SEPARABILITY IN ALGEBRA AND CATEGORY THEORY

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Abstract. Separable field extensions are essentially known since the 19th century and their formal definition was given by Ernst Steinitz in 1910. In this survey we first recall this notion and equivalent characterisations. Then we outline how these were extended to more general structures, leading to separable algebras (over rings), Frobenius algebras, (non associative) Azumaya algebras, coalgebras, Hopf algebras, and eventually to separable functors. The purpose of the talk is to demonstrate that the development of new notions and definitions can lead to simpler formulations and to a deeper understanding of the original concepts. The formalism also applies to algebras and coalgebras over semirings and S-acts (transition systems).

Contents

Preliminaries

The investigation of the interplay between (roots of) polynomials over the rationals and (automorphisms of) extensions of the rationals was started by Joseph-Louis Lagrange (1736 -1813). Through the work of Évariste Galois (1811-1832) it became evident that this was the key to an interesting deep relationship between finite extensions of the rationals and the theory of finite groups. The first presentation of Galois' ideas in a textbook was given 1866 by Joseph Alfred Serret (1819-1885). At that time, the abstract notion of a *field* was not yet available in algebra. The (German) name *Körper* (Engl. *body*) was coined by Richard Dedekind in 1871, meaning substructures of the complex numbers. His intention of choosing this name was to signify – as in the natural sciences, geometry, and human society – a system with certain completeness, seclusion, and perfection. He provided the fundamentals for *linear algebra* over a *Körper* like linear dependence, basis, dimensions as well as trace and norm of finite extensions.

In 1893, Heinrich M. Weber [78] defined the abstract notion of a *Körper* (as a set allowing for addition and multiplication subject to certain conditions) to give the

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right frame to Galois theory. Eliakim H. Moore used in [59] the English word *field* as synonym for *endliche Körper* in the sense of Weber. Nowadays *field* and *Körper* are synonyms without finiteness restrictions.

The abstract *theory of fields* was initiated by Ernst Steinitz in 1910 [72]. Systematically developing the axioms of (commutative) fields he introduced the notions of *prime field*, *algebraic closure*, and *field extensions of first (resp. second) kind* which later on were called *separable* (resp. *inseparable*) extensions in van der Waerden's textbook from 1930.

New perspectives in the study of field extensions were obtained by applying *tensor products*. These arose (under different labels) in the late 19th century in physics and mathematics for vector spaces over the real or complex numbers (e.g. Gibbs [5]). The crucial step in extending tensor products to *abelian groups* was made by Hassler Whitney in [79], 1938. From this, tensor products of modules over rings (and fields) are obtained by suitable coequalisers.

In this survey, we shall first recall the properties of separable field extensions, and then generalisations deduced from them. Recall that a field extension $L:K$ is called *separable* provided every element of L is a root of an irreducible polynomial with coefficients in K which does not have multiple roots.

For any normal extension $L:K$, the automorphism group $G := \mathrm{Aut}(L:K)$ acts on L thus making L a module over the group algebra $K[G]$ and a comodule over the dual coalgebra $K[G]^*$. Then $L:K$ is separable provided $L \simeq K[G]$ as $K[G]$-module.

For any field extension $L:K$, L is a vector space over K and any L-vector space Y is given by a K-linear map $\varrho : L \otimes_K Y \to Y$. Furthermore, for any K-vector space X, $L \otimes_K X$ is an L-vector space by the action

$$m \otimes X : L \otimes_K L \otimes_K X \to L \otimes_K X, \quad a \otimes b \otimes x \mapsto ab \otimes x,$$

where $m : L \otimes_K L \to L$ denotes the multiplication in L. Denoting by $\mathbb{V}_k$ the category of vector spaces over any field k, this gives rise to the functors

$$\begin{aligned} L \otimes_K - : &\quad \mathbb{V}_K \to \mathbb{V}_K, \quad X \mapsto L \otimes_K X, \\ \phi_L : &\quad \mathbb{V}_K \to \mathbb{V}_L, \quad X \mapsto (L \otimes_K X, m \otimes X), \\ U_L : &\quad \mathbb{V}_L \to \mathbb{V}_K, \quad (Y, L \otimes_K Y \to Y) \mapsto Y, \end{aligned}$$

and ϕ_L (the *free functor*) is left adjoint to U_L (the *forgetful functor*). Now, separability of $L:K$ is equivalent to require that, for any $V, N \in \mathbb{V}_L$, the canonical map

$$\mathrm{Hom}_L(V, N) \to \mathrm{Hom}_K(U_L(V), U_L(N))$$

is a (naturally) splitting monomorphism.

The setting just described can readily be transferred to general categories, replacing the functor $L \otimes_K -$ by a monad on any category $\mathbb{A}$ and defining separable and Frobenius functors (and (co)monads). This is done in Section 2 and, for the convenience of the reader, basic notions from category theory are recalled in Appendix 7. In the ensuing sections, the categorical notions are applied to generalise field extensions by replacing

(i) the field L by a (non-) associative algebra A (Section 3);
(ii) the base field K by a (non-) commutative ring R (Subsection 3.11);
(iii) the field L by a coalgebra C (Section 4);
(iv) the base field K by a semiring R (Section 5);
(v) the base field K by a set A (Section 6);
(vi) the bialgebra $K[G]$ by any Hopf algebra (Subsection 4.7).

In the theory of separability for algebras, *derivations* also play a mayor role (e.g. in [22, 46]). However, this aspect is not addressed in the present survey.

1. Separable and normal fields

A *field* is a triple $K = (K, +, \cdot\,)$ where K is a set, $(K, +)$ is an abelian group with neutral element 0,

$$\cdot : K \times K \to K \tag{1.1}$$

is a bilinear map making $(K \setminus \{0\}, \cdot\,)$ an abelian group with unit 1, and the relation between $+$ and $\cdot$ is given by distributivity.

$K[X]$ will denote the ring of polynomials with coefficients in K.

A *morphism* between two fields K, K' is a map $K \to K'$ respecting addition, multiplication, the neutral element, and the unit.

As prototypes of fields we have the rationals $\mathbb{Q}$ and the factor rings $\mathbb{Z}/p\mathbb{Z}$ of the integers $\mathbb{Z}$, for any prime number p. There are no field morphisms between these two types.

1.1. **Field extensions.** A subset K of a field L is called a subfield, if K contains 1 and is closed under subtraction and division in L. This setting is also called a *field extension* and denoted by $L\!:\!K$. By an *intermediate field* M of $L\!:\!K$ we mean a subfield of L containing K as a subfield. For a subset $S \subset L$, the smallest intermediate field of $L\!:\!K$ containing S is denoted by $K(S)$, and for $a \in L$ we write $K(\{a\}) = K(a)$.

$L : K$ is called a *finite extension*, if $\dim_K L \leq \infty$ and this dimension is the *degree* $[L\!:\!K]$ of the extension.

An element $a \in L$ is called *algebraic over* K, if there is a non-vanishing polynomial $f \in K[X]$ with $f(a) = 0$. The monic polynomial of smallest degree with this property is called the *minimal polynomial* of a over K, denoted by $\mathrm{Min}(a : K)$; its degree is $[K(a)\!:\!K]$.

