FROM GALOIS FIELD EXTENSIONS TO GALOIS COMODULES

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Given a finite automorphism group G of a field extension $E \supset K$, E can be considered as module over the group algebra $K[G]$. Moreover, E can also be viewed as a comodule over the bialgebra $K[G]^*$ and here a canonical isomorphism involving the subfield fixed under the action of G arises. This isomorphism and its consequences were extended and studied for group actions on commutative rings, for actions of Hopf algebras on noncommutative algebras, then for corings with grouplike elements and eventually to comodules over corings. The purpose of this note is to report about this development and to give the reader some idea about the notions and results involved in this theory (without claiming to be comprehensive).

1. Preliminaries

To begin with we recall the algebraic structures for which Galois type conditions are applied. We follow the notation in [9]. Throughout R will denote a commutative associative ring with unit.

1.1. Algebras and modules. A, or more precisely $(A, \mu, 1_A)$, stands for an associative R-algebra with multiplication $\mu : A \otimes_R A \to A$ and unit 1_A. Right A-modules are defined as R-modules M with an action $\varrho_M : M \otimes_R A \to M$.

For the category of right A-modules we write $\mathbf{M}_A$ and denote the morphisms between $M, N \in \mathbf{M}_A$ by $\mathrm{Hom}_A(M, N)$. It is well known that A is a projective generator in $\mathbf{M}_A$.

1.2. Coalgebras and comodules. An *R-coalgebra* is a triple (C, Δ, ε) where C is an R-module, $\Delta : C \to C \otimes_R C$ is the coproduct and $\varepsilon : C \to R$

is the counit. Right C-comodules are R-modules M with a coaction $\varrho^M : M \to M \otimes_R C$.

The category of right C-comodules is denoted by $\mathbf{M}^C$ and the morphisms between $M, N \in \mathbf{M}^C$ are written as $\mathrm{Hom}^C(M,N)$. As a right comodule, C is a subgenerator in $\mathbf{M}^C$, that is, every right C-comodule is a subcomodule of a C-generated comodule. Note that $\mathbf{M}^C$ need not have projectives even if R is a field.

Left (co)modules and their categories are defined and denoted in an obvious way.

1.3. Bialgebras and Hopf modules. An *R-bialgebra* is a quintuple $(B, \Delta, \varepsilon, \mu, 1_B)$ where (B, Δ, ε) is an R-coalgebra and $(B, \mu, 1_B)$ is an R-algebra such that Δ is an algebra morphism (equivalently μ is a coalgebra morphism).

An R-module M that is a right B-module by $\varrho_M : M \otimes_R B \to M$ and a right B-comodule by $\varrho^M : M \to M \otimes_R B$ is called a *right B-Hopf module* provided for any $m \in M$ and $b \in B$, $\varrho^M(mb) = \varrho^M(m)\Delta(b)$. The category of all right B-Hopf modules is denoted by $\mathbf{M}_B^B$. The module $B \otimes_R B$ allows for a right B-Hopf module structure and with this it is a subgenerator in $\mathbf{M}_B^B$. For $M \in \mathbf{M}_B^B$ the *coinvariants* are defined as

$$M^{coB} = \{m \in M \mid \varrho^M(m) = m \otimes_R 1_B\} \simeq \mathrm{Hom}_B^B(A, M).$$

An R-bialgebra B is called a *Hopf algebra* if there is an antipode, that is, an R-linear map $S : B \to B$ which is the inverse of the identity of B with respect to the convolution product in $\mathrm{End}_R(B)$ (see also 2.5).

For any R-algebra A which is finitely generated and projective as R-module, the dual $A^* = \mathrm{Hom}_R(A, R)$ can be considered as an R-coalgebra with natural comultiplication and counit. Here we are interested in the following special case.

1.4. Group algebras and their dual. Let G be a finite group of order $n \in \mathbb{N}$ and $R[G]$ the group algebra, that is, $R[G]$ is a free R-module with basis the group elements $\{g_1, \ldots, g_n\}$ and the product given by the group multiplication. Furthermore, $R[G]$ is an R-coalgebra with coproduct induced by $\Delta(g) = g \otimes g$ and counit $\varepsilon(g) = 1_R$, for $g \in G$. With these structures $R[G]$ is an R-bialgebra, and even a Hopf algebra with antipode S induced by $S(g) = g^{-1}$ for $g \in G$.

The R-dual $R[G]^* = \mathrm{Hom}_R(R[G], R)$ is also a Hopf algebra. The multiplication of $f, g \in R[G]^*$ is given by $f * g(x) = f(x)g(x)$ for $x \in G$. To describe the coalgebra structure let $\{p_g\}_{g\in G} \subset R[G]^*$ be the dual basis to $\{g\}_{g\in G}$. Then coproduct and counit are defined by

$$\Delta(p_g) = \sum_{kh=g} p_k \otimes p_h, \quad \varepsilon(P_g) = \delta_{1,g}.$$

The antipode S of $R[G]^*$ is induced by $S(p_g) = p_{g^{-1}}$ for $g \in G$.

1.5. Comodule algebras and relative Hopf modules. Let B be an R-bialgebra. An R-algebra A is called *right B-comodule algebra* if A is a right B-comodule by $\varrho^A : A \to A \otimes_R B$ such that ϱ^A is an algebra morphisms.

A *right (A, B)-Hopf module* is an R-module M which is a right A-module and a right B-comodule by $\varrho^M : M \to M \otimes_R B$ such that for all $m \in M$ and $a \in A$, $\varrho^M(ma) = \varrho^M(m)\varrho^A(a)$. The category of these modules is denoted by $\mathbf{M}_A^H$ and it has $A \otimes_R H$ as a subgenerator. For $M \in \mathbf{M}_A^H$ the coinvariants are defined as

$$M^{coB} = \{m \in M \mid \varrho^M(m) = m \otimes_R 1_B\} \simeq \mathrm{Hom}_A^B(A, M).$$

Note that in the above construction the right (A, B)-Hopf modules may be replaced by the category $\mathbf{M}(B)_A^D$ of *right (A, D)-Hopf modules* where D is a right B-module coalgebra and the objects are right D-comodules which are also right A-modules satisfying some compatibility condition. Then $A \otimes_R D$ is a subgenerator $\mathbf{M}(B)_A^D$ (see [13], [18]).

