Modules whose hereditary pretorsion classes are closed under products

John E. van den Berg and Robert Wisbauer

Abstract

A module M is called *product closed* if every hereditary pretorsion class in $\sigma[M]$ is closed under products in $\sigma[M]$. Every module which is locally of finite length is product closed and every product closed module is semilocal. Let $M \in R$-Mod be product closed and projective in $\sigma[M]$. It is shown that (1) M is semiartinian; (2) if M is finitely generated then M satisfies the DCC on fully invariant submodules; (3) if M is finitely generated and every hereditary pretorsion class in $\sigma[M]$ is M-dominated, then M has finite length. If the ring R is commutative it is proven that M is product closed if and only if M is locally of finite length. An example is provided of a product closed module with zero socle.

1991 Mathematics Subject Classification: primary 16S90.

It was shown by Beachy and Blair [2, Proposition 1.4, p. 7 and Corollary 3.3, p. 25] that the following three conditions on a ring R with identity are equivalent:

(1) every hereditary pretorsion class in R-Mod is closed under arbitrary (and not just finite) direct products, or equivalently, every left topologizing filter on R is closed under arbitrary (and not just finite) intersections;

(2) every left R-module M is finitely annihilated, meaning $(0 : M) = (0 : X)$ for some finite subset X of M;

(3) R is left artinian.

In this paper we shall attempt to describe those modules M with the property that every hereditary pretorsion class in the Grothendieck category $\sigma[M]$ is closed under products in $\sigma[M]$. A main theorem demonstrates that if M is a finitely generated product closed module such that M is projective in $\sigma[M]$ and every hereditary pretorsion class in $\sigma[M]$ is M-dominated (meaning, every hereditary pretorsion class in $\sigma[M]$ is subgenerated by an M-generated module), then M has finite length. This result extends Beachy and Blair's characterization of left artinian rings. Their proof is based on two results due to Beachy [1, Proposition 1, p. 449 and Proposition 5, p. 451], but the techniques used by Beachy are not easily generalized in a manner useful for our purposes. We have thus had to develop new methods.

1 Preliminaries

The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. Throughout the paper R will denote an associative ring with identity and R-Mod the category of unital left R-modules. If $N, M \in R$-Mod we write $N \leq M$ [resp. $N \lesssim M$] if N is a submodule of M [resp. N is embeddable in M]. If X, Y are nonempty subsets of M we define $(X : Y) = \{r \in R : rY \subseteq X\}$. For subsets X, Y of R we define $(X :_l Y) = \{r \in R : rY \subseteq X\}$.

We recall some of the basic definitions and results of torsion theory. The reader is refered to [3], [4], [12] and [13] for background information on hereditary pretorsion classes.

We say $N \in R$-Mod is *subgenerated* by a nonempty class $\mathcal{C}$ in R-Mod if N is isomorphic to a submodule of a homomorphic image of a direct sum of modules in $\mathcal{C}$. We denote by $\sigma[\mathcal{C}]$ the class of all modules which are subgenerated by $\mathcal{C}$. If $\mathcal{C} = \{M\}$ is a singleton we write $\sigma[M]$ in place of $\sigma[\{M\}]$. A nonempty class of modules in R-Mod which is closed under direct sums, homomorphic images and submodules is called a *hereditary pretorsion class*; $\sigma[\mathcal{C}]$ is the smallest such class containing $\mathcal{C}$. Every hereditary pretorsion class in R-Mod is of the form $\sigma[M]$ for some $M \in R$-Mod.

Given any hereditary pretorsion class $\mathcal{T}$ in R-Mod and $N \in R$-Mod, the submodule

$$\mathcal{T}(N) := \mathrm{Tr}\,(\mathcal{T}, N) = \sum \{\mathrm{Im}\, f \mid f \in \mathrm{Hom}\,(L, N) \text{ for some } L \in \mathcal{T}\}$$

is the unique largest submodule of N belonging to $\mathcal{T}$. For each ring R the collection of all hereditary pretorsion classes in R-Mod is a complete lattice under the relation of inclusion.