The extension $L\!:\!K$ is called *algebraic*, if all $a \in L$ are algebraic over K. The field K is called *algebraically closed*, if K has no proper algebraic extension.

1.2. **K-morphisms.** If $L\!:\!K$ and $Q\!:\!K$ are field extensions, and $\varphi : L \to Q$ is a field homomorphism such that $\varphi(k) = k$ for all $k \in K$, then φ is called a *K-morphism* (also *K-isomorphism*, e.g. in [13]).

A theorem of Dedekind (from 1871, [47]) says that a family of pairwise distinct K-morphisms $L \to Q$ are linearly independent over Q (as elements of $\mathrm{Hom}_K(L, Q)$).

Evidently, the bijective K-morphisms $L \to L$ form a group, the *automorphism group* of $L\!:\!K$,

$$\mathrm{Aut}(L\!:\!K) = \{\varphi : L \to L \,|\, \varphi \text{ is a } K\text{-isomorphism}\},$$

which acts on L. If $L\!:\!K$ is algebraic, its orbits on L are finite (consisting of roots of some irreducible polynomial).

For any subset (subgroup) $H \subseteq \mathrm{Aut}(L\!:\!K)$, the invariant elements

$$\mathrm{Fix}(L\!:\!H) := \{a \in L \,|\, h(a) = a \text{ for any } h \in H\}$$

form an intermediate field of $L\!:\!K$.

The main concern of studying field extensions is to find roots for polynomials and the next two results give a basic answer to this problem.

1.3. **Splitting fields.** *Let K be a field.*

(i) *(Kronecker 1887): For any polynomial $f \in K[X]$, there is a smallest extension $L\!:\!K$ such that f splits completely in L into a product of linear polynomials and a constant.*

L is generated as a field by the roots of f and is unique up to K-isomorphisms. It is called the splitting field of f over K.

(ii) *(Steinitz 1910): For any set of polynomials $S \subseteq K[X]$, there is a smallest extension $L : K$ such that any $f \in S$ splits completely in L into a product of linear polynomials and a constant.*

For $S = K[X]$, this gives an algebraic extension $\widehat{K}$ of K which is algebraically closed.

As easily seen, the number of roots of an irreducible $f \in K[X]$ may be smaller then the dimension of the splitting field $L\!:\!K$. These numbers are equal for the

1.4. **Separable polynomials.** An irreducible polynomial $f \in K[X]$ is called *separable*, if it has no double roots in an algebraic closure $\widehat{K}$ of K. An algebraic field extension $L\!:\!K$ is said to be *separable* provided, for every $a \in L$, the minimal polynomial $\mathrm{Min}(a\!:\!K)$ is separable.

For an irreducible $f \in K[X]$, the following are equivalent:

(a) *f is separable;*

(b) *for any field extension $L\!:\!K$ and $a \in L$, $(X - a)^2$ does not divide f in $L[X]$;*

(c) *there is an extension $L\!:\!K$ such that f has $\deg(f)$ roots in L;*

(d) *the number of distinct roots of f is equal to the degree of f.*

1.5. **Normal extensions.** An algebraic field extension $L\!:\!K$ is called *normal* if, for every $a \in L$, the minimal polynomial $\mathrm{Min}(a\!:\!K)$ splits completely into linear factors over L. In this case, $L\!:\!M$ is normal for each intermediate field M of $L\!:\!K$.

Let K be a field with algebraic closure $\widehat{K}$ and L an intermediate field of $\widehat{K}\!:\!K$. Then the following are equivalent:

(a) *$L\!:\!K$ is normal;*

(b) *L is the splitting field of a set of polynomials in $K[X]$;*

(c) *for any $\varphi \in \mathrm{Aut}(\widehat{K}\!:\!K)$, $\varphi(L) \subseteq L$.*

Combining the preceding notions, we get extensions of particular interest:

1.6. **Galois extensions.** A field extension $L : K$ is said to be *Galois* provided it is separable and normal; $\mathrm{Aut}(L\!:\!K)$ is called its *Galois group*.

For a finite field extension $L\!:\!K$, the following are equivalent:

(a) *$L\!:\!K$ is Galois;*

(b) *L is the splitting field of a separable polynomial in $K[X]$;*

(c) *$[L\!:\!K] = |\mathrm{Aut}(L\!:\!K)|$;*

(d) *$\mathrm{Fix}(L\!:\!\mathrm{Aut}(L\!:\!K)) = K$.*

In view of the observations made above it is not difficult now to prove our next result. It shows the close connection between the structures of groups and field extensions. For example, it implies that polynomials of degree 5 need not be *solvable by radicals* since the corresponding Galois groups need not be *solvable*, that is, need not have a subnormal series with abelian factor groups.

1.7. **Galois correspondence.** *For a finite Galois extension $L\!:\!K$, there is an (order reversing) bijection between the sets (lattices)*

$$\mathbb{F} := \{\textit{intermediate fields of } L\!:\!K\},$$
$$\mathbb{G} := \{\textit{subgroups of } \mathrm{Aut}(L\!:\!K)\},$$

given by

$$\mathrm{Aut}(L\!:\!-) : \mathbb{F} \to \mathbb{G}, \quad M \mapsto \mathrm{Aut}(L\!:\!M),$$
$$\mathrm{Fix}(L:-) : \mathbb{G} \to \mathbb{F}, \quad H \mapsto \mathrm{Fix}(L, H).$$

Notice that a Galois extension $L\!:\!K$ need not be finite dimensional. In case it is not, there is a bijective correspondence between $\mathbb{F}$, the intermediate fields of $L:K$, and those subgroups of $\mathrm{Aut}(L:K)$, which are closed in the topology on $\mathrm{Aut}(L:K)$ induced by the pointwise convergence on L. Then $\mathrm{Aut}(L:K)$ is a projective limit of the finite groups $\mathrm{Aut}(M:K)$, where the intermediate fields M are finite Galois extensions of K (see [44]).

1.8. **Trace form.** For any finite field extension $L:K$, the K-endomorphism ring of the vector space L, $\mathrm{End}_K(L)$, is isomorphic to the $n \times n$-matrix ring over K, $n = [L\!:\!K]$. The trace function on this ring (sum of diagonal elements) is K-linear and thus provides a K-linear map $\mathrm{Tr} : \mathrm{End}_K(L) \to K$ satisfying

$$\mathrm{Tr}(f \circ g) = \mathrm{Tr}(g \circ f) \text{ for any } f, g \in \mathrm{End}_K(L).$$

The (left) multiplication by any $a \in L$,

$$\lambda_a : L \to L, \quad x \mapsto ax,$$

is in $\mathrm{End}_K(L)$ and for $a, b \in L$, $\lambda_{ab} = \lambda_a \circ \lambda_b$, thus yielding an (injective) K-algebra morphism

$$\lambda : L \to \mathrm{End}_K(L), \quad a \mapsto \lambda_a.$$

Composing these maps we get the K-linear map

$$\mathrm{tr} : L \xrightarrow{\lambda} \mathrm{End}_K(L) \xrightarrow{\mathrm{Tr}} K, \tag{1.2}$$

with the properties

$$\mathrm{tr}(ab) = \mathrm{Tr}(\lambda_a \circ \lambda_b) = \mathrm{Tr}(\lambda_b \circ \lambda_a) = \mathrm{tr}(ba). \tag{1.3}$$

