $\alpha\in Hom_R(M,N)$, $\alpha\neq 0$.

Other interesting applications will occur in §4.

We call a submodule $K\subset M$ a kernel submodule (or annihilator submodule) if it is the kernel of an endomorphism of M. For the next theorem we need:

(3.6) LEMMA For an M-injective module M the following properties are equivalent:

(a) M has the ascending chain condition on kernel submodules;

(b) $S=End_R(M)$ has the descending chain condition on cyclic right ideals.

Proof: This follows from the fact that for any $t\in S$ we have $Hom_R(M/Ker\,t,M) = tS$ (compare Harada-Ishii [13], Prop.1).

(3.7) THEOREM Let M be an R-module with finite (Goldie) dimension and every essential submodule rational in M. Consider the conditions:

(i) $Hom_R(M,N)\neq 0$ for all $N\subset M$ and $S=End_R(M)$ is semiprime;

(ii) for every essential submodule $N\subset M$ there is a monomorphism $M \to N$;

(iii) $T=End_R(\hat{M})$ is the (semisimple artinian) classical left quotient ring of S and for every essential submodule $L\subset M$ we have $L\cdot T=\hat{M}$.

For these conditions the implications (i)⇒(ii)⇒(iii) are true.

Proof: (i)⇒(ii) From (3.5) we know that T is the maximal left ring of quotients of S and by Proposition (2.1) T is semisimple artinian. Thus S is a left finite-dimensional, non-singular ring. By assumption it is also semiprime and hence S is a Goldie ring and has a classical left ring of quotients which must be equal to T. According to Lemma (3.6) $\hat{M}$ has acc on kernel submodules; since $S\subset T$ this implies acc on kernel submodules for M.

Now assume L to be an essential submodule of M (compare Goldie [10], Theorem 2): $Hom_R(M,L)$ is not a nil ideal in S (=Goldie ring) and hence there is $\alpha_1 \in Hom_R(M,N)$ with $Ker\,\alpha_1 = Ker\,\alpha_1^2$.
If $Ker\alpha_1 \neq 0$, we chose an $\alpha_2 \in Hom_R(M, N \cap Ker\alpha_1)$ with $Ker\alpha_2 = Ker\alpha_2^2$. Continuing this way we get a direct sum of submodules

$$M\alpha_1 \oplus M\alpha_2 \oplus \ldots \oplus M\alpha_k$$

with $\alpha_{k+1} \neq 0$ as long as $N \cap Ker\alpha_1 \cap \ldots \cap Ker\alpha_k \neq 0$.
Since M is finite dimensional and N is essential in M we finally must have $Ker\alpha_1 \cap Ker\alpha_2 \cap \ldots \cap Ker\alpha_k = 0$ for some $k \in \mathbb{N}$. Then $\beta = \alpha_1 + \alpha_2 + \ldots \alpha_k \in Hom_R(M,N)$ is a monomorphism.

(ii) ⇒ (iii) Since M is finite dimensional, for any monomorphism $f \in S$ Mf is essential in M: Assume $Mf \cap K = 0$ for $K \subset M$. Then the sum $Kf^1 + Kf^2 + Kf^3 + \ldots$ is direct and, if $K \neq 0$, $Kf^r \neq 0$ for all $r \in \mathbb{N}$. So we must have K=0.
By (ii) we know that for any essential $L \subset M$ there is a monomorphism $M \to N$ with essential image in N (and M). Now Theorem (1.12) tells us that T is the classical ring of quotients of S.
For an essential $L \subset M$, the M-injective hull $\hat{L}$ is equal to $\hat{M}$. The L-injective hull of L can be written an $L \cdot End_R(\hat{L}) = L \cdot T$. Since a copy of M is contained in L (by (ii)), the L-injective hull is in fact M-injective and hence $L \cdot T = \hat{M}$.

(3.8) COROLLARY Assume the R-module M to be a self-generator with finite (Goldie) dimension and every essential submodule rational in M. Then the following properties are equivalent:

(a) $S=End_R(M)$ is a semiprime ring;

(b) for every essential submodule $L\subset M$ there is a monomorphism $M \to L$;

(c) $T=End_R(\hat{M})$ is the classical left quotient ring of S (and is semisimple artinian).