Under weak (projectivity) conditions, for all the structures considered above the related (co)module categories can be understood as module categories over some algebra subgenerated by a suitable module. We refer to [24] for more details. All this settings are subsumed as special cases of

1.6. Corings and comodules. An A-coring is a triple $(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ where $\mathcal{C}$ is an (A, A)-bimodule with coproduct $\underline{\Delta} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and counit $\underline{\varepsilon} : \mathcal{C} \to A$. Associated to this there are the right and left dual rings $\mathcal{C}^* = \mathrm{Hom}_A(\mathcal{C}, A)$ and ${}^*\mathcal{C} = {}_A\mathrm{Hom}(\mathcal{C}, A)$ with the convolution products.

A *right $\mathcal{C}$-comodule* is a right A-module M together with an A-linear $\mathcal{C}$-coaction $\varrho^M : M \to M \otimes_A \mathcal{C}$. These comodules form a category which we denote by $\mathbf{M}^{\mathcal{C}}$. It is an additive category with coproducts and cokernels, and $\mathcal{C}$ is a subgenerator in it. The functor $-\otimes_A \mathcal{C} : \mathbf{M}_A \to \mathbf{M}^{\mathcal{C}}$ is right adjoint to the forgetful functor by the isomorphisms, for $M \in \mathbf{M}^{\mathcal{C}}$ and $X \in \mathbf{M}_A$,

$$\mathrm{Hom}^{\mathcal{C}}(M, X \otimes_A \mathcal{C}) \to \mathrm{Hom}_A(M, X),\ f \mapsto (I_X \otimes \underline{\varepsilon}) \circ f,$$

with inverse map $\quad h \mapsto (h \otimes I_{\mathcal{C}}) \circ \varrho^M$.

Notice that for any monomorphism (injective map) $f : X \to Y$ in $\mathbf{M}_A$, the colinear map $f \otimes I_{\mathcal{C}} : X \otimes_A \mathcal{C} \to Y \otimes_A \mathcal{C}$ is a monomorphism in $\mathbf{M}^{\mathcal{C}}$ but need not be injective. In case ${}_A\mathcal{C}$ is flat, monomorphisms in $\mathbf{M}^{\mathcal{C}}$ are injective maps and in this case $\mathbf{M}^{\mathcal{C}}$ is a Grothendieck category (see 18.14 in [9]).

Any right $\mathcal{C}$-comodule (M, ϱ^M) allows for a left ${}^*\mathcal{C}$-module structure by putting $f \rightharpoonup m = (I_M \otimes f) \circ \varrho^M(m)$, for any $f \in {}^*\mathcal{C}$, $m \in M$. This yields a faithful functor $\Phi : \mathbf{M}^{\mathcal{C}} \to {}_{^*\mathcal{C}}\mathbf{M}$ which is a full embedding if and only if the map

$$\alpha_K : K \otimes_A \mathcal{C} \to \mathrm{Hom}_A({}^*\mathcal{C}, K), \quad n \otimes c \mapsto [f \mapsto nf(c)],$$

is injective for any $K \in \mathbf{M}_A$. This is called the *left α-condition* on $\mathcal{C}$ and it holds if and only if ${}_A\mathcal{C}$ is locally projective. In this case $\mathbf{M}^{\mathcal{C}}$ can be identified with $\sigma[{}_{^*\mathcal{C}}\mathcal{C}]$, the full subcategory of ${}_{^*\mathcal{C}}\mathbf{M}$ whose objects are subgenerated by $\mathcal{C}$.

1.7. A as a $\mathcal{C}$-comodule. An element g of an A-coring $\mathcal{C}$ is called a *grouplike element* if $\underline{\Delta}(g) = g \otimes g$ and $\underline{\varepsilon}(g) = 1_A$. Such a grouplike element g exists if and only if A is a right or left $\mathcal{C}$-comodule, by the coactions

$$\varrho^A : A \to \mathcal{C},\ a \mapsto ga, \quad {}^A\varrho : A \to \mathcal{C},\ a \mapsto ag.$$

Write A_g or ${}_gA$ to consider A with the right or left comodule structure induced by g. Given an A-coring $\mathcal{C}$ with a grouplike element g and $M \in \mathbf{M}^{\mathcal{C}}$, the *$g$-coinvariants* of M are defined as the R-module

$$M_g^{co\mathcal{C}} = \{m \in M \mid \varrho^M(m) = m \otimes g\} = \mathrm{Ke}\,(\varrho^M - (- \otimes g)),$$

and there is an isomorphism

$$\mathrm{Hom}^{\mathcal{C}}(A_g, M) \to M_g^{co\mathcal{C}}, \quad f \mapsto f(1_A).$$

The bijectivity of this map is clear by the fact that any A-linear map with source A is uniquely determined by the image of 1_A. As special cases we have the coinvariants

(1) $\mathrm{End}^{\mathcal{C}}(A_g) \simeq A_g^{co\mathcal{C}} = \{a \in A_g \,|\, ga = ag\}$, the centraliser of g in A.

(2) For any $X \in \mathbf{M}_A$, $(X \otimes_A \mathcal{C})^{co\mathcal{C}} \simeq \mathrm{Hom}^{\mathcal{C}}(A_g, X \otimes_A \mathcal{C}) \simeq X$, and for $X = A$,

$$\mathcal{C}^{co\mathcal{C}} \simeq \mathrm{Hom}^{\mathcal{C}}(A_g, \mathcal{C}) \simeq \mathrm{Hom}_A(A_g, A) \simeq A,$$

which is a left A- and right $\mathrm{End}^{\mathcal{C}}(A_g)$-morphism.