For separable extension $L\!:\!K$, with algebraic closure $\widehat{K}\!:\!K$, and $a \in L$,

$$\mathrm{tr}(a) = \sum \{\sigma(a) \mid \sigma : L \to \widehat{K} \text{ is a } K\text{-morphism}\}, \tag{1.4}$$

and it follows from Dedekind's theorem (see 1.2) that $\mathrm{tr}(a) \neq 0$.
If $L\!:\!K$ is Galois, $a \in L$, and $G = \mathrm{Aut}(L\!:\!K)$,

$$\mathrm{tr}(a) = \sum\nolimits_{\sigma \in G} \sigma(a). \tag{1.5}$$

1.9. **The dual space.** For any field extension $L\!:\!K$, $L^* = \mathrm{Hom}_K(L, K)$ is an L-vector space with the action of $b \in L$ on $f \in L^*$ given by $b \cdot f(-) := f(b-)$.
Fixing $a \in L$, $\mathrm{tr}(a-) : L \to K$, $x \mapsto \mathrm{tr}(ax)$, is in L^* and the map

$$\psi : L \to L^*, \quad a \mapsto \mathrm{tr}(a-),$$

is L-linear since $b \cdot \mathrm{tr}(ax) = \mathrm{tr}(b(ax)) = \mathrm{tr}((ab)x)$, that is,

$$b \cdot \mathrm{tr}(a-) = \mathrm{tr}(ba-).$$

If $L\!:\!K$ is finite and separable, $\mathrm{tr}(a)$ is nonzero for $a \in L$ (see (1.4)) and hence ψ is injective, and in fact bijective, since $\dim_K L = \dim_K L^*$.

These are the ingredients for the

1.10. **Proposition.** *For a finite field extension $L:K$, there are equivalent:*

(a) *$L:K$ is separable;*

(b) *$\mathrm{tr} : L \to K$ is not zero;*

(c) *$\psi : L \to L^*$, $a \mapsto \mathrm{tr}(a-)$, is an isomorphism (and is L-linear).*

Notice that so far our results are essentially obtained with the techniques available in the 19th century. The way of looking at the material was greatly expanded when the following notion came into play.

1.11. **Tensor product of modules.** Let R be a commutative ring. The *tensor product* of R-modules N and M is a pair $(N \otimes_R M, \pi)$ where $N \otimes_R M$ is an R-module and $\pi : N \times M \to L \otimes_R M$ is a R-module homomorphism such that, for any R-module G and R-bilinear map $\beta : N \times M \to G$, there exists a unique R-module homomorphism $\widetilde{\beta} : N \otimes_R M \to G$ with commutative diagram

$$\begin{tikzcd} N \times M \arrow[r, "\pi"] \arrow[dr, "\beta"'] & N \otimes_R M \arrow[d, "\widetilde{\beta}"] \\ & G. \end{tikzcd}$$

The tensor product of R-linear maps $f : N \to N'$ and $g : M \to M'$ between R-modules, is defined as $f \otimes g : N \otimes_R M \to N' \otimes_R M'$ by putting

$$f \otimes g(n \otimes m) := f(n) \otimes g(m), \qquad \text{for } n \in N,\ m \in M,$$

and showing that this defines in fact a map on $N \otimes_R M$. For the identity $1_N : N \to N$, it is customary just to write $1_N \otimes g = N \otimes g$. We often delete the suffix of $\otimes_R$ (in particular in formulas) and just write $\otimes$ if no ambiguity arises.

1.12. **Tensor product and field extensions.** For a field extension $L : K$, the product $m : L \otimes_K L \to L$ and unit $\iota : K \to L$ are K-linear maps with commutative diagrams

$$(1.6) \qquad \begin{tikzcd} L \otimes L \otimes L \arrow[r, "m \otimes L"] \arrow[d, "L \otimes m"'] & L \otimes L \arrow[d, "m"] \\ L \otimes L \arrow[r, "m"] & L, \end{tikzcd} \qquad \begin{tikzcd} K \otimes L \arrow[r, "\iota \otimes L"] \arrow[dr, "="'] & L \otimes L \arrow[d, "m"] & L \otimes K \arrow[l, "L \otimes \iota"'] \arrow[dl, "="] \\ & L & \end{tikzcd} .$$

For two field extensions $L:K$ and $Q:K$, a multiplication on $Q \otimes_K L$ is defined by putting, for $q_1, q_2 \in Q$ and $l_1, l_2 \in L$,

$$(q_1 \otimes l_1) \cdot (q_2 \otimes l_2) = q_1 q_2 \otimes l_1 l_2.$$

This makes $Q \otimes_K L$ a K-algebra which need not be a field, in particular, it may allow for nilpotent elements.

The set of all nilpotent elements of a commutative finite dimensional K-algebra A is called the *nil radical* and is denoted by $\mathrm{Nil}(A)$. The structure theorem says that $\mathrm{Nil}(A) = 0$ if and only if A is a (finite) direct product of fields. The relevance of nilpotency for separability becomes evident in the following result (e.g. [87, Section 28]).

1.13. **Proposition.** *Let $L:K$ be an algebraic field extension and assume $a \in L$ is not separable over K. Then there exists some field extension $Q:K$ such that $L \otimes_K Q$ has non-zero nilpotent elements.*

Proof. Since over fields with zero characteristic all extensions are separable, we have to consider the case $\mathrm{char}(K) = p \neq 0$. Let $f \in K[X]$ be the minimal polynomial of a over K and Q its splitting field. Then there exists a polynomial

$$h(X) = \sum_{i=1}^{r} b_i X^i \quad \text{with } \ b_i \in Q,\ b_r \neq 0, \quad \text{and } \ f(X) = h(X)^p.$$

Since $r < n$, the elements $1, a, \cdots, a^r$ are linearly independent over K, hence

$$0 \neq c = \sum\nolimits_{j=1}^{r} a^j \otimes b_j \in L \otimes_K Q, \quad \text{and}$$

$$c^p = (\sum_{j=1}^{r} a^j \otimes b_j)^p = \sum_{j=1}^{r} a^{jp} \otimes b_j^p = \sum_{j=1}^{r} (b_j a^j)^p \otimes 1 = h(a)^p \otimes 1 = 0.$$

So $c \in L \otimes_K Q$ is non-zero and nilpotent □

1.14**. Remark.** A more general version of Proposition 1.13 can be given in terms of radicals, e.g. [69, Chapitre 2]: for any K-algebra A, define the *prime radical* $\mathrm{rad}(A)$ as intersection of all prime ideals of A. Then, for any field extension $L\!:\!K$,

$$\mathrm{rad}(A \otimes_K L) \cap A = \mathrm{rad}(A),$$

and if $L\!:\!K$ is separable,

$$\mathrm{rad}(A \otimes_K L) = \mathrm{rad}(A) \otimes_K L.$$

Notice that for commutative algebras A, $\mathrm{rad}(A) = \mathrm{Nil}(A)$.