If $\mathcal{T}$ and $\mathcal{T}'$ are hereditary pretorsion classes in R-Mod the *extension of* $\mathcal{T}'$ *by* $\mathcal{T}$ is defined as

$$\begin{aligned}\mathcal{T}:\mathcal{T}' = \{&N \in R\text{-Mod} \mid \text{there exists an exact sequence } 0 \to A \to N \to B \to 0,\\ &\text{where } A \in \mathcal{T} \text{ and } B \in \mathcal{T}'\}.\end{aligned}$$

It is easily verified that $\mathcal{T} : \mathcal{T}'$ is a hereditary pretorsion class containing both $\mathcal{T}$ and $\mathcal{T}'$ and $(\mathcal{T} : \mathcal{T}')(M)/\mathcal{T}(M) = \mathcal{T}'(M/\mathcal{T}(M))$ for all $M \in R$-Mod. Observe that $\mathcal{T}$ is idempotent in the sense that $\mathcal{T} : \mathcal{T} = \mathcal{T}$ precisely if $\mathcal{T}$ is closed under extensions and thus a hereditary torsion class.

The transfinite product $\mathcal{T}^\alpha$ (α an ordinal) is defined recursively as follows:

$$\begin{aligned}\mathcal{T}^1 &= \mathcal{T}\\ \mathcal{T}^{\alpha+1} &= \mathcal{T}^\alpha : \mathcal{T}\\ \mathcal{T}^\beta &= \bigvee_{\alpha<\beta} \mathcal{T}^\alpha \text{ if } \beta \text{ is a limit ordinal.}\end{aligned}$$

If α is the smallest ordinal for which $\mathcal{T}^{\alpha+1} = \mathcal{T}^\alpha$ then $\overline{\mathcal{T}} := \mathcal{T}^\alpha$ is the unique smallest hereditary torsion class containing $\mathcal{T}$ (see [4, Proposition VI.1.5, p. 137 and Corollary VI.3.4, p. 142]).

Each hereditary pretorsion class $\mathcal{T}$ in R-Mod is a Grothendieck category; coproducts, quotient objects and subobjects in $\mathcal{T}$ are the same as in R-Mod because of the defining closure properties of a hereditary pretorsion class [12, 15.1((1),(2)), p. 118]. It follows that the hereditary pretorsion classes of the category $\mathcal{T}$ are precisely the hereditary pretorsion classes of R-Mod which are contained in $\mathcal{T}$. This means that the set of hereditary pretorsion classes of $\mathcal{T}$, when viewed as a lattice, coincides with an interval in the lattice of all hereditary pretorsion classes of R-Mod. If $\{N_i \mid i \in \Gamma\}$ is a family of modules in $\mathcal{T}$ then

$$\prod_{i\in\Gamma}^{\mathcal{T}} N_i := \mathcal{T}(\textstyle\prod_{i\in\Gamma} N_i) = \mathrm{Tr}\,(\mathcal{T}, \prod_{i\in\Gamma} N_i)$$

is the product of $\{N_i \mid i \in \Gamma\}$ in $\mathcal{T}$ and if $N \in \mathcal{T}$ then

$$E^{\mathcal{T}}(N) := \mathcal{T}(E(N)) = \mathrm{Tr}\,(\mathcal{T}, E(N))$$

is the injective hull of N in $\mathcal{T}$.

2 Main results

Let $\mathcal{T}$ be a hereditary pretorsion class in R-Mod and $N \in R$-Mod. We call a submodule N' of N, *$\mathcal{T}$-dense* if $N/N' \in \mathcal{T}$. The set $\mathcal{L}(N, \mathcal{T})$ of all $\mathcal{T}$-dense submodules of N is a filter in the lattice theoretic sense on the lattice of submodules of N (see [13, 9.7, p. 60]). We shall adopt the following notation:

$$N^{\mathcal{T}} = \bigcap\{N' \leq N : N/N' \in \mathcal{T}\} = \bigcap \mathcal{L}(N, \mathcal{T}).$$

In general, $N^{\mathcal{T}}$ is not a $\mathcal{T}$-dense submodule of N.

Theorem 1 *The following assertions are equivalent for a left R-module M:*
(i) for every hereditary pretorsion class $\mathcal{T}$ in $\sigma[M]$ and $\{N_i \mid i \in \Gamma\} \subseteq \mathcal{T}$, $\prod_{i\in\Gamma}^{\sigma[M]} N_i \in \mathcal{T}$;
(ii) for every hereditary pretorsion class $\mathcal{T}$ in $\sigma[M]$ and $N \in \sigma[M]$ the set of $\mathcal{T}$-dense submodules of N is closed under arbitrary intersections, or equivalently, $N^{\mathcal{T}}$ is a $\mathcal{T}$-dense submodule of N, i.e., $N^{\mathcal{T}} \in \mathcal{L}(N, \mathcal{T})$;
(iii) for every hereditary pretorsion class $\mathcal{T}$ in $\sigma[M]$ and finitely generated $N \in \sigma[M]$ the set of $\mathcal{T}$-dense submodules of N is closed under arbitrary intersections.