Given any right B-module M, $M \otimes_B A$ is a right $\mathcal{C}$-comodule via the coaction

$$\varrho^{M\otimes_B A} : M \otimes_B A \to M \otimes_B A \otimes_A \mathcal{C} \cong M \otimes_B \mathcal{C}, \quad m \otimes a \mapsto m \otimes ga.$$

This yields a functor $-\otimes_B A : \mathbf{M}_B \to \mathbf{M}^{\mathcal{C}}$. Right adjoint to this is the *g-coinvariants functor* $\mathrm{Hom}^{\mathcal{C}}(A_g, -) : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_B$.

For $N \in \mathbf{M}_B$ the unit of the adjunction is given by

$$N \to (N \otimes_B A)^{co\mathcal{C}}, \quad n \mapsto n \otimes 1_A,$$

and for $M \in \mathbf{M}^{\mathcal{C}}$, the counit reads

$$M^{co\mathcal{C}} \otimes_B A \to M, \quad m \otimes a \mapsto ma.$$

1.8. Coring of a projective module. For R-algebras A, B, let P be a (B, A)-bimodule that is finitely generated and projective as a right A-module. Let $p_1, \ldots, p_n \in P$ and $\pi_1, \ldots, \pi_n \in P^* = \mathrm{Hom}_A(P, A)$ be a dual basis for P_A. Then the (B, B)-bimodule $P \otimes_A P^*$ is an algebra by the isomorphism

$$P \otimes_A P^* \to \mathrm{End}_A(P), \quad p \otimes f \mapsto [q \mapsto pf(q)],$$

and the (A, A)-bimodule $P^* \otimes_B P$ is an A-coring with coproduct and counit

$$\underline{\Delta} : P^* \otimes_B P \to (P^* \otimes_B P) \otimes_A (P^* \otimes_B P), \quad f \otimes p \mapsto \textstyle\sum_i f \otimes p_i \otimes \pi_i \otimes p,$$

$$\underline{\varepsilon} : P^* \otimes_B P \to A, \quad f \otimes p \mapsto f(p).$$

As a special case, for the (A, A)-bimodule $P = A^n$, $n \in \mathbb{N}$, $P^* \otimes_A P$ can be identified with the $n \times n$-matrices $M_n(A)$ over A, endowed with an A-coring structure (matrix coring).

1.9. The Sweedler coring. Given an R-algebra morphism $\phi : B \to A$, the tensor product $\mathcal{C} = A \otimes_B A$ is an A-coring with coproduct

$$\underline{\Delta} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C} \simeq A \otimes_B A \otimes_B A, \quad a \otimes a' \mapsto a \otimes 1_A \otimes a',$$

and counit $\underline{\varepsilon}(a \otimes a') = aa'$. $\mathcal{C}$ is called the *Sweedler A-coring* associated to the algebra (or ring) morphism $\phi : B \to A$. Clearly $1_A \otimes 1_A$ is a grouplike element in $\mathcal{C}$.

Since A is finitely generated and projective as right A-module, in view of 1.7 this is a special case of 1.8.

1.10. Entwining structures. Given an R-algebra A and an R-coalgebra C one may think about compatibility conditions between these two structures. This led to the notion of a (right-right) *entwining structure* which is given by an *entwining map*, that is, an R-module map $\psi : C \otimes_R A \to A \otimes_R C$ satisfying the conditions

(1) $\psi \circ (I_C \otimes \mu) = (\mu \otimes I_C) \circ (I_A \otimes \psi) \circ (\psi \otimes I_A)$,
(2) $(I_A \otimes \Delta) \circ \psi = (\psi \otimes I_C) \circ (I_C \otimes \psi) \circ (\Delta \otimes I_A)$,
(3) $\psi \circ (I_C \otimes \iota) = \iota \otimes I_C$,
(4) $(I_A \otimes \varepsilon) \circ \psi = \varepsilon \otimes I_A$.

Associated to any entwining structure (A, C, ψ) is the category of (right-right) (A, C, ψ)-*entwined modules* denoted by $\mathbf{M}_A^C(\psi)$. An object $M \in \mathbf{M}_A^C(\psi)$ is a right A-module with multiplication ϱ_M and a right C-comodule with coaction ϱ^M satisfying

$$\varrho^M \circ \varrho_M = (\varrho_M \otimes I_C) \circ (I_M \otimes \psi) \circ (\varrho^M \otimes I_A),$$

and morphisms in $\mathbf{M}_A^C(\psi)$ are maps which are right A-module as well as right C-comodule morphisms.

Entwining structures were introduced in [7] in the context of gauge theory on noncommutatice spaces. It then turned out that they are instances of corings since - with the data given above - $A \otimes_R C$ is an A-coring with (A, A)-bimodule struture

$$b(a' \otimes c)a = ba'\psi(c \otimes a), \text{ for } a, a', b \in A,\ c \in C,$$

coproduct $\underline{\Delta} = I_A \otimes \Delta$ and counit $\underline{\varepsilon} = I_A \otimes \varepsilon$ (see 32.6 in [9]). With this correspondence the category $\mathbf{M}_A^C(\psi)$ can be identified with the comodule category $\mathbf{M}^{A \otimes_R C}$.

1.11. Bialgebras and corings (see 33.1 in [9]). Let $(B, \Delta_B, \varepsilon_B)$ be an R-bialgebra. Then $B \otimes_R B$ is a B-coring by the coproduct $\underline{\Delta} : I_B \otimes \Delta_B$, the counit $\underline{\varepsilon} = I_B \otimes \varepsilon_B$, and the (B, B)-bimodule structure

$$a(c \otimes d)b = (ac \otimes d)\Delta_B(b) \text{ where } a, b, c, d \in B.$$

With this structure the right B-Hopf modules can be identified with the right $B \otimes_R B$-comodules, that is, $\mathbf{M}_B^B = \mathbf{M}^{B \otimes_R B}$. Clearly $1_B \otimes 1_B$ is a grouplike element in $B \otimes_R B$ and the ring of $B \otimes_R B$-covariants of B is isomorphic to R.