For M=R this is substantially one of Goldie's theorems.

A module M is called compressible, if every submodule of M contains an isomorphic copy of M.

(3.9) COROLLARY Let M be a compressible R-module in which every submodule is rational.
Then $T=End_R(\hat{M})$ is the quotient skew field of $S=End_R(M)$ and for every submodule $K\subset M$ we have $K\cdot T=\hat{M}$.

This is actually again the situation of Corollary (2.2). Modules satisfying the conditions of (3.9) are called critically compressible and the first assertion of the corollary is Lemma 2 in Zelmanowitz [30].

Analogous to Theorem (2.5), we want to study the $\mathfrak{T}_M$-torsion theory for a subgenerator in R-MOD:

(3.10) THEOREM Assume M to be a subgenerator in R-MOD with every essential submodule rational in M. Then

(1) R is $\mathfrak{T}_M$-torsion-free (since $R\subset M^k$);

(2) $Q_{\mathfrak{T}}(R)=\hat{R}$, the (M-)injective hull of R;

(3) $\hat{M}$ is a generator in $\hat{R}$-MOD;

(4) $T=\mathrm{End}_R(\hat{M})=\mathrm{End}_{\hat{R}}(\hat{M})$ is a regular, left self-injective ring and $\hat{M}_T$ is a finitely generated, projective T-module;

(5) $\hat{R}\cong\mathrm{End}_T(\hat{M}_T)$, hence $\hat{R}$ is regular and R is non singular;

(6) if M is finite dimensional, then T and $\hat{R}$ are semisimple artinian;

(7) if M is finite dimensional and $S=\mathrm{End}_R(M)$ (or R) is semiprime, then T (resp. $\hat{R}$) is the classical ring of quotients of S (resp. R).

Proof: (2) Because of (1), $Q_{\mathfrak{T}}(R)$ is the M-injective hull of R; since M is subgenerator, $Q_{\mathfrak{T}}(R)$ is even R-injective. (3) M generates $\hat{R}$ and by part (3) of Theorem (1.9), $\hat{M}$ generates $\hat{R}$; $\hat{R}$ gets a ring structure by Theorem (1.11). (4) T is regular and self-injective by Theorem (3.5), and $\mathrm{End}_R(\hat{M})=\mathrm{End}_{\hat{R}}(\hat{M})$ was shown in Theorem (1.11). The rest of (4) and the isomorphism in (5) are consequences of (3). (6) If M is finite dimensional, then T is semisimple artinian ((2.1)) and the same is true for $\hat{R}$ because of (5). (7) If M is finite dimensional, then S and R are finite dimensional non singular rings; in case they are semiprime they allow classical rings of quotients (Goldie theorem) which must be equal to T resp. $\hat{R}$.

(3.11) COROLLARY Let M be an R-module whose essential submodules are rational in M and $T=\mathrm{End}_R(\hat{M})$. The following properties are equivalent:

(a) M is a subgenerator in R-MOD;

(b) $\hat{M}_T$ is a finitely generated T-module and ${}_RM$ is faithful.

Proof: (a)⇒(b) was seen in (3.10). (b)⇒(a) M and $\hat{M}$ are faithful R-modules. Since $\hat{M}_T$ is finitely generated as T-module, there is a monomorphism $R\to\hat{M}^k$, $k\in\mathbb{N}$, which means $R\in\sigma[\hat{M}]=\sigma[M]$.

4. The central closure of a (nonassociative) ring

Let A be a not necessarily associative ring and $\Omega(A)$ (or Ω) its multiplication ring, i.e. the subring of $\mathrm{End}_{\mathbb{Z}}(A)$ generated by the left and right multiplications in A and the identity in $\mathrm{End}_{\mathbb{Z}}(A)$. Consider A as module over $\Omega(A)$. The $\Omega(A)$-submodules are just the two-sided ideals in A. $\mathrm{End}_\Omega(A)$ is the centroid $c(A)$ of A. It coincides with the centre $C(A)$ of A if A has a unit element.