Proof. (i)⇒(ii) $N/N^{\mathcal{T}} \lesssim \prod_{N'\in\mathcal{L}(N,\mathcal{T})} N/N'$. Since $N \in \sigma[M]$, $N/N^{\mathcal{T}} \in \sigma[M]$, so $N/N^{\mathcal{T}} \subseteq \mathrm{Tr}\,(\sigma[M], \prod_{N'\in\mathcal{L}(N,\mathcal{T})} N/N') = \prod_{N'\in\mathcal{L}(N,\mathcal{T})}^{\sigma[M]} N/N'$. Inasmuch as $N/N' \in \mathcal{T}$ for all $N' \in \mathcal{L}(N, \mathcal{T})$, we must have $N/N^{\mathcal{T}} \in \mathcal{T}$, so $N^{\mathcal{T}} \in \mathcal{L}(N, \mathcal{T})$.

(ii)⇒(iii) is obvious.

(iii)⇒(i) Let $\mathcal{T}$ be a hereditary pretorsion class in $\sigma[M]$ and $\{N_i \mid i \in \Gamma\} \subseteq \mathcal{T}$. Take $x = \{x_i\}_{i\in\Gamma} \in \prod_{i\in\Gamma}^{\sigma[M]} N_i = \mathrm{Tr}\,(\sigma[M], \prod_{i\in\Gamma} N_i)$. Put $N = {}_RR/(0 : x) \cong Rx$ and $L_i = (0 : x_i)/(0 : x)$ for each $i \in \Gamma$. Note that N is finitely generated. Inasmuch as $N/L_i \cong Rx_i \leq N_i \in \mathcal{T}$, L_i is a $\mathcal{T}$-dense submodule of N for all $i \in \Gamma$. Since $\bigcap_{i\in\Gamma}(0 : x_i) = (0 : x)$, $\bigcap_{i\in\Gamma} L_i = 0$, so by (iii), $N \in \mathcal{T}$. We conclude that $\prod_{i\in\Gamma}^{\sigma[M]} N_i \in \mathcal{T}$, as required. □

We shall call $M \in R$-Mod *product closed* if it satisfies the equivalent assertions in Theorem 1.

Remark 2 *Observe that if $M \in R$-*Mod *is product closed then so is every module in $\sigma[M]$.*

Recall that $M \in R$-Mod is said to be *locally artinian* [resp. *locally of finite length*] if every finitely generated submodule of M is artinian [resp. has finite length].

Proposition 3 *Every locally artinian left R-module is product closed.*

Proof. Suppose $M \in R$-Mod is locally artinian. Let $\mathcal{T}$ be a hereditary pretorsion class in $\sigma[M]$ and $N \in \sigma[M]$ with N finitely generated. Since every module in $\sigma[M]$ is locally artinian, N must be artinian, so every nonempty set of submodules of N has a minimal element. Assertion (iii) of Theorem 1 thus holds. $\square$

Remark 4 *(i) The converse to Proposition 3 is not valid as shown in Example 11. However, we shall prove in Theorem 16 that if M is a finitely generated product closed module such that M is projective in $\sigma[M]$ and satisfies a 'weak generator' type property, then M has finite length.*
(ii) Every semisimple left R-module is locally artinian and therefore product closed by Proposition 3.
(iii) Every torsion abelian group is a locally artinian $\mathbb{Z}$-module and therefore product closed.

We now establish some general properties of product closed modules.

Proposition 5 *If a left R-module M is product closed then every cogenerator for $\sigma[M]$ is a subgenerator for $\sigma[M]$.*

Proof. Let C be a cogenerator for $\sigma[M]$. If N is an arbitrary object in $\sigma[M]$ then $N \lesssim \prod_{\Gamma}^{\sigma[M]} C$ for some index set Γ. Since M is product closed we have by Theorem 1(i) that $\prod_{\Gamma}^{\sigma[M]} C \in \sigma[C]$. We conclude that $N \in \sigma[C]$, so C is a subgenerator for $\sigma[M]$. $\square$

We shall denote by $\mathcal{SOC}$ the hereditary pretorsion class consisting of all semisimple left R-modules. More generally, if $M \in R$-Mod we shall denote by $\mathcal{SOC}_M$ the hereditary pretorsion class of all semisimple modules in $\sigma[M]$. Observe that if $N \in \sigma[M]$ then $N^{\mathcal{SOC}_M}$ equals $J(N)$ the intersection of all maximal proper submodules of N.