1.12. Comodule algebras and corings (see 33.2 in [9]). Let $(B, \Delta_B, \varepsilon_B)$ be an R-bialgebra. Then for a right B-comodule algebra A, $A \otimes_R B$ is an A-coring with coproduct $\underline{\Delta} = I_A \otimes \Delta_B$, counit $\underline{\varepsilon} = I_A \otimes \varepsilon_B$, and (A, A)-bimodule structure

$$a(c \otimes b)d = (ac \otimes b)\varrho^A(d), \text{ for } a, c, d \in A \text{ and } b \in B.$$

Here the right relative (A, B)-Hopf modules are just the right $A \otimes_R B$-comodules, that is, $\mathbf{M}_A^B = \mathbf{M}^{A \otimes_R B}$.

1.13. Cointegrals. An (A, A)-bilinear map $\delta : \mathcal{C} \otimes_A \mathcal{C} \to \mathcal{C}$ is called a *cointegral in* $\mathcal{C}$ if

$$(I_{\mathcal{C}} \otimes \delta) \circ (\Delta \otimes I_{\mathcal{C}}) = (\delta \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{C}} \otimes \Delta).$$

Cointegrals are characterised by the fact that for any $M \in \mathbf{M}^{\mathcal{C}}$, the map

$$(I_M \otimes \delta) \circ (\varrho^M \otimes I_{\mathcal{C}}) : M \otimes_A \mathcal{C} \to M$$

is a comodule morphism (or by the corresponding property for left $\mathcal{C}$-comodules).

In [10], Section 5, these maps are related to the counit for the adjoint pair of functors $- \otimes_A \mathcal{C}$ and the forgetful functor. For R-coalgebras C over a commutative ring R with C_R locally projective, a cointegral is precisely a C^*-balanced R-linear map $C \otimes_R C \to R$ (e.g., 6.4 in[9]).

Recall some properties of relative injectivity from [27], Section 2:

1.14. Relative injectivity. Let $M \in \mathbf{M}^{\mathcal{C}}$ and $S = \mathrm{End}^{\mathcal{C}}(M)$.

M is $(\mathcal{C}, A)$*-injective* provided the structure map $\varrho^M : M \to M \otimes_A \mathcal{C}$ is split by a $\mathcal{C}$-morphism $\lambda : M \otimes_A \mathcal{C} \to M$.

M is called *strongly* $(\mathcal{C}, A)$*-injective* if this λ is $\mathcal{C}$-colinear and S-linear. Given a subring $B \subseteq S$, M is said to be B*-strongly* $(\mathcal{C}, A)$*-injective* if λ is $\mathcal{C}$-colinear and B-linear.

M is called *fully* $(\mathcal{C}, A)$*-injective* if there is a cointegral $\delta_M : \mathcal{C} \otimes_A \mathcal{C} \to \mathcal{C}$ such that ϱ^M is split by $(I_M \otimes \delta_M) \circ (\varrho^M \otimes I_{\mathcal{C}})$.

The notions for left $\mathcal{C}$-comodules are defined symmetrically.

For R-coalgebras C, B-strongly (C, R)-injective comodules are named B*-equivariantly* C*-injective* (see Definition 5.1 in [20]).

1.15. Fully $(\mathcal{C}, A)$-injective comodules. *Let $M \in \mathbf{M}^{\mathcal{C}}$ with $S = \mathrm{End}^{\mathcal{C}}(M)$.*

(1) M is fully $(\mathcal{C}, A)$-injective if and only if

$$(I_M \otimes \widetilde{\delta}_M) \circ \varrho^M = I_M \quad \textit{where} \quad \widetilde{\delta}_M = \delta_M \circ \Delta : \mathcal{C} \to A.$$

(2) $\mathcal{C}$ is a fully $(\mathcal{C}, A)$-injective right (left) comodule if and only if $\mathcal{C}$ is a coseparable coring.

(3) Let M be fully $(\mathcal{C}, A)$-injective. Then:

- *(i) Every comodule in $\sigma[M]$ is fully $(\mathcal{C}, A)$-injective.*
- *(ii) If M is a subgenerator in $\mathbf{M}^{\mathcal{C}}$ then $\mathcal{C}$ is a coseparable coring.*
- *(iii) For any subring $B \subset S$ and $X \in \mathbf{M}_B$, $X \otimes_B M$ is fully $(\mathcal{C}, A)$-injective.*
- *(iv) If M_A is finitely generated and projective, then M^* is a fully $(\mathcal{C}, A)$-injective left $\mathcal{C}$-comodule.*

2. Galois extensions and comodules

Classical Galois theory studies the action of a finite automorphism group G on a field E and then considers E as extension of the subfield of the elements which are left unchanged by the action of G. This can be understood as a comodule situation (compare [19], Chapter 8).

2.1. Galois field extension. Let G be a finite automorphism group of a field extension $E \supset K$ and let $F = E^G$ be the fixed field of G. Thus the group algebra $K[G]$ acts on E and so its dual, the Hopf algebra $H = \mathrm{Hom}_K(K[G], K) = K[G]^*$ coacts on E.

To describe this let $G = \{g_1, \dots, g_n\}$ and choose $\{b_1, \dots, b_n\} \subset E$ as a basis of the F-vectorspace E. Denote by $\{p_1, \dots, p_n\} \subset K[G]^*$ the dual basis to $\{g_1, \dots, g_n\} \subset K[G]$. Then E is a right $K[G]^*$-comodule by the coaction

$$\varrho^E : E \to E \otimes_K K[G]^*, \quad a \mapsto \sum_{i=1}^{n} (g_i \cdot a) \otimes p_i,$$

and we can define the *Galois map*

$$\gamma : E \otimes_F E \to E \otimes_K K[G]^*, \quad a \otimes b \mapsto \sum_{i=1}^{n} a(g_i \cdot b) \otimes p_i.$$

For any $w = \sum_j a_j \otimes b_j \in \operatorname{Ke} \gamma$, we have $\sum_{j,i} a_j(g_i \cdot b_j) \otimes p_i = 0$ and by the independence of the $p_1, \dots, p_n$, $\sum_j a_j(g_i \cdot b_j) = 0$ for all i. Now Dedekind's lemma on the independence of automorphisms implies that all $a_j = 0$ and thus $w = 0$. This shows that γ is injective and for dimension reasons it is in fact bijective.