From these observations we get:

1.15**. Proposition.** *For a finite field extension $L : K$, the following properties are equivalent:*

(a) $L\!:\!K$ *is separable;*
(b) *for any field extension $Q : K$,* $\mathrm{Nil}(L \otimes_K Q) = 0$*;*
(c) $\mathrm{Nil}(L \otimes_K L) = 0$*;*
(d) L *is projective as an $L \otimes_K L$-module;*
(e) $m : L \otimes_K L \to L$ *is split by some (L, L)-bimodule morphism* $\delta : L \to L \otimes_K L$*;*
(f) *there exist $e \in L \otimes_K L$ with $ae = ea$, for any $a \in L$, and $m(e) = 1$ (choose $e = \delta(1)$, separability idempotent).*

By the structure theorem mentioned above, $\mathrm{Nil}(L \otimes_K L) = 0$ implies that $L \otimes_K L$ is a semisimple algebra and hence all $L \otimes_K L$-modules are projective. The last two characterisations are just module theoretic variations of projectivity in the category of $L \otimes L$-modules.

The map δ in (e) opens the view to a new structure:

1.16**. Coproduct on** L**.** If $L : K$ is separable, then $\delta : L \to L \otimes_K L$ in (e) of the above proposition is left and right L-linear and this is expressed by commutativity of the diagrams

(1.7)
$$\begin{tikzcd} L \otimes L \arrow[r, "m"] \arrow[d, "\delta\otimes L"'] & L \arrow[d, "\delta"] \\ L \otimes L \otimes L \arrow[r, "L\otimes m"] & L \otimes L\,, \end{tikzcd} \qquad \begin{tikzcd} L \otimes L \arrow[r, "L\otimes\delta"] \arrow[d, "m"'] & L \otimes L \otimes L \arrow[d, "m\otimes L"] \\ L \arrow[r, "\delta"] & L \otimes L\,. \end{tikzcd}$$

These relations between m and δ are called *Frobenius conditions*.

Moreover, $m \circ \delta = 1_L$, and an easy argument shows that these imply commutativity of the diagram (*coassociativity*, compare (1.6))

(1.8)
$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L \otimes L \arrow[d, "L\otimes\delta"] \\ L \otimes L \arrow[r, "\delta\otimes L"] & L \otimes L \otimes L\,. \end{tikzcd}$$

There is yet another way to define a coproduct on L.

1.17**. Dual of algebra (coalgebra).** For any finite field extension $L:K$, applying $(-)^* = \mathrm{Hom}_K(-,K)$ to $m: L\otimes_K L \to L$ and $\iota : K \to L$, yields K-linear maps

$$L^* \xrightarrow{m^*} (L\otimes_K L)^* \simeq L^* \otimes_K L^*, \qquad L^* \xrightarrow{\iota^*} K.$$

Given a K-isomorphism $\psi : L \to L^*$ leads to K-linear maps

$$\delta : L \to L\otimes_K L, \qquad \varepsilon : L \to K, \tag{1.9}$$

by the diagrams

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\psi"'] & L\otimes_K L \\ L^* \arrow[r, "m^*"] & L^*\otimes_K L^* , \arrow[u, "\psi^{-1}\otimes\psi^{-1}"'] \end{tikzcd} \qquad \begin{tikzcd} L \arrow[d, "\psi"'] \arrow[dr, "\varepsilon"] & \\ L^* \arrow[r, "\iota^*"] & K, \end{tikzcd} \tag{1.10}$$

and exploiting associativity of m and unitality yields commutativity of the diagrams (dual to (1.6))

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L\otimes L \arrow[d, "L\otimes\delta"] \\ L\otimes L \arrow[r, "\delta\otimes L"] & L\otimes L\otimes L , \end{tikzcd} \qquad \begin{tikzcd} L \arrow[dr, "="'] & L\otimes L \arrow[l, "L\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes L"] & L \\ & L \arrow[u, "\delta"] \arrow[ur, "="'] & \end{tikzcd} , \tag{1.11}$$

which describe *coassociativity* and *counitality*, respectively.

In case $\psi : L \to L^*$ is even L-linear (e.g. Proposition 1.10), the Frobenius conditions (1.7) are satisfied for (L, m, δ).

1.18**. Tensor functor.** For a field extension $L:K$, the L-vector spaces can be defined as K-vector spaces V with K-linear maps $\varrho : L\otimes_K V \to V$ subject to associativity and unitality conditions, that is, commutativity of the diagrams

$$\begin{tikzcd} L\otimes L\otimes V \arrow[r, "m\otimes V"] \arrow[d, "L\otimes\varrho"'] & L\otimes V \arrow[d, "\varrho"] \\ L\otimes V \arrow[r, "\varrho"] & V, \end{tikzcd} \qquad \begin{tikzcd} L\otimes V \arrow[r, "\varrho"] & V \\ K\otimes V \arrow[u, "\iota\otimes V"] \arrow[ur, "="'] & \end{tikzcd} .$$

In particular, $L\otimes_K V$ is always an L-vector space by

$$m\otimes V : L\otimes L\otimes V \to L\otimes V,$$

and this leads to the *extension of scalars* functor

$$\Phi_L : \mathbb{V}_K \to \mathbb{V}_L, \qquad V \mapsto (L\otimes_K V, m\otimes V),$$
$$V \xrightarrow{f} V' \mapsto L\otimes_K V \xrightarrow{L\otimes f} L\otimes_K V';$$

there is a *restriction of scalars* (or *forgetful*) functor

$$U_L : \mathbb{V}_L \to \mathbb{V}_K, \quad (V,\varrho) \mapsto V,$$

leaving morphisms unchanged. These form an *adjoint pair* of functors by the bijection, for $V\in\mathbb{V}_K$, $N\in\mathbb{V}_L$,

$$\mathrm{Hom}_L(L\otimes_K V, N) \simeq \mathrm{Hom}_K(V, U_L(N)), \quad f \mapsto f\circ(\iota\otimes V).$$

It is customary to write $U_L(N) = N$ if no confusion occurs.