$\sigma_\Omega[A]$ denotes the subcategory of $\Omega(A)$-MOD subgenerated by the $\Omega(A)$-module A (see §1). The A-injective hull $\hat{A}$ in $\sigma_\Omega[A]$ is generated by A, i.e. $\hat{A}=A\cdot\mathrm{Hom}_\Omega(A,\hat{A})$. We are going to interpret the rational torsion theory of §3 in this context.

According to (3.1) an ideal $U\subset A$ is called <u>rational in A</u>, if $\mathrm{Hom}\,(A/U,\hat{A})=0$. $Q_r(A)$ denotes the quotient module of A in the rational torsion theory. Since A is torsion-free in this theory we have

$$A\subset Q_r(A)\subset\hat{A}$$

for the $\Omega(A)$-modules A, $Q_r(A)$ and $\hat{A}$.
The following observation allows us to apply the results of §3 to semiprime rings:

(4.1) <u>LEMMA</u> Let A be a semiprime ring. Then

(1) Any essential ideal is rational in A;
(2) $Q_r(A)=\hat{A}$, the A-injective hull of A in $\sigma_\Omega[A]$;
(3) $\mathrm{End}_\Omega(\hat{A})=\varinjlim \mathrm{Hom}_\Omega(U,A)$, U essential in A;
(4) $\mathrm{End}_\Omega(\hat{A})$ is a commutative ring.

Proof: (1) Take an essential ideal $U\subset A$, $U\subset V\subset A$, and $\alpha\in\mathrm{Hom}_\Omega(V/U,A)$. Then $(V/U)\alpha$ is an ideal in A and $(U\cap(V/U)\alpha)^2=0$; hence $U\cap(V/U)\alpha=0$ since A is semiprime and $(V/U)\alpha=0$ since U is essential in A, i.e., $\alpha=0$. Thus $\mathrm{Hom}_\Omega(V/U,A)=0$ and by (3.1) U is rational in A.
(2) is a consequence of (1) (see (1.5)).
(3) is a special case of Theorem (1.9).
(4) Take $f,g\in\mathrm{End}_\Omega(\hat{A})$ and $U=Af^{-1}\cap Ag^{-1}\cap A$. U is essential in A and the ideal generated by U^2 is also essential, hence rational in A, since for any ideal $K\subset A$ $0\neq(U\cap K)^2\subset U^2\cap K$.

Now, for $a,b\in U$ we have

$$(ab)fg=(a(bf))g=(ag)(bf)=(ab)gf.$$

This means $U^2(fg-gf)=0$ and fg-gf is zero on the rational ideal generated by U^2, i.e. fg=gf.

Remark: For a torsion class $\mathfrak{T}$ in $\sigma_\Omega[A]$ the ring $End_\Omega(Q_\mathfrak{T}(A))$ is commutative if for any $\mathfrak{T}$-dense ideal $U\subset A$ the ideal generated by U^2 is also $\mathfrak{T}$-dense in A.

(4.2) THEOREM Let A be a semiprime ring and $\hat{A}$ the A-injective hull of A in $\sigma_\Omega[A]$. Then

(1) $T=End_\Omega(\hat{A})=Hom_\Omega(A,\hat{A})$ is a commutative, regular, selfinjective ring;

(2) the centroid $c(A)=End_\Omega(A)$ is a subring of T;

(3) if A is a finite dimensional $\Omega(A)$-module, then T is semisimple artinian;

(4) $\hat{A}=A\cdot Hom_\Omega(A,\hat{A})=A\cdot T$ allows a ring structure and A is a subring of the ring $\hat{A}$;

(5) $\hat{A}$ is a semiprime ring with centroid $c(\hat{A})=T$;

(6) $\hat{A}$ is a self-injective module over its multiplication ring $\Omega(\hat{A})$.

Proof: (1), (2), (3) are application of Theorem (3.4) and Lemma (4.1).

(4) For $a,b\in A$ and $s,t\in T$ we define

$$(as)\cdot(bt) = (ab)st.$$

As readily checked this determines a multiplication on $\hat{A}=A\cdot T$ and A is a subring of $\hat{A}$.