Notice that the coinvariants of the $K[G]^*$-comodule E are

$$\{a \in E \mid \sum_{i=1}^{n} (g_i \cdot a) \otimes p_i = a \otimes \varepsilon\} = E^G,$$

since for each such $a \in E$ and $g_i \in G$, $g_i \cdot a = (g_i \cdot a)p_i(g_i) = a\varepsilon(g_i) = a$.

The definition of Hopf Galois extensions goes back to Chase-Harrison-Rosenberg [11] where the classical Galois theory of fields was extended to groups acting on commutative rings. This was generalised in Chase-Sweedler [12] to coactions of Hopf algebras on commutative R-algebras and then, in Kreimer-Takeuchi [17], to coactions on noncommutative R-algebras.

2.2. Comodule algebras. Let H be a Hopf R-algebra and A a right H-comodule algebra with structure map $\varrho^A : A \to A \otimes_R H$ and $B = A^{coH}$. Then $B \subset A$ is called *right H-Galois* if the following map is bijective:

$$\gamma : A \otimes_B A \to A \otimes H, \quad a \otimes b \mapsto (a \otimes 1)\varrho^A(b).$$

For examples and more information about such extensions we refer to [19], Section 8. Further investigation on such structures were done in particular by Doi, Takeuchi and Schneider [14], [15], [21], [22], [23].

Generalising results about the action of an affine algebraic group scheme on an affine scheme the following theorem was proved in [21]. This shows (again) that H-Galois extensions are closely related to modules inducing equivalences.

Schneider's Theorem. *Let H be a Hopf algebra over a field R with bijective antipode. Then for a right H-comodule algebra A and $B = A^{coH}$ the following are equivalent:*

(a) $B \subset A$ is a H-Galois extension and A is faithfully flat as a left B-module;

(b) $B \subset A$ is a H-Galois extension and A is faithfully flat as a right B-module;

(c) $- \otimes_B A : \mathbf{M}_B \to \mathbf{M}_A^H$ is an equivalence;

(d) $A \otimes_B - : {}_B\mathbf{M} \to {}_A\mathbf{M}^H$ is an equivalence.

Notice that the above theorem shows a left right symmetry which will not be maintained in (most of) the subsequent generalisations.

As mentioned in 1.5, the (A, H)-Hopf modules can be generalised to (A, D)-Hopf modules where D is a right H-module coalgebra yielding the category $\mathbf{M}(H)_A^D$. If there is a *grouplike element* $x \in D$, then A is in $\mathbf{M}(H)_A^D$ and for any $M \in \mathbf{M}(H)_A^D$ coinvariants can be defined as $\mathrm{Hom}_A^D(A, M)$. Then $B = \mathrm{Hom}_A^D(A, A)$ is a subring of A and the inclusion $B \hookrightarrow A$ is called a *right Hopf-Galois extension* provided the canonical map

$$A \otimes_B A \to A \otimes_R D, \quad a \otimes b \mapsto (a \otimes x)\varrho^A(b)$$

is bijective. For this setting an extension of Schneider's Theorem is proved by Menini and Zuccoli (see Theorem 3.29 in [18]).

2.3. Coalgebra-Galois extensions. Let C be an R-coalgebra and A an R-algebra and a right C-comodule with coaction $\varrho^A : A \to A \otimes_R C$. Define the *coinvariants* of A as

$$B = \{b \in A \mid \text{for all } a \in A,\ \varrho^A(ba) = b\varrho^A(a)\}.$$

The extension $B \hookrightarrow A$ is called a *coalgebra-Galois extension* (or a *C-Galois extension*) if the following left A-module, right C-comodule map is bijective:

$$\gamma : A \otimes_B A \to A \otimes_R C, \qquad a \otimes a' \mapsto a\varrho^A(a').$$

Notice that here the definition of covariants does not require the existence of a grouplike element in C and thus coalgebra-Galois extensions are defined for arbitrary coalgebras. This notion was introduced in [6], following their appearance as generalised principal bundles in [7]. The main geometric motivation for this was the need for principal bundles with coalgebras playing the role of a structure group. The main result Theorem 2.7 in [6] shows how coalgebra Galois extensions are related to entwining structures.

Theorem. *Let R be a field and A a C-Galois extension of B (as defined above). Then there exists a unique entwining map $\psi : C \otimes_R A \to A \otimes_R C$ such that $A \in \mathbf{M}_A^C(\psi)$ with structure map ϱ^A.*

2.4. Galois corings. Let $\mathcal{C}$ be an A-coring with a grouplike element g and $B = A_g^{co\mathcal{C}}$. Following Definition 5.3 in [4], $(\mathcal{C}, g)$ is called a *Galois coring* if the canonical map

$$\chi : A \otimes_S A \to \mathcal{C}, \quad a \otimes a' \mapsto aga',$$

is an isomorphism (of corings). It was pointed out in [26] that this can be seen as the evaluation map

$$\mathrm{Hom}^{\mathcal{C}}(A_g, \mathcal{C}) \otimes_S A \to \mathcal{C}, \quad f \otimes a \mapsto f(a).$$

The following assertions are equivalent (4.6 in [26])*:*

(a) $(\mathcal{C}, g)$ is a Galois coring;

(b) for every $(\mathcal{C}, A)$-injective comodule $N \in \mathbf{M}^{\mathcal{C}}$, the evaluation

$$\mathrm{Hom}^{\mathcal{C}}(A_g, N) \otimes_B A \to N, \quad f \otimes a \mapsto f(a),$$

is an isomorphism.