1.19. Splitting of U_L. Let $L : K$ be a finite separable field extension. For (V, ϱ), (N, ϱ') in $\mathbb{V}_L$, the forgetful functor $U_L : \mathbb{V}_L \to \mathbb{V}_K$ provides the canonical map (writing $U_L(V) = V$ etc.)

$$\Phi^U_{V,N} : \mathrm{Hom}_L(V, N) \to \mathrm{Hom}_K(U_L(V), U_L(N)) = \mathrm{Hom}_K(V, N). \tag{1.12}$$

Since (L, m, δ) satisfies the Frobenius conditions, it is not difficult to show that for the L-linear map

$$\omega : V \xrightarrow{\iota\otimes V} L \otimes V \xrightarrow{\delta\otimes V} L \otimes L \otimes V \xrightarrow{L\otimes\varrho} L \otimes V, \tag{1.13}$$

one has $\varrho \circ \omega = 1_V$. Now define the map (natural in V and N)

$$\Psi^U_{V,N} : \mathrm{Hom}_K(V, N) \to \mathrm{Hom}_L(V, N),$$
$$V \xrightarrow{f} N \quad \mapsto \quad V \xrightarrow{\omega} L \otimes_K V \xrightarrow{L\otimes f} L \otimes_K N \xrightarrow{\varrho'} N.$$

If f happens to be L-linear one easily sees that $\Psi^U_{V,N}(f) = f$. This proves one implication of the

1.20. Proposition. *For a field extension $L\!:\!K$ there are equivalent:*

(a) *$L\!:\!K$ is separable;*

(b) *$\Phi^U_{V,N}$ is a naturally split monomorphism.*

1.21. Group algebra and its dual. Let G be any group with unit e and K a field (or commutative ring). The group algebra $G[K]$ is defined as a K-vector space with basis set G and product (multiplication) with unit

$$m_G : K[G] \otimes_K K[G] \to K[G],\ g \otimes h \mapsto gh, \quad \iota : K \to K[G],\ k \mapsto k\cdot e,$$

$K[G]$ also allows for a coproduct with counit

$$\delta_G : K[G] \to K[G] \otimes_K K[G],\ g \mapsto g \otimes g, \quad \varepsilon : K[G] \to K,\ g \mapsto \delta_{g,e},$$

where $\delta_{g,e}$ denotes the Kronecker symbol. One has

$$(1_{K[G]} \otimes \varepsilon)\delta_G = 1_{K[G]} = (\varepsilon \otimes 1_{K[G]})\delta_G,$$

and for $u, v \in K[G]$, $\delta_G(uv) = \delta_G(u) \cdot \delta_G(v)$.

This shows that $(K[G], m_G, \delta_G)$ is a K-*bialgebra*.

If G is a finite group, product and coproduct of $K[G]$ are - by the functor $(\quad)^* = \mathrm{Hom}_K(-, K)$ - transferred to (the dual) coproduct and product on $K[G]^*$ (compare 1.17) thus making $(K[G]^*, m^*_G, \delta^*_G)$ also a K-bialgebra.

1.22. Action induced by $\mathrm{Aut}(L\!:\!K)$. For a field extension $L\!:\!K$, put $G = \mathrm{Aut}(L\!:\!K)$. By the action of G, L becomes a $K[G]$-module,

$$\varrho : K[G] \otimes_K L \to L, \quad g \otimes a \mapsto g(a).$$

If $[L : K]$ is finite, then G is finite, say with elements $g_1, \ldots, g_n$ which form a basis for $K[G]$. Choosing $p_1, \ldots, p_n \in K[G]^*$ as a dual K-base for the bialgebra $K[G]^*$ (see 1.21) yields a morphism

$$\tilde{\eta} : K \to \sum K[G]^* \otimes_K K[G], \quad 1 \mapsto \sum p_i \otimes g_i,$$

and leads to the K-linear map (coaction)

$$\omega : L \xrightarrow{\tilde{\eta}\otimes L} K[G]^* \otimes K[G] \otimes L \xrightarrow{K[G]^*\otimes\varrho} K[G]^* \otimes L, \quad a \mapsto \sum\nolimits_i p_i \otimes g_i(a).$$

Composing $\omega \otimes L$ with $K[G]^* \otimes m$ yields the map

$$\beta : L \otimes_K L \to K[G]^* \otimes_K L, \quad a \otimes b \mapsto \sum_i p_i \otimes g_i(a)b.$$

These constructions can be applied to prove:

1.23. **Proposition.** *Let $L : K$ be a field extension with $[L : K] = n$, $n \in \mathbb{N}$, and $G = \mathrm{Aut}(L{:}K)$. With the notation from above, the following are equivalent:*

(a) *$L{:}K$ is separable and normal (Galois extension);*

(b) *for some $a \in L$, $\{g(a)\}_{g \in G}$ is a K-basis of L;*

(c) *for some $a \in L$, the map $K[G] \to L$, $g \mapsto g(a)$, is an isomorphism of $K[G]$-modules (*Normal basis theorem*);*

(d) *β is an isomorphism.*

The assertions are essentially derived from the fact that finite separable extensions can be generated by a single (primitive) element (e.g. [57, 8.1.2]). For a detailed presentation of (Hopf) Galois extension the reader may consult [77, Chapter 4].

Summarising, to any field extension $L : K$, we have attached the endofunctors $L \otimes_K -$ and the adjoint pair of functors (ϕ_L, U_L) and, as shown in Proposition 1.20, $L{:}K$ is separable precisely when for U_L, the canonical map

$$\Phi^U_{V,N} : \mathrm{Hom}_L(V, N) \to \mathrm{Hom}_K(U_L(V), U_L(N))$$

is a naturally split monomorphism. In the next section we shall consider functors and endofunctors for any categories and follow the above pattern to define *separable* (and *Frobenius*) functors.

2. Separable and Frobenius functors

In this section, $\mathbb{A}$ and $\mathbb{B}$ denote any categories. For basic definitions and notations we refer to the Appendix, Section 7.

2.1. **Definition.** A functor $F : \mathbb{A} \to \mathbb{B}$ between categories is said to be *separable* provided, for any $A, A' \in \mathbb{A}$, the canonical map

$$\Phi^F_{A,A'} : \mathrm{Hom}_{\mathbb{A}}(A, A') \to \mathrm{Hom}_{\mathbb{B}}(F(A), F(A'))$$

is a naturally split monomorphism, that is, there is a map

$$\Psi^F_{A,A'} : \mathrm{Mor}_{\mathbb{B}}(F(A), F(A')) \to \mathrm{Mor}_{\mathbb{A}}(A, A'),$$

natural in A, A', with $\Psi^F_{A,A'} \circ \Phi^F_{A,A'} = 1_{\mathbb{A}}$.

Clearly, for a separable functor F, $\Phi^F_{A,A'}$ is always injective, that is, F is a faithful functor. Of course, every equivalence functor F is separable. In this context, adjoint pairs of functors are of particular interest (e.g. [68, 18, 12]).

2.2. **Adjoint pairs.** *Let $\mathsf{L} \dashv \mathsf{R} : \mathbb{B} \to \mathbb{A}$ be an adjoint pair of functors with unit $\tilde{\eta} : 1_{\mathbb{A}} \to \mathsf{RL}$ and counit $\tilde{\varepsilon} : \mathsf{LR} \to 1_{\mathbb{B}}$.*

(i) *L is separable if and only if $\tilde{\eta}$ is a split monomorphism;*

(ii) *R is separable if and only if $\tilde{\varepsilon}$ is a split epimorphism.*

The situation for field extensions is subsumed by (co-)monads (see [12, 2.9]). Recall that for a monad F, $\phi_F \dashv U_F$ with counit $\varepsilon_F : \phi_F U_F \to 1_{\mathbb{A}_F}$, and for a comonad G, $U^G \dashv \phi^G$ with unit $\eta^G : 1_{\mathbb{A}^G} \to \phi^G U^G$.