Theorem 6 *Every product closed left R-module M is semilocal, that is to say, $M/J(M)$ is semisimple.*

Proof. By Theorem 1(ii), $M/M^{\mathcal{SOC}_M} \in \mathcal{SOC}_M$. But, as noted above, $M^{\mathcal{SOC}_M} = J(M)$, so $M/J(M)$ is semisimple. $\square$

A module $N \in \sigma[M]$ is called *M-singular* if $N \cong L/K$ for some $L \in \sigma[M]$ and essential submodule K of L. The class of all M-singular left R-modules is a hereditary pretorsion class in $\sigma[M]$ which we shall denote by $\mathcal{S}_M$ (see [12, 17.3, p. 138 and 17.4, p. 139]). We call M *polyform* if $\mathcal{S}_M(M) = 0$, i.e., M is $\mathcal{S}_M$-torsion-free. If $N \in \sigma[M]$ it is clear that every essential submodule of N is $\mathcal{S}_M$-dense in N, i.e., $\{N' : N' \text{ is an essential submodule of } N\} \subseteq \mathcal{L}(N, \mathcal{S}_M)$ so $\mathcal{SOC}_M(N) = \bigcap\{N' : N' \text{ is an essential submodule of } N\} \supseteq \bigcap \mathcal{L}(N, \mathcal{S}_M) = N^{\mathcal{S}_M}$.

Proposition 7 *Every polyform product closed left R-module has essential socle.*

Proof. Suppose $M \in R$-Mod is polyform and product closed. Since M is by definition $\mathcal{S}_M$-torsion-free, every $\mathcal{S}_M$-dense submodule of M is essential in M. It follows that $\mathcal{SOC}_M(M) = \bigcap \mathcal{L}(M, \mathcal{S}_M) = M^{\mathcal{S}_M}$. Since M is product closed, $M^{\mathcal{S}_M}$ is $\mathcal{S}_M$-dense and hence essential in M. $\square$

Recall that $M \in R$-Mod is said to be *semiartinian* if $M \in \mathcal{SOC}^{\alpha}$ for some ordinal α, or equivalently, if every nonzero factor module of M has nonzero socle (see [12, 32.6, p. 270]).

Our next objective is to prove that if M is product closed and has the property that M is projective in $\sigma[M]$ then M is semiartinian.

Lemma 8 *The following assertions are equivalent for a left R-module M:*
(i) M is semiartinian;
(ii) M/U has nonzero socle for all proper fully invariant submodules U of M.

Proof. (i)⇒(ii) is obvious.

(ii)⇒(i) Let α be the smallest ordinal for which $\mathcal{SOC}^{\alpha+1}(M) = \mathcal{SOC}^{\alpha}(M)$ (this ordinal is the so-called *Loewy length* of M). Observe that $U = \mathcal{SOC}^{\alpha}(M)$ is a fully invariant submodule of M. Inasmuch as $\mathcal{SOC}(M/U) = \mathcal{SOC}^{\alpha+1}(M)/\mathcal{SOC}^{\alpha}(M) = 0$, it follows from (ii) that $M/U = 0$, whence $M = \mathcal{SOC}^{\alpha}(M)$ and M is semiartinian. $\square$

Lemma 9 *Suppose M is a left R-module which is projective in $\sigma[M]$ and U is any nonzero fully invariant submodule of M. Then:*
(i) M/U is projective in $\sigma[M/U]$;
(ii) $\sigma[M/U] \neq \sigma[M]$.

Proof. (i) follows easily from the fact that if $A \in \sigma[M/U]$ and $f \in \mathrm{Hom}(M, A)$ then f factors through M/U.