Notice that here the canonical isomorphism can be extended to related isomorphisms for the class of all relative injective comodules.

The following is a one-sided generalization of Schneider's theorem (see 4.8 in [26]).

The Galois Coring Structure Theorem.

(1) The following are equivalent:

(a) $(\mathcal{C}, g)$ is a Galois coring and ${}_BA$ is flat;

(b) ${}_A\mathcal{C}$ is flat and A_g is a generator in $\mathbf{M}^{\mathcal{C}}$.

(2) The following are equivalent:

(a) $(\mathcal{C}, g)$ *is a Galois coring and* ${}_BA$ *is faithfully flat;*

(b) ${}_A\mathcal{C}$ *is flat and* A_g *is a projective generator in* $\mathbf{M}^{\mathcal{C}}$*;*

(c) ${}_A\mathcal{C}$ *is flat and* $\mathrm{Hom}^{\mathcal{C}}(A_g, -) : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_B$ *is an equivalence with inverse* $- \otimes_B A : \mathbf{M}_B \to \mathbf{M}^{\mathcal{C}}$ (cf. 1.7).

If the base ring A is injective as right A-module, then $\mathcal{C}$ is injective as right $\mathcal{C}$-comodule and thus (see 4.9 in [26]) we obtain the

Corollary. *Assume* A *to be a right self-injective ring and let* $\mathcal{C}$ *be an* A*-coring with grouplike element* g.

(1) The following are equivalent:

(a) $(\mathcal{C}, g)$ *is a Galois coring;*

(b) for every injective comodule $N \in \mathbf{M}^{\mathcal{C}}$*, the evaluation*

$$\mathrm{Hom}^{\mathcal{C}}(A_g, N) \otimes_B A \to N, \quad f \otimes a \mapsto f(a),$$

is an isomorphism.

(2) The following are equivalent:

(a) $(\mathcal{C}, g)$ *is a Galois coring and* ${}_BA$ *is (faithfully) flat;*

(b) ${}_BA$ *is (faithfully) flat and for every injective comodule* $N \in \mathbf{M}^{\mathcal{C}}$*, the following evaluation map is an isomorphism:*

$$\mathrm{Hom}^{\mathcal{C}}(A_g, N) \otimes_B A \to N, \quad f \otimes a \mapsto f(a).$$

2.5. Hopf algebras. Given an R-bialgebra B, by definition the B-coring $B \otimes_R B$ is Galois provided the canonical map

$$\gamma : B \otimes_R B \to B \otimes_R B, \quad a \otimes b \mapsto (a \otimes 1)\Delta(b)$$

is an isomorphism. Since bijectivity of this map is equivalent to the existence of an antipode (see 15.2 in [9]) we have:

For a bialgebra B *the following are equivalent:*

(a) $B \otimes_R B$ *is a Galois* B*-coring;*

(b) B *is a Hopf algebra (has an antipode);*

(c) $\mathrm{Hom}_B^B(B, -) : \mathbf{M}_B^B \to \mathbf{M}_R$ *is an equivalence (with inverse* $- \otimes_R B$*).*

If (any of) these conditions hold, B *is a projective generator in* $\mathbf{M}_B^B$.

The notion of Galois corings was extended to comodules by El Kaoutit and Gómez-Torrecillas in [16], where to any bimodule ${}_SP_A$ with P_A finitely generated and projective, a coring $P^* \otimes_S P$ was associated (see 1.8) and it was shown that the map

$$\varphi : \mathrm{Hom}_A(P, A) \otimes_S P \simeq \mathrm{Hom}^{\mathcal{C}}(P, \mathcal{C}) \otimes_S P \to \mathcal{C}$$

is a coring morphism provided P is also a right $\mathcal{C}$-comodule and $S = \mathrm{End}^{\mathcal{C}}(P)$.

In [9], 18.25, such comodules P are termed *Galois comodules* provided φ is bijective, and it is proved in [9], 18.26, that this condition implies that the functors $\mathrm{Hom}_A(P, -) \otimes_S P$ and $- \otimes_A \mathcal{C}$ from $\mathbf{M}_A$ to $\mathbf{M}^{\mathcal{C}}$ are isomorphic.

2.6. Galois comodules. Let P be a right $\mathcal{C}$-comodule such that P_A is finitely generated and projective and let $S = \mathrm{End}^{\mathcal{C}}(P)$. Then P is called a *Galois comodule* if the evaluation map

$$\mathrm{Hom}^{\mathcal{C}}(P, \mathcal{C}) \otimes_S P \to \mathcal{C},\ f \otimes m \mapsto f(m),$$

is an isomorphism of right $\mathcal{C}$-comodules.

Considering $P^* \otimes_S P$ as an A-coring (via 1.8), the following are equivalent:

(a) P is a Galois comodule;

(b) there is a (coring) isomorphism

$$P^* \otimes_S P \to \mathcal{C}, \quad \xi \otimes m \mapsto \sum (\xi \otimes I_{\mathcal{C}})\, \varrho^P(m);$$

(c) for every $(\mathcal{C}, A)$-injective comodule $N \in \mathbf{M}^{\mathcal{C}}$, the evaluation

$$\mathrm{Hom}^{\mathcal{C}}(P, N) \otimes_S P \to N, \quad f \otimes m \mapsto f(m),$$

is a (comodule) isomorphism;

(d) for every right A-module X, the map

$$\mathrm{Hom}_A(P, X) \otimes_S P \to X \otimes_A \mathcal{C}, \quad g \otimes m \mapsto (g \otimes I_{\mathcal{C}})\, \varrho^P(m),$$

is a (comodule) isomorphism.

The next theorem - partially proved in [16] - shows which additional conditions on a Galois comodule are sufficient to make it a (projective) generator in $\mathbf{M}^{\mathcal{C}}$ (see 18.27 in [9]).

The Galois comodule structure theorem.

(1) The following are equivalent:

(a) P is a Galois comodule and ${}_SP$ is flat;

(b) ${}_A\mathcal{C}$ is flat and P is a generator in $\mathbf{M}^{\mathcal{C}}$.