2.3. **Definitions.** A monad (F, m, ι) on $\mathbb{A}$ is called *separable* if the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ is separable.

A comonad (G, δ, ε) on $\mathbb{A}$ is called *coseparable* if the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$ is separable.

2.4. **Separability for monads and comonads.**

(1) *For a monad (F, m, ι) on $\mathbb{A}$, the following are equivalent:*

   (a) *(F, m, ι) is separable;*

   (b) *there exists a natural transformation $\delta : F \to FF$ with $m \cdot \delta = 1_F$ and (Frobenius condition)*
$$Fm \cdot \delta F = \delta \cdot m = mF \cdot F\delta;$$

   (c) *$\varepsilon_F : \phi_F U_F \to 1_{\mathbb{A}_F}$ is a split epimorphism.*

(2) *For a comonad (G, δ, ε) on $\mathbb{A}$, the following are equivalent:*

   (a) *(G, δ, ε) is coseparable;*

   (b) *there exists a natural transformation $m : GG \to G$ with $m \cdot \delta = 1_G$ and*
$$mG \cdot G\delta = \delta \cdot m = Gm \cdot \delta G;$$

   (c) *$\eta^G : 1_{\mathbb{A}^G} \to \phi^G U^G$ is a split monomorphism.*

2.5. **Separability of adjoints.** Consider an adjoint pair of endofunctors $G \dashv F : \mathbb{A} \to \mathbb{A}$ with unit $\eta : 1_{\mathbb{A}} \to FG$ and counit $\varepsilon : GF \to 1_{\mathbb{A}}$. Assume (G, δ, ε) to be a comonad on $\mathbb{A}$ and denote by (F, m, ι) the corresponding monad (see Theorem 7.12). Then there are pairs of adjoint (free and forgetful) functors,

$$\mathbb{A} \xrightarrow{\phi_F} \mathbb{A}_F,\ \mathbb{A}_F \xrightarrow{U_F} \mathbb{A}, \quad \text{with unit } \eta_F \text{ and counit } \varepsilon_F,$$
$$\mathbb{A}^G \xrightarrow{U^G} \mathbb{A},\ \mathbb{A} \xrightarrow{\phi^G} \mathbb{A}^G, \quad \text{with unit } \eta^G \text{ and counit } \varepsilon^G,$$

(1) ϕ^G is separable if and only if ϕ_F is separable;

(2) U^G is separable if and only if U_F is separable.

The isomorphism $\psi : L \to L^*$ for finite separable field extensions $L : K$ in Proposition 1.10 can be understood as natural isomorphism between the functors ϕ_L and $\mathrm{Hom}_K(L, -)$ from $\mathbb{V}_K$ to $\mathbb{V}_L$. This is the basic property of Frobenius algebras. More generally, the role of adjoint pairs of functors for Frobenius extensions was highlighted by K. Morita in [60]. The key to this approach is the

2.6. **Definition.** A functor $F : \mathbb{A} \to \mathbb{B}$ is said to be *Frobenius* provided it has a right adjoint $G : \mathbb{B} \to \mathbb{A}$ which is also left adjoint to F.

Similar to separability, Frobenius (co)monads are defined by properties of the corresponding forgetful functors.

2.7. **Definition.** A monad (F, m, ι) on $\mathbb{A}$ is said to be a *Frobenius monad* provided the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ is Frobenius.

A comonad (G, δ, ε) is said to be a *Frobenius comonad* provided the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$ is Frobenius (see 2.6).

As a first characterisation we note (see [54, Proposition 3.11]):

2.8. **Proposition.** *A monad (F, m, ι) on $\mathbb{A}$ with a right adjoint comonad (G, δ, ε) is Frobenius if and only if the functors F and G are isomorphic as left F-module functors.*

Obviously, every Frobenius monad may also be seen as a Frobenius comonad and hence it suffices to talk about Frobenius monads. We collect various characterisations of such functors (e.g. [73], [54, Theorem 3.13]).

Recall from Theorem 7.12 that, in the situation of Proposition 2.8, the categories of F-modules and of G-comodules are isomorphic.

2.9. **Frobenius monads.** *For a monad* (F, m, ι) *on* $\mathbb{A}$*, there are equivalent:*

(a) (F, m, ι) *is a Frobenius monad;*

(b) *the free functor* $\phi_F : \mathbb{A} \to \mathbb{A}_F$ *is Frobenius;*

(c) F *admits a comonad structure* (F, δ, ε) *and - equivalently -*

  (i) $Fm \cdot \delta F = \delta \cdot m = mF \cdot F\delta$ *(Frobenius conditions);*

  (ii) $F \dashv F$ *by unit and counit,*

$$1_{\mathbb{A}} \xrightarrow{\iota} F \xrightarrow{\delta} FF, \quad FF \xrightarrow{m} F \xrightarrow{\varepsilon} 1_{\mathbb{A}};$$

  (iii) *an isomorphism of categories*

$$Q : \mathbb{A}_F \to \mathbb{A}^F \text{ with } U^F \cdot Q = U_F \text{ and } Q \cdot \phi_F \simeq \phi^F.$$

The following shows how close separable monads are to Frobenius monads.

2.10. **Corollary.** *Let Let* (F, m, ι) *be a monad on* $\mathbb{A}$ *and* $\delta : F \to FF$ *a coassociative coproduct such that* (F, m, δ) *satisfies the Frobenius condition (e.g. 2.4). Then:*

(1) (F, m, ι, δ) *is a separable monad if and only if* $m \cdot \delta = 1_{\mathbb{A}}$.

(2) (F, m, ι, δ) *is a Frobenius monad if and only if* (F, δ) *allows for a counit* $\varepsilon : F \to 1_{\mathbb{A}}$.

3. Separable and Frobenius algebras

We sketch the application of the categorical results from the preceding section for various special cases, beginning with the categories $\mathbb{M}_R$ of modules over a commutative ring R. Unless stated otherwise our algebras will be associative with unit.

3.1. **R-algebras.** A triple (A, m, ι) is called an *R-algebra* provided A is an R-module and there are R-linear maps (product and unit),

$$m : A \otimes_R A \to A, \quad \iota : R \to A,$$

satisfying the conditions to make the functor $A \otimes_R - : \mathbb{M}_R \to \mathbb{M}_R$ a monad. The induced module category is just the category ${}_A\mathbb{M}$ of left A-modules with the free and forgetful functors, $\phi_A : \mathbb{M}_R \to {}_A\mathbb{M}$ and $U_A : {}_A\mathbb{M} \to \mathbb{M}_R$.

The functor $- \otimes_R A$ leads to the category $\mathbb{M}_A$ of right A-modules and corresponding constructions.