(5) Assume $J^2=0$ for an ideal J in $\hat{A}$. Then $(J\cap A)^2=0$ and therefore $J\cap A=0$ since A is semiprime. A being essential in A this means $J=0$. From the definition of the multiplication in $\hat{A}$ and the commutativity of T we deduce $T=\mathrm{End}_{\Omega(A)}(\hat{A})=\mathrm{End}_{\Omega(\hat{A})}(\hat{A})=c(\hat{A})$.

(6) follows from (5) and the fact that A is self-injective as $\Omega(A)$-module.

(4.1) and (4.2) show that for a semiprime ring A $\mathrm{End}_{\Omega}(\hat{A})$ is just the extended centroid of A (introduced in [17], [7]) and $\hat{A}$ -with the multiplication given above- is the central closure of A.
Theorem (4.2) extends Theorem (2.5) and (2.15) of Baxter-Martindale [4].

We will be concerned with rings, whose non-zero ideals have non-zero intersection with the centre. This is clearly a generalization of commutative associative rings and like for these we can show:

(4.3) PROPOSITION Let A be a ring with centre $C(A)$ and $U\cap C(A)\neq 0$ for any ideal $U\neq 0$. Then the following properties are equivalent:

(a) A is semiprime;

(b) every essential ideal is rational in A and A has no absolute zero divisors.

Proof: (a)⇒(b) was seen in (4.1).

(b)⇒(a) Assume U to be an ideal in A with $U^2=0$. Then there is an ideal $K\subset A$ for which $U \oplus K$ is essential and hence rational in A and $U(U \oplus K)=0$. Any element $0\neq u\in U\cap C(A)$ defines a non-zero homomorphism $A\to Au\subset U$ with rational kernel $U \oplus K$. This means $U\cap C(A)=0$ and therefore $U=0$.

(4.4) <u>THEOREM</u> Let A be a semiprime ring with centre $C(A)$. If for every non-zero ideal $U\subset A$ $U\cap C(A)\neq 0$, then $End_\Omega(\hat{A})$ is the maximal ring of quotients of the centroid $End_\Omega(A)$.

Proof: The given condition implies that $Hom_\Omega(A,U)\neq 0$ for every non-zero ideal $U\subset A$, and the assertion follows from Theorem (3.5).

For the finite dimensional case we get the following two-sided analogue of <u>Goldie's theorem</u> on semiprime rings:

(4.5) <u>THEOREM</u> Let A be a semiprime ring with unit, $C(A)$ the centre of A and $T=End_\Omega(\hat{A})=C(\hat{A})$ the centre of $\hat{A}$. The following conditions are equivalent:

(a) A has finite (Goldie) dimension as $\Omega(A)$-module and one of the equivalent properties:

1) for every non-zero ideal $U\subset A$ $U\cap C(A)\neq 0$;

2) for every essential ideal $V\subset A$ there is a monomorphism $A\to V$;

3) for every essential ideal $V\subset A$ $V\cap C(A)$ contains a regular element;

4) for every essential ideal $V\subset A$ we have $V\cdot T=\hat{A}$ and T is the classical quotient ring of $C(A)$;

(b) $\hat{A}$ is (finite) direct sum of simple rings and T is the classical quotient ring of $C(A)$.

<u>Remark</u>: Since the classical quotient ring of $C(A)$ is a flat $C(A)$-module we have in (4.5) $\hat{A}=A\cdot T \simeq A\otimes_{C(A)}T$.

Proof: (a.1)$\Rightarrow$(a.2) Under the given condition T is the maximal ring of quotients of $C(A)$ by Theorem (3.5). Since T is commutative and regular, $C(A)$ is semiprime and we can apply (i)$\Rightarrow$(ii) of Theorem (3.7).

(a.2)$\Leftrightarrow$(a.3) Given a monomorphism $f:A\to V$, $1f$ is a regular element in $V\cap C(A)$.

(a.3)$\Rightarrow$(a.4) corresponds to the implication (ii)$\Rightarrow$(iii) of Theorem (3.7).