(ii) is proved in [11, Lemma 2.8, p. 3623]. □

Theorem 10 *Let M be a product closed left R-module. If M is projective in $\sigma[M]$ then M is semiartinian.*

Proof. A cogenerator for $\sigma[M]$ is given by $C = \bigoplus_{i \in \Gamma} E^{\sigma[M]}(S_i)$ where $\{S_i \mid i \in \Gamma\}$ is a representative set of simple modules in $\sigma[M]$. It follows from Proposition 5 that C is a subgenerator for $\sigma[M]$. Since M is projective in $\sigma[M]$, we must have $M \lesssim \bigoplus_\Lambda C$ for some index set Λ. If $M = 0$ there is nothing to prove. If $M \neq 0$ then $\mathcal{SOC}(M) \neq 0$ because $\bigoplus_\Lambda C$ has essential socle. Now let U be any proper fully invariant submodule of M. By Lemma 9(i), M/U is projective in $\sigma[M/U]$. Inasmuch as $M/U \in \sigma[M]$, M/U is also product closed. The above argument, applied to M/U in place of M, shows that M/U has nonzero socle. We conclude from Lemma 8 that M is semiartinian. □

Example 11 *It is known [10, Lemma 6, p. 24] that if R is an arbitrary left chain ring then every hereditary pretorsion class $\mathcal{T}$ in R-*Mod *has one of two forms:*

$$\begin{aligned} \mathcal{T} &= \{N \in R\text{-Mod} \mid IN = 0\};\ \textit{or} \\ \mathcal{T} &= \{N \in R\text{-Mod} \mid (0 : x) \supset I \textit{ for all } x \in N\} \end{aligned}$$

*for some ideal I of R. The lattice of hereditary pretorsion classes in R-*Mod *thus constitutes a chain. Furthermore, if R is a domain and every ideal of R is idempotent, then every hereditary pretorsion class in R-*Mod *is, in fact, a hereditary torsion class [6, Theorem 28, p. 5539].*

*Now suppose that R is a left chain domain whose only proper nonzero ideal is the Jacobson radical $J(R)$. (The existence of such rings is established in [9, Proposition 16, p. 1112] and [8, Theorem 9, p. 104].) It follows that there are exactly two nontrivial proper hereditary pretorsion classes in R-*Mod*:*

$$\begin{aligned} \mathcal{T}_1 &= \{N \in R\text{-Mod} \mid J(R)N = 0\},\ \textit{and} \\ \mathcal{T}_2 &= \{N \in R\text{-Mod} \mid (0 : x) \neq 0 \textit{ for all } x \in N\}. \end{aligned}$$

Observe that $\mathcal{T}_1$ *consists of all the semisimple modules in* R-Mod*, i.e.,* $\mathcal{T}_1(M) = \mathcal{SOC}(M)$ *for all* $M \in R$-Mod*. Note also that* $\mathcal{T}_1$ *is closed under arbitrary direct products in* R-Mod *because it consists precisely of all those left* R*-modules which are annihilated by the ideal* $J(R)$*. Observe that* $\mathcal{T}_2$ *consists of all modules in* R-Mod *which are not cofaithful. (Recall that* $N \in R$-Mod *is said to be* cofaithful *if* $(0 : X) = 0$ *for some finite subset* X *of* N*; this is equivalent to* N *being a subgenerator for* R-Mod*.)*

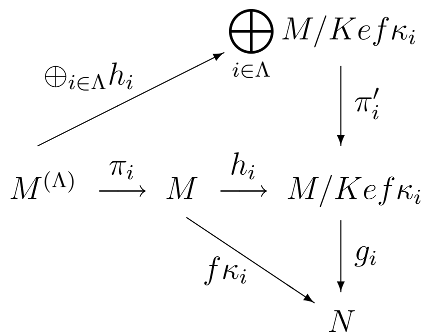
Take $N \in \mathcal{T}_2 \backslash \mathcal{T}_1$ *and put* $M = N/\mathcal{T}_1(N)$*. Since* $\mathcal{T}_1$ *is a hereditary* torsion *class and* $N \notin \mathcal{T}_1$*,* M *is a nonzero module with* $\mathcal{SOC}(M) = 0$*. Clearly,* $\mathcal{T}_2 = \sigma[M]$*. Since* $\mathcal{T}_1$ *is the only nontrivial hereditary pretorsion class contained in* $\sigma[M]$ *and* $\mathcal{T}_1$ *is closed under arbitrary direct products, assertion (i) of Theorem 1 is clearly satisfied. We conclude that* M *is product closed. Observe that* M *cannot be semiartinian for* $\mathcal{SOC}(M) = 0$*.*