The functor $A \otimes_R -$ has a right adjoint $\mathrm{Hom}_R(A, -) : \mathbb{M}_R \to \mathbb{M}_R$, with unit and counit, for $X, Y \in \mathbb{M}_R$,

$$\begin{aligned} \eta : &\quad X \to \mathrm{Hom}_R(A, A \otimes_R X), \quad x \mapsto [a \mapsto a \otimes x], \\ \varepsilon : &\quad A \otimes \mathrm{Hom}_R(A, X) \to X, \quad a \otimes f \mapsto f(a), \end{aligned}$$

and the bijection

$$\mathrm{Hom}_R(A \otimes X, Y) \to \mathrm{Hom}_R(X, \mathrm{Hom}_R(A, Y)), \; f \mapsto [x \mapsto f(- \otimes x)]. \tag{3.1}$$

By Theorem 7.12, $\mathrm{Hom}_R(A, -)$ is a comonad on $\mathbb{M}_R$ with coproduct and counit,

$$\mathrm{Hom}_R(A, -) \xrightarrow{\mathrm{Hom}(m,-)} \mathrm{Hom}_R(A \otimes A, -) \simeq \mathrm{Hom}_R(A, \mathrm{Hom}_R(A, -)),$$

$$\mathrm{Hom}_R(A, -) \xrightarrow{\mathrm{Hom}(\iota,-)} 1_{\mathbb{M}_R}.$$

If A is finitely generated and projective as R-module, $\mathrm{Hom}_R(A,-) \simeq A^* \otimes_R -$. In this case, $A^* \otimes_R -$ is left and right adjoint to $A \otimes_R -$.

Comonads based on tensor functors are usually called *coalgebras.*

For an R-algebra A, the opposite algebra A^o is defined as the same R-module with opposite multiplication. Then the (A,A)-bimodules can be considered as (left) modules over the algebra $A^e := A \otimes_R A^o$. In particular, A is a left A^e-module by the action

$$\mu_A : (A \otimes A^o) \otimes A \to A, \quad a \otimes b \otimes c \mapsto acb. \tag{3.2}$$

and its endomorphism ring $\mathrm{End}_{A^e}(A)$ is just its center $Z(A)$.

Now Definition 2.1 yields:

3.2. **Definition.** An R-algebra (A, m, ι) is called *separable* if the induced monad on $\mathbb{M}_R$ is separable, that is, $U_A : \mathbb{M}_A \to \mathbb{M}_R$ is a separable functor, which means that, for $M, N \in {}_A\mathbb{M}$, the canonical map

$$\Phi^A_{M,N} : \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_R(U_A(M), U_A(N))$$

is a (naturally) split monomorphism.

This generalises the characterisation of separable field extensions in 1.19 and we derive from 2.4:

3.3. **Separable algebras.** *For an R-algebra A, the following are equivalent:*

(a) *A is a separable algebra;*

(b) *$m : A \otimes_R A \to A$ splits as an $A \otimes_R A^o$-module morphism (by some $\delta : A \to A \otimes_R A$);*

(c) *there exists $e \in A \otimes_R A^o$ with $(a \otimes 1)e = (1 \otimes a)e$ for $a \in A$ and $m(e) = 1$ (choose $e = \delta(1)$, separability idempotent);*

(d) *every A-epimorphism which splits as R-module map splits as an A-module map (A is (A,R)-semisimple).*

To show (b)⇒(a), the splitting of $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_R(M,N)$, for $M, N \in {}_A\mathbb{M}$, is given by sending any R-morphism $g : M \to N$ to the composite

$$M \xrightarrow{\iota \otimes M} A \otimes M \xrightarrow{\delta \otimes M} A \otimes A \otimes M \xrightarrow{A \otimes \varrho_M} A \otimes M \xrightarrow{A \otimes g} A \otimes N \xrightarrow{\varrho_N} N.$$

Characterisation (c) is possible, since the A^e-linear map δ is uniquely determined by the image of 1_A yielding the *separability idempotent* $\delta(1_A)$ which is often used to prove properties of these algebras.

The condition in (b), reducing the property to the splitting of a single linear map, was used by M. Auslander and O. Goldman in their paper [5] and before a special case of this was considered by G. Azumaya in [6].

3.4. **Definition.** A separable R-algebra (A, m, ι) is said to be *strongly separable* if the separability idempotent $e \in A \otimes_R A^o$ (in Theorem 3.3(c)) is symmetric, that is,

$$e = \sum\nolimits_i e_i \otimes f_i = \sum\nolimits_i f_i \otimes e_i.$$

Such algebras were considered by A. Hattori in [32] and more observations on these were made, among others, by L. Kadison and A. Stolin [37], and M. Aguiar in [4].

One question of interest is which of the properties of field extension can be obtained for algebras A over a field K. For a finite dimensional A, the K-endomorphism ring

$\mathrm{End}_K(A)$ is a matrix ring, canonically isomorphic to $A \otimes_K A^*$, and so we get the K-linear map

$$\mathrm{tr} : A \xrightarrow{\lambda} \mathrm{End}_K(A) \simeq A \otimes A^* \xrightarrow{\mathrm{ev}} K, \quad a \mapsto \mathrm{Tr}(\lambda_a), \tag{3.3}$$

where ev denotes the evaluation map, and $\mathrm{End}_K(A) \to K$ just gives the trace map Tr. For $a, b, c \in A$, we have

$$\mathrm{tr}(ab) = \mathrm{tr}(ba), \quad \mathrm{tr}(a(bc)) = \mathrm{tr}((ab)c),$$

that is, tr is a *symmetric* and *balanced linear form.* It is called *non-degenerate* provided the K-linear map

$$A \to A^*, \; a \mapsto \mathrm{tr}(a-),$$

is a K-linear isomorphism.

We note that this construction can also be made for algebras A over commutative rings R provided A is finitely generated and projective as R-module.

3.5. **Separable algebras over fields.** *Let A be a finite dimensional algebra A over a field K.*

(1) The following are equivalent:

(a) *A is a separable K-algebra (Definition 3.2);*

(b) *for every field extension $L : K$, $A \otimes_K L$ is a (finite) direct product of simple algebras.*

(c) *for any field extension $L : K$, $\mathrm{rad}(A \otimes_K L) = 0$:*

(d) *$A \otimes_K A^o$ is a direct product of simple algebras.*

(2) Furthermore, the following are equivalent:

(a) *A is a strongly separable K-algebra (Definition 3.4);*

(b) *$\mathrm{tr} : A \to K$ is a non-degenerate linear form.*

The equivalence of (b) and (c) in (1) is due to the fact that for any finite dimensional algebra the prime radical is zero if and only if the algebra is semisimple. For the equivalences stated in (2) we refer to [4, Theorem 3.1]. Notice that here it is not enough to require tr to be non-zero (as it is for field extensions).

3.6. **Definition.** A separable R-algebra A with center $Z(A) = R$ is called *central separable* or *Azumaya algebra.*

The class of these structures can be characterised in the following way.

3.7. **Azumaya algebras.** *For a central R-algebra (A, m, ι), there are equivalent:*

(a) *A is a separable R-algebra;*

(b) *there is an A^e-linear map $\delta : A \to A \otimes_R A$ with $m \cdot \delta = 1_A$;*

(c) *A induces an equivalence of categories,*

$$A \otimes_R - : \mathbb{M}_R \to {}_{A^e}\mathbb{M}, \;\; X \mapsto (A \otimes_R X, m \otimes X);$$

(d) *A as a left A^e-module is a generator in ${}_{A^e}\mathbb{M}$.*

We note that in any full module category over a ring, generators are finitely generated and projective over their endomorphism rings (e.g. [88, 18.8]). Thus an equivalence as given in (c) always forces this kind of finiteness condition on the R-module structure of A.