(a.4)$\Rightarrow$(b) Let X be an essential ideal in $\hat{A}$. For any $\Omega(A)$-submodule K of $\hat{A}$, $K\cdot T$ is an ideal in $\hat{A}$ and hence $X\cap K\cdot T\neq 0$.

Assume $0\neq x=a_1t_1+\ldots+a_kt_k$ for $x\in X$, $a_i\in K$, $t_i\in T$. Then there exists a regular element $s\in C(A)$ with $t_is\in C(A)$ for $i\leq k$ and hence $xs=a_1t_1s+\ldots+a_kt_ks\in X\cap K$. We conclude that X is essential as $\Omega(A)$-submodule and (a.4) implies $X=X\cdot T=\hat{A}$. Thus $\hat{A}$ has no non-trivial essential (two-sided) ideals, i.e., $\hat{A}$ is direct sum of simple ideals.

(b)⇒(a.1) Since $T=\mathrm{End}_{\Omega}(\hat{A})$ has no infinite sequence of orthogonal idempotents, $\hat{A}$ and A are finite dimensional $\Omega(A)$-modules. For any non-zero ideal $U\subset A$, $U\cdot T$ is an ideal in $\hat{A}$, hence direct summand in $\hat{A}$ and $U\cdot T\cap T\neq 0$. If $0\neq u_1t_1+\ldots+u_kt_k=q\in T$ for $u_i\in U$, $t_i\in T$, then there exists a regular element $s\in C(A)$ with $t_is\in C(A)$ for $i\leq k$ and $qs\in C(A)$. This means $0\neq u_1t_1s+\ldots+u_kt_ks =$ $= qs\in U\cap C(A)$.

(4.6) <u>COROLLARY</u> For a prime ring A with unit and centre C(A) the following properties are equivalent:

(a) for any non-zero ideal $U\subset A$ $U\cap C(A)\neq 0$;

(b) for any non-zero ideal $U\subset A$ there is a monomorphism $A\to U$;

(c) $\hat{A}$ is a simple ring and its centre is the quotient field of C(A).

Proof: A prime ring is finite dimensional and every ideal is rational in it. Thus (4.6) follows from (4.5). It is also a special case of Corollary (2.2).

In (4.6), A is a critically compressible module over $\Omega(A)$ and hence $\Omega(A)$ is a weakly primitive ring in the sense of Zelmanowitz [30]. This generalizes the fact that the multiplication ring of a simple ring is primitive.

For associative rings the result of (4.6) was proved in Delale [6], Proposition II.9.6. (see also Verschoren [24], Theorem I.3.19). Notice that in (4.5) and (4.6) $\hat{A}=A\cdot T$ is a central extension of A as considered, e.g., in Amitsur [2].

The next theorem will describe the case when A is a subgenerator in $\Omega(A)$-MOD, that is we have one of the following equivalent conditions:

(i) $\Omega(A) \in \sigma_\Omega[A]$;

(ii) $\Omega(A)\subset A^k$, $k\in\mathbb{N}$;

(iii) $\sigma_\Omega[A]=\Omega(A)$-MOD.

If A is semiprime, then $\Omega(A)$ is torsion-free in the torsion theory defined by $\hat{A}$ (because of (ii)). The quotient module $Q_{\mathfrak{T}}(\Omega)$ is just the (Ω-) injective hull $\hat{\Omega}$ of Ω and can be made into a ring by Theorem (1.11). We readily deduce from Theorem (3.10):

(4.7) THEOREM Let A be a semiprime ring. If A is a subgenerator in $\Omega(A)$-MOD, then

(1) $\hat{A}$ is a generator in $\hat{\Omega}$-MOD ($\Omega=\Omega(A)$);

(2) $T=End_\Omega(\hat{A})=End_{\hat{\Omega}}(\hat{A})$ is commutative, regular and self-injective;

(3) $\hat{A}_T$ is a finitely generated, projective T-module;

(4) $\hat{\Omega}\cong End_T(\hat{A}_T)$ is regular and Ω is non singular.