Let $M \in R$-Mod. A hereditary pretorsion class $\mathcal{T}$ in $\sigma[M]$ is said to be *M-dominated* if $\mathcal{T}$ has an M-generated subgenerator. The set of all M-dominated hereditary pretorsion classes in $\sigma[M]$ is closed under arbitrary joins. This is a consequence of the join operation in the lattice of all hereditary pretorsion classes: if $\{\mathcal{T}_i : i \in \Gamma\}$ is a family of hereditary pretorsion classes in R-Mod and each $\mathcal{T}_i = \sigma[M_i]$ with $M_i \in R$-Mod, then $\bigvee_{i\in\Gamma} \mathcal{T}_i = \sigma[\bigoplus_{i\in\Gamma} M_i]$. Observe that if M is a generator for $\sigma[M]$ then every hereditary pretorsion class in $\sigma[M]$ is M-dominated.

The following result shows that an M-dominated hereditary pretorsion class $\mathcal{T}$ in $\sigma[M]$ is determined by the set of all $\mathcal{T}$-dense submodules of M.

Proposition 12 *Let* M *be a left* R*-module. If* $\mathcal{T}$ *is an* M*-dominated hereditary pretorsion class in* $\sigma[M]$ *then* $\mathcal{T}$ *is subgenerated by the class of all* $\mathcal{T}$*-torsion factor modules of* M*.*

Proof. Let N be an M-generated subgenerator for $\mathcal{T}$. There exists an epimorphism $f : M^{(\Lambda)} \to N$. For each $i \in \Lambda$ let $\pi_i : M^{(\Lambda)} \to M$ and $\kappa_i : M \to M^{(\Lambda)}$ denote the canonical projection and embedding. Take $i \in \Lambda$. Factor $f\kappa_i$ through $M/Kef\kappa_i$ as $f\kappa_i = g_i h_i$ for suitable homomorphisms $h_i : M \to M/Kef\kappa_i$ and $g_i : M/Kef\kappa_i \to N$. Observe that $Kef\kappa_i$ is a $\mathcal{T}$-dense submodule of M. Let $\pi'_i : \bigoplus_{i\in\Lambda} M/Kef\kappa_i \to M/Kef\kappa_i$ denote the canonical projection. Consider the following commutative diagram:



Note that $f = \sum_{i\in\Lambda} f\kappa_i\pi_i = \sum_{i\in\Lambda} g_i h_i \pi_i$. Since $h_i\pi_i = \pi'_i(\oplus_{i\in\Lambda}h_i)$ for all $i \in \Lambda$, it follows that $f = \sum_{i\in\Lambda} g_i h_i \pi_i = \sum_{i\in\Lambda} g_i\pi'_i(\oplus_{i\in\Lambda}h_i) = \left(\sum_{i\in\Lambda} g_i\pi'_i\right)(\oplus_{i\in\Lambda}h_i)$. Thus f factors through $\bigoplus_{i\in\Lambda} M/Kef\kappa_i$. We conclude that N is generated by $\{M/Kef\kappa_i : i \in \Lambda\}$, whence $\mathcal{T} = \sigma[\bigoplus_{i\in\Lambda} M/Kef\kappa_i]$. □

If $M \in R$-Mod and $\mathcal{T}$ is an arbitrary hereditary pretorsion class in $\sigma[M]$ then clearly $\sigma[M/M^{\mathcal{T}}] \supseteq \sigma[\{M/N : N \in \mathcal{L}(M,\mathcal{T})\}]$. The previous proposition tells us that the right hand side of this containment coincides with $\mathcal{T}$ in the case where $\mathcal{T}$ is M-dominated. If M is product closed then $\mathcal{T} \supseteq \sigma[M/M^{\mathcal{T}}]$. The next result follows immediately.

Corollary 13 *Let M be a product closed left R-module. If $\mathcal{T}$ is an M-dominated hereditary pretorsion class in $\sigma[M]$ then $\mathcal{T} = \sigma[M/M^{\mathcal{T}}]$.*

Recall that an element c of a complete upper semilattice L is said to be *compact* if $c \leq \bigvee X$ implies $c \leq \bigvee Y$ for some finite subset Y of X, whenever $X \subseteq L$. If L is chosen to be the complete lattice of all hereditary pretorsion classes of R-Mod, then the compact elements of L are precisely those hereditary pretorsion classes which possess a finitely generated subgenerator (see [3, Proposition 2.16, p. 21]). We shall speak of a hereditary pretorsion class as compact if it is a compact element in the lattice of all hereditary pretorsion classes.