On the other hand, there are simple algebras over rings (and fields) which are not finitely generated over their centers (which are fields). To include this type of algebras in our theory we proceed in the following way.

The basic idea of the above characterisations is to relate internal properties of an algebra A (as given in (a)) with properties in a suitable category (as in (e)). The category ${}_{A^e}\mathbb{M}$ has coproducts and cokernels, products and kernels (Grothendieck category). We restrict our considerations to a (smallest) subcategory with similar properties which contains the A^e-module A: So we form a full category $\sigma_{A^e}[A]$ by taking as objects all direct sums of copies of A, all homomorphic images and submodules of the resulting modules. This gives us again a Grothendieck category (see [87]).

3.8. **Definition.** A central R-algebra (A, m, ι) is called an *Azumaya ring* provided the A^e-module A is a projective generator in $\sigma_{A^e}[A]$.

These algebras were also investigated by J.P. Delale in [21] under the name *algèbres affines*, by G. Azumaya in [7] under the name *separable rings*, and D.G. Burkholder called them *Azumaya rings* in [16]. The notion was extended to non-associative algebras in [84]. Any simple algebra A is a simple A^e-module and hence every module in $\sigma_{A^e}[A]$ is semisimple, in fact a direct sum of copies of A, and hence A is a projective generator in $\sigma_{A^e}[A]$, i.e., A is an Azumaya ring.

Generators in $\sigma_{A^e}[A]$ are flat as modules over their endomorphism rings and we get the following characterisations (see [87, 26.4]).

3.9. **Azumaya rings.** *For a central R-algebra (A, m, ι), the following are equivalent:*

(a) *A is an Azumaya ring;*

(b) *A induces an equivalence of categories,*

$$A \otimes_R - : \mathbb{M}_R \to \sigma_{A^e}[A],\ X \mapsto (A \otimes_R X, m_a \otimes X);$$

(c) *A is a generator in $\sigma_{A^e}[A]$ and A is faithfully flat over R;*

(d) $\mathrm{Hom}_{A^e}(A, -) : \sigma_{A^e}[A] \to \mathbb{M}_R$ *is an equivalence of categories.*

As easily seen, a central algebra A is an Azumaya algebra if and only if A is an Azumaya ring and A is finitely generated as an R-module. In this case one has $\sigma_{A^e}[A] = {}_{A^e}\mathbb{M}$.

So far we have considered algebras with unit. The missing of an identity element changes the picture in some aspects.

3.10. **Algebras without units.** For a non-unital R-algebra (A, m), $A \otimes_R -$ is no longer a monad, A^e may have no unit and hence need neither be projective nor a generator in ${}_{A^e}\mathbb{M}$. Furthermore, $\mathrm{End}_{A^e}(A)$ is no longer the center of the algebra A, it is called the *centroid* $C(A)$ of A (e.g. [87, 2.7]). If $A = A^2$, then $C(A)$ is a commutative R-algebra and A is a $C(A)$-algebra which is called *central* if $C(A) \simeq R$. The definition of Azumaya rings also hold for non-unital algebras. However, their characterisations considered above are to be modified (see [87, Chapter 7]). Non-unital separable algebras may be defined by the (A, A)-bimodule splitting of $m : A^e \to A$. They are investigated, for example, by Taylor in [76] to study a *bigger Brauer group*.

Most of the preceding results in this section depend heavily on the commutativity of the base ring (or field) R, which allows the twist map $X \otimes_R Y \to Y \otimes_R X,\ x \otimes y \mapsto y \otimes x$. This is, for example, needed to define multiplication on the tensor product $A \otimes_R A^o$. Yet some constructions are still possible in more general situations.

3.11. **Non-commutative base ring.** Let (A, m, ι) be a ring ($\mathbb{Z}$-algebra). For a ring extension $B \to A$, $A \otimes_B - : {}_B\mathbb{M} \to {}_B\mathbb{M}$ defines a monad on ${}_B\mathbb{M}$, and if this is separable, the extension is said to be *separable.* As for algebras, this is the case if $m : A \otimes_B A \to A$ splits as an (A, A)-bimodule map. The investigation of this case was initiated by K. Hirata and S. Sugano in [34]. Several properties of separable algebras are maintained but here $A \otimes_B A$ need not have a ring structure and hence the results from subsection 3.3 need modification.

For example, for the separable extension, A need not be a projective generator for the (A, A)-bimodules. To replace this, K. Hirata in [33], suggested a stronger condition for separability (and K. Sugano coined its name [74]):

3.12**. Definition.** A ring extension $B \to A$ is called *H-separable* if, for some $n \in \mathbb{N}$, $A \otimes_B A$ is a direct summand of A^n as an (A, A)-submodule.

As observed in [33, Theorem 2.2], any H-separable extension is a separable extension. These structures are investigated in a series of papers by K. Hirata, K. Sugano, T. Nakamoto, Y. Kurata, S. Morimoto, S., F. Kasch and B. Pareigis [33, 74, 61, 39, 45], and others.

3.13**. Separability and modules.** The notion of separability applies to any functors. In particular, one may ask when functors related to bimodules have this property. Early papers considering this question were Sugano [75] and Cunningham [19].

3.14**. Definition.** A bimodule ${}_AM_B$ over any rings A, B is said to be *separable* provided the functor

$$\mathrm{Hom}_A(M, -) : {}_A\mathbb{M} \to {}_B\mathbb{M}, \quad X \mapsto \mathrm{Hom}_A(M, X), \tag{3.4}$$

is a separable functor.

Since $M \otimes_B -$ is left adjoint to $\mathrm{Hom}_A(M, -)$, it follows from 2.2 that the latter is separable if and only if, for any $X \in {}_A\mathbb{M}$, the evaluation map

$$\mathrm{ev}_X : M \otimes_B \mathrm{Hom}_A(M, X) \to X, \quad m \otimes f \mapsto f(m),$$

is a naturally split epimorphism.

If M is finitely generated and projective as A-module, the condition can be reduced to require that the evaluation

$$\mathrm{ev}_A : M \otimes_B \mathrm{Hom}_A(M, A) \to A, \quad m \otimes f \mapsto f(m), \tag{3.5}$$

is naturally split as an (A, A)-bimodule map. This situation was investigated in [75] and further results in this direction are given in [19] and [86].

3.15**. Remark.** The functor $\mathrm{Hom}_A(M, -)$ in (3.4) can be restricted to the category $\sigma[{}_AM]$, the full subcategory of ${}_A\mathbb{M}$ subgenerated by ${}_AM$ (e.g. [88]). Then it still has the functor $M \otimes_B - : {}_B\mathbb{M} \to \sigma[M]$ as a left adjoint and the results from Section 2 apply. However, separability of $\mathrm{Hom}_A(M, -)$ does not imply that ev_A from 3.5 is surjective (unless $A \in \sigma[{