(2) The following are equivalent:

(a) M is a Galois comodule and ${}_SP$ is faithfully flat;

(b) ${}_A\mathcal{C}$ is flat and P is a projective generator in $\mathbf{M}^{\mathcal{C}}$;

(c) ${}_A\mathcal{C}$ is flat and $\mathrm{Hom}^{\mathcal{C}}(P,-) : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_S$ is an equivalence with the inverse $-\otimes_S P : \mathbf{M}_S \to \mathbf{M}^{\mathcal{C}}$.

These Galois comodules are further investigated in Brzeziński [5] and their relevance for descent theory, vector bundles, and non-commutative geometry is pointed out there. In particular *principal comodules* are considered, that is, Galois comodules in the above sense which are projective as modules over their endomorphism rings. Related questions are, for example, also considered by Caenepeel, De Groot and Vercruysse in [10].

3. General Galois comodules

Recall that for a Galois $\mathcal{C}$-comodule P in the sense of 2.6 (where P_A is finitely generated and projective) the functors $-\otimes_A\mathcal{C}$ and $\mathrm{Hom}_A(P,-)\otimes_S P$ are isomorphic. In [27] it is suggested to take this property as definition without further condition on the A-module structure of P.

Throughout this section let $\mathcal{C}$ be an A-coring, $P \in \mathbf{M}^{\mathcal{C}}$ and $S = \mathrm{End}^{\mathcal{C}}(P)$, $T = \mathrm{End}_A(P)$.

3.1. Galois comodules. We call P a *Galois comodule* if

$$-\otimes_A \mathcal{C} \simeq \mathrm{Hom}_A(P,-)\otimes_S P \text{ as functors} : \mathbf{M}_A \to \mathbf{M}^{\mathcal{C}}.$$

The following are equivalent ([27], 2.1)*:*

(a) P is a Galois comodule;

(b) $\mathrm{Hom}_A(P,-)\otimes_S P$ is right adjoint to the forgetful functor $\mathbf{M}^{\mathcal{C}} \to \mathbf{M}_A$, that is, for $K \in \mathbf{M}_A$ and $M \in \mathbf{M}^{\mathcal{C}}$, there is a (bifunctorial) isomorphism

$$\mathrm{Hom}^{\mathcal{C}}(M, \mathrm{Hom}_A(P,K)\otimes_S P) \to \mathrm{Hom}_A(M,K);$$

(c) *for any* $K \in \mathbf{M}_A$ *there is a functorial isomorphism of comodules*

$$\mathrm{Hom}_A(P,K) \otimes_S P \to K \otimes_A \mathcal{C},\ g \otimes p \mapsto (g \otimes I_{\mathcal{C}})\varrho^P(p);$$

(d) *for every* $(\mathcal{C}, A)$*-injective* $N \in \mathbf{M}^{\mathcal{C}}$,

$$\mathrm{Hom}^{\mathcal{C}}(P,N) \otimes_S P \to N,\ f \otimes p \mapsto f(p),$$

is an isomorphism (in $\mathbf{M}^{\mathcal{C}}$*).*

These comodules have good properties (see 2.2 in [27]):

3.2. Isomorphisms for Galois comodules. *Let* $P \in \mathbf{M}^{\mathcal{C}}$ *be a Galois comodule.*

(1) *For any* $(\mathcal{C}, A)$*-injective* $N \in \mathbf{M}^{\mathcal{C}}$*, there is a canonical isomorphism*

$$\mathrm{Hom}^{\mathcal{C}}(P,N) \to \mathrm{Hom}^{\mathcal{C}}(P, \mathrm{Hom}^{\mathcal{C}}(P,N) \otimes_S P).$$

(2) *For any* $K \in \mathbf{M}_A$*, there is a canonical isomorphism*

$$\mathrm{Hom}_A(P,K) \to \mathrm{Hom}^{\mathcal{C}}(P, \mathrm{Hom}_A(P,K) \otimes_S P).$$

(3) *There are right* $\mathcal{C}$*-comodule isomorphisms*

$$\mathrm{Hom}^{\mathcal{C}}(P,\mathcal{C}) \otimes_S P \simeq \mathcal{C} \simeq \mathrm{Hom}_A(P,A) \otimes_S P.$$

(4) *There is a* T*-linear isomorphism*

$$T \otimes_S P \to P \otimes_A \mathcal{C}, \quad t \otimes p \mapsto (t \otimes I_{\mathcal{C}})\varrho^P(p).$$

(5) *For any* $K \in \mathbf{M}_A$ *and index set* Λ,

$$\mathrm{Hom}^{\mathcal{C}}(P, (K \otimes_A \mathcal{C})^{\Lambda}) \otimes_S P \simeq \mathrm{Hom}_A(P,K)^{\Lambda} \otimes_S P \simeq K^{\Lambda} \otimes_A \mathcal{C}.$$

It is clear from the definition that $(\mathcal{C}, A)$-injective modules are of particular interest in this setting (see 2.3 in [27]):

3.3. $(\mathcal{C}, A)$**-injective modules.** *Let* P *be a Galois comodule.*

(1) *For* $N \in \mathbf{M}^{\mathcal{C}}$ *the following are equivalent:*
   (a) N *is* $(\mathcal{C}, A)$*-injective;*
   (b) $\mathrm{Hom}^{\mathcal{C}}(P, \varrho^N) : \mathrm{Hom}^{\mathcal{C}}(P,N) \to \mathrm{Hom}^{\mathcal{C}}(P, N \otimes_A \mathcal{C})$ *is a coretraction in* $\mathbf{M}_S$.

(2) *For* P *the following are equivalent:*
   (a) P *is* $(\mathcal{C}, A)$*-injective;*

(b) the inclusion $S \hookrightarrow T$ is split by a right S-linear map.

(3) For P the following are equivalent:

(a) P is strongly $(\mathcal{C}, A)$-injective;

(b) the inclusion $S \hookrightarrow T$ is split by an (S, S)-bilinear map.