Proposition 14 *Let M be a product closed left R-module. If M is finitely generated then all M-dominated hereditary pretorsion classes in $\sigma[M]$ are compact. Consequently, there is no strictly ascending chain of M-dominated hereditary pretorsion classes in $\sigma[M]$.*

Proof. Let $\mathcal{T}$ be an M-dominated hereditary pretorsion class in $\sigma[M]$. By Corollary 13, $\mathcal{T} = \sigma[M/M^{\mathcal{T}}]$. Since $M/M^{\mathcal{T}}$ is finitely generated, $\mathcal{T}$ is compact.

The second assertion of the proposition is the consequence of a routine and purely lattice theoretic argument: a complete upper semilattice satisfies the ACC if and only if every element in the upper semilattice is compact. □

Proposition 15 *Let M be a finitely generated product closed left R-module with the property that M is projective in $\sigma[M]$. Then M satisfies the DCC on fully invariant submodules.*

Proof. Suppose U_1 and U_2 are fully invariant submodules of M with $U_1 \supset U_2$. We claim that $\sigma[M/U_1] \subset \sigma[M/U_2]$. To see this note first that M/U_2 is projective in $\sigma[M/U_2]$ by Lemma 9(i). Using the fact that M is projective in $\sigma[M]$ and U_1 is fully invariant, it is easily shown that U_1/U_2 is a fully invariant submodule of M/U_2. We conclude from Lemma 9(ii), that $\sigma[M/U_1] \neq \sigma[M/U_2]$, as claimed.

Now suppose, contrary to the proposition, that $U_1 \supset U_2 \supset U_3 \supset \ldots$ is a strictly descending chain of fully invariant submodules of M. The above argument shows that this induces a strictly ascending chain $\sigma[M/U_1] \subset \sigma[M/U_2] \subset \sigma[M/U_3] \subset \ldots$ of hereditary pretorsion classes in $\sigma[M]$. But each $\sigma[M/U_i]$ is M-dominated and this contradicts Proposition 14. □

The following result is a partial converse to Proposition 3.

Theorem 16 *Let M be a finitely generated product closed left R-module with the property that M is projective in $\sigma[M]$ and every hereditary pretorsion class in $\sigma[M]$ is M-dominated. Then M has finite length.*

Proof. Let $\mathcal{L}$ be the class of all modules in $\sigma[M]$ which are locally of finite length. It is easily shown that $\mathcal{L}$ is a hereditary *torsion* class in $\sigma[M]$. Consider $M^{\mathcal{L}} \leq M$. Note that M and hence $M^{\mathcal{L}}$ is semiartinian by Theorem 10. It follows from the hypothesis and Proposition 14 that the lattice of all hereditary pretorsion classes in $\sigma[M]$ satisfies the ACC. It follows that if $\mathcal{T}$ is an arbitrary hereditary pretorsion class in $\sigma[M]$ then $\overline{\mathcal{T}} = \mathcal{T}^\alpha$ for some *finite* ordinal α. In particular then, $M^{\mathcal{L}} = \mathcal{SOC}^n(M^{\mathcal{L}})$ for some $n \in \mathbb{N}$. Suppose $M^{\mathcal{L}} \neq 0$. Then $M^{\mathcal{L}}$ has a maximal proper submodule L, say. Since $M^{\mathcal{L}}/L$ is simple, $M^{\mathcal{L}}/L \in \mathcal{L}$. Since $M/M^{\mathcal{L}}, M^{\mathcal{L}}/L \in \mathcal{L}$ and $\mathcal{L}$ is closed under extensions, we must have $M/L \in \mathcal{L}$, so $L \supseteq M^{\mathcal{L}}$, a contradiction. We conclude that $M^{\mathcal{L}} = 0$, i.e., $M \in \mathcal{L}$. Since M is finitely generated it must have finite length. □

Remark 17 *This identifies a possibly serious shortcoming in the previous theorem.*

The previous results show that if M is a finitely generated product closed module which is projective in $\sigma[M]$, then M enjoys the following properties:
(1) M is semilocal (Theorem 6);
(2) M is semiartinian (Theorem 10);
(3) all M-dominated hereditary pretorsion classes in $\sigma[M]$ are compact (Proposition 14);
(4) M satisfies the DCC on fully invariant submodules (Proposition 15).

It is conceivable that the above properties might be enough to force the module M to have finite length, but I don't see a proof. If such a proof can be found then the requirement in Theorem 16 that 'every hereditary pretorsion class in $\sigma[M]$ is M-dominated' can be dispensed with and a more satisfying result obtained. The aforementioned requirement seems to be strong and looks rather artificial, it's a disappointing feature of Theorem 16. Of course it might be that the requirement is necessary, but then we need to produce an example of a finitely generated product closed module which is projective in $\sigma[M]$ but which is not of finite length. Finding such a module looks like a difficult task.

If, in Theorem 16, the module M is chosen to be ${}_RR$, we obtain Beachy and Blair's result [2, Proposition 1.4, p. 7 and Corollary 3.3, p. 25]:

Corollary 18 *The following assertions are equivalent for a ring R:*
(i) ${}_RR$ is product closed, i.e., every hereditary pretorsion class in R-Mod *is closed under direct products;*
(ii) R is left artinian.

Proof. (ii)⇒(i) follows from Proposition 3.

(i)⇒(ii) The product closed module $M = {}_RR$ is a progenerator for R-Mod and therefore satisfies the conditions of Theorem 16. □

Theorem 19 *Let R be a commutative ring. The following assertions are equivalent for a left R-module M:*
(i) M is product closed;
(ii) M is locally artinian.

Proof. (ii)⇒(i) follows from Proposition 3.

(i)⇒(ii) It clearly suffices to show that every cyclic submodule of M is artinian. Let $N \leq M$ be cyclic. Then $N \cong {}_R(R/I)$ for some ideal I of R. Note that $\sigma[N]$ corresponds with the module category R/I-Mod and N is a progenerator for $\sigma[N]$. Consequently, N must satisfy the conditions of Theorem 16. We conclude that N is artinian. □

References

[1] J.A. Beachy, *On quasi-artinian rings*, J. London Math. Soc. (2)**3** (1971), 449-452.

[2] J.A. Beachy and W.D. Blair, *Finitely annihilated modules and orders in artinian rings*, Comm. Algebra **6**(1) (1978), 1-34.

[3] J.S. Golan, *Linear Topologies on a Ring: An Overview*, Pitman Research Notes in Mathematics Series, No. 159. Longman Scientific and Technical, Harlow (1987).

[4] B. Stenström, *Rings of Quotients*, Grundlehren mathematischen Wissenschaften, Series No. 237. Springer-Verlag, New York, Heidelberg, Berlin (1975).

[5] J.E. van den Berg, *When multiplication of topologizing filters is commutative*, J. Pure and Applied Algebra **140** (1999), 87-105.

[6] J.E. van den Berg, *When every torsion preradical is a torsion radical*, Comm. Algebra **27**(11), 5527-5547 (1999).

[7] J.E. van den Berg, *Primeness described in the language of torsion preradicals*, Semigroup Forum, to appear.

[8] J.E. van den Berg and J.G. Raftery, *Every algebraic chain is the congruence lattice of a ring*, J. Algebra **162**(1) (1993), 95-106.

[9] J.E. van den Berg and J.G. Raftery, *On rings (and chain domains) with restricted completeness conditions on topologizing filters*, Comm. Algebra **22**(4) (1994), 1103-1113.

[10] A.M.D. Viola-Prioli and J.E. Viola-Prioli, *Rings whose kernel functors are linearly ordered*, Pacific J. Math. **132**(1), 21-34 (1988).

[11] A.M.D. Viola-Prioli, J.E. Viola-Prioli and R. Wisbauer, *Module categories with linearly ordered closed subcategories*, Comm. Algebra **22** (1994), 3613-3627.

[12] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading (1991).

[13] R. Wisbauer, *Modules and Algebras: Bimodule Structure and Group Actions on Algebras*, Pitman Monographs 81, Longman (1996).

John van den Berg, School of Mathematics, Statistics and Information Technology, University of Natal Pietermaritzburg, Private Bag X01, Scottsville 3209, South Africa.
vandenberg@nu.ac.za
Robert Wisbauer, Mathematics Institute, University of Düsseldorf, D-40225 Düsseldorf, Germany.
wisbauer@math.uni-duesseldorf.de