(4.8) COROLLARY For a semiprime ring A the following assertions are equivalent:

(a) A is a subgenerator in $\Omega(A)$-MOD;

(b) $\hat{A}_T$ is a finitely generated module over $T=End_\Omega(\hat{A})$.

Proof: Corollary (3.11).

In general, there is a difference between $\Omega(\hat{A})$, the multiplication ring of $\hat{A}$, and $\hat{\Omega}$, the maximal quotient ring of $\Omega=\Omega(A)$. Under the conditions of (4.7) we have the inclusions

$$\Omega\subset\Omega(\hat{A})\subset End_T(\hat{A}_T)=\hat{\Omega}.$$

In case $\hat{A}$ is direct sum of simple rings (and $\hat{A}_T$ finitely generated) $\Omega(\hat{A})$ is semisimple artinian, hence self-injective and we conclude

$$\Omega(\hat{A})=End_T(\hat{A}_T)=\hat{\Omega},$$

which means that A is an Azumaya algebra over T. So we state the following as a final theorem:

(4.9) THEOREM For a semiprime ring A with unit and centre C(A) the following properties are equivalent:

(a) A is a subgenerator in $\Omega(A)$-MOD, A has finite (Goldie) dimension over $\Omega(A)$ and $U\cap C(A)\neq 0$ for any non-zero ideal U;

(b) $T=End_\Omega(\hat{A})$ is the classical quotient ring of C(A) and $\hat{A}$ has one of the following equivalent properties:

1) $\hat{A}$ is direct sum of simple rings and $\hat{A}_T$ is a finitely generated T-module;

2) $\Omega(\hat{A})$, the multiplication ring of $\hat{A}$, is semisimple artinian and finitely generated as T-module;

3) $\hat{A}$ is Azumaya algebra over T.

Proof: We have seen in (4.5) that, under the conditions of (a), T is the classical quotient ring of C(A).

(a)⇒(b.1) $\hat{A}$ is direct sum of simple rings by (4.5) and finitely generated as T-module by (4.7).

(b.1)⇒(b.2)⇒(b.3) was shown above.

(b.3)⇒(a) From (4.8) we know that A is a subgenerator in $\Omega(A)$-MOD and the rest follows from Theorem (4.5).

(4.10) <u>COROLLARY</u> For a prime ring A with unit and centre C(A) the following conditions are equivalent:

(a) A is a subgenerator in $\Omega(A)$-MOD and $U \cap C(A) \neq 0$ for any non-zero ideal $U \subset A$;

(b) $\hat{A}$ is a finite dimensional central simple algebra and $T = End_{\Omega}(\hat{A}) = C(\hat{A})$ is the quotient field of C(A).

(4.11) <u>EXAMPLES</u>

(i) Rings A (with unit) whose non-zero ideals intersect the centre non-trivially:

1) A is Azumaya algebra;

2) A is self-generator as $\Omega(A)$-module;

3) A is an associative semiprime PI-ring;

4) A is a (purely) alternative prime ring with $3A \neq 0$ (Slater [22]);

5) A is a semiprime NPI Jordan ring (Rowen [20], Theorem (2.5));

6) A is centrally admissible (Rowen [21], Theorem (3.3)).

(ii) Rings A (with unit) which are subgenerators in $\Omega(A)$-MOD:

1) examples 1) and 4) above;

2) A is a finitely generated module over its centre;

3) A is an associative prime PI-ring;

4) A is a (nonassociative) prime NPI-ring of finite degree (Rowen [2o], Theorem (1.5)).

(iii) An example of an associative prime ring A for which $T=End_{\Omega}(\hat{A})$ is a transcendental field extension of the quotient field of the centre of A is given in Delale [5].

(iv) For an associative prime ring A with unit the following construction for $T=End_{\Omega}(\hat{A})$ is given in Delale [6], Proposition II.9.5.2: Take $E=\{(a,b)\in A\times A \mid b\neq 0 \text{ and } axb=bxa \text{ for all } x\in A\}$.

Define an equivalence relation on E by

$$(a,b)\sim(a',b') \text{ iff } axb'=bxa' \text{ for all } x\in A.$$

Then the equivalence classes by this relation allow the structure of a field which is isomorphic to T.

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