(4) For P the following are equivalent:

(a) P is fully $(\mathcal{C}, A)$-injective;

(b) $\mathcal{C}$ is a coseparable A-coring.

Notice that so far we did not make any assumptions neither on the A-module nor on the S-module structure of P. Of course special properties of this type influence the behaviour of Galois comodules. For the S-module structure we get (see 4.8 in [27]):

3.4. Module properties of ${}_SP$. *Let $P \in \mathbf{M}^{\mathcal{C}}$ be a Galois comodule.*

(1) If ${}_SP$ is finitely generated, then ${}_A\mathcal{C}$ is finitely generated.

(2) If ${}_SP$ is finitely presented, then ${}_A\mathcal{C}$ is finitely presented.

(3) If ${}_SP$ is projective, then ${}_A\mathcal{C}$ is projective.

(4) If ${}_TP$ is finitely generated and ${}_SP$ is locally projective, then ${}_A\mathcal{C}$ is locally projective.

(5) If ${}_SP$ is flat, then ${}_A\mathcal{C}$ is flat and P is a generator in $\mathbf{M}^{\mathcal{C}}$.

(6) If ${}_SP$ is faithfully flat, then ${}_A\mathcal{C}$ is flat and P is a projective generator in $\mathbf{M}^{\mathcal{C}}$.

If ${}_A\mathcal{C}$ is flat as an A-module then $\mathbf{M}^{\mathcal{C}}$ is a Grothendieck category (see 18.14 in [9]) and the endomorphism ring of any semisimple right $\mathcal{C}$-comodule is a (von Neumann) regular ring. This implies part of the next proposition (see 4.11 in [27]).

3.5. Semisimple Galois comodules. *Assume ${}_A\mathcal{C}$ to be flat. For a semisimple right $\mathcal{C}$-comodule P, the following are equivalent:*

(a) P is a Galois comodule;

(b) P is a generator in $\mathbf{M}^{\mathcal{C}}$;

(c) $\mu_{\mathcal{C}} : \mathrm{Hom}^{\mathcal{C}}(P, \mathcal{C}) \otimes_S P \to \mathcal{C}$ is surjective.

In this case $\mathcal{C}$ is a right semisimple coring (and ${}_A\mathcal{C}$ is projective).

Recall that $P^* \otimes_S P$ has a coring structure provided P_A is finitely generated and projective (see 1.8). Moreover, $P^* = \mathrm{Hom}_A(P, A)$ is a left $\mathcal{C}$-comodule canonically and we have a left-right symmetry for Galois comodules (see 5.3 in [27]):

3.6. Galois comodules with P_A f.g. projective. *Assume P_A to be finitely generated and projective. Then the following are equivalent:*

(a) P *is a Galois right* $\mathcal{C}$*-comodule;*

(b) $\mathrm{Hom}^{\mathcal{C}}(P, \mathcal{C}) \otimes_R P \simeq \mathcal{C}$ *as right* $\mathcal{C}$*-comodule;*

(c) P^* *is a Galois left* $\mathcal{C}$*-comodule;*

(d) ${}^{\mathcal{C}}\mathrm{Hom}(P^*, \mathcal{C}) \otimes P^* \simeq \mathcal{C}$ *as left* $\mathcal{C}$*-comodule;*

(e) $P^* \otimes_S P \simeq \mathcal{C}$ *as* A*-corings.*

In case A is a $\mathcal{C}$-comodule, that is, there is a grouplike element $g \in \mathcal{C}$, and $S = \mathrm{End}^{\mathcal{C}}(A)$, it is a Galois (right) comodule (($\mathcal{C}, g$) is a Galois coring) if and only if the map

$$A \otimes_S A \to \mathcal{C}, \quad a \otimes a' \mapsto aga',$$

is an isomorphism. Under the given conditions, $A \otimes_S A$ has a canonical coring structure (Sweedler coring, 1.9) and the map is a coring isomorphisms (see 28.18 in [9]).

At various places we have observed a nice behaviour of strongly $(\mathcal{C}, A)$-injective comodules. For Galois comodules this property is symmetric in the following sense - an observation also proved in [5], Theorem 7.2.

3.7. Strongly $(\mathcal{C}, A)$-injective Galois comodules. *Let P be a Galois comodule with P_A finitely generated and projective. Then the following are equivalent:*

(a) P *is strongly* $(\mathcal{C}, A)$*-injective;*

(b) P^* *is strongly* $(\mathcal{C}, A)$*-injective;*

(c) the inclusion $S \hookrightarrow T$ *is split by an* (S, S)*-bilinear map.*

Proof. This follows from 3.3 and symmetry.

Finally we consider various conditions which imply that a Galois comodule induces an equivalence (see 5.7 in [27]).

3.8. Equivalences. *Let $P \in \mathbf{M}^{\mathcal{C}}$ be a Galois comodule with P_A finitely generated and projective. Then*

$$\mathrm{Hom}^{\mathcal{C}}(P, -) : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_S$$

is an equivalence with inverse functor $- \otimes_S P$ provided that

(i) P is strongly $(\mathcal{C}, A)$-injective, or

(ii) P^* is $(\mathcal{C}, A)$-injective and ${}_SP$ is flat, or

(iii) P^* is coflat and ${}_SP$ is flat, or

(iv) $\mathcal{C}$ is a coseparable coring.

3.9. Remarks. (1) Entwining structures can be considered as corings and hence the assertions in 3.3 may be compared with Lemma 4.1 and Remarks 4.2 and 5.3 in Schauenburg and Schneider [20].

(2) *Weak Galois corings* are considered in [25], 2.4. For such corings the action of A on $\mathcal{C}$ is not required to be unital.

(3) For a deeper study of weak entwining and weak coalgebra-Galois extensions the reader may consult Brzeziński, Turner and Wrightson [8].

(4) For recent investigation of the Galois theory for Hopf algebroids we refer to Böhm [1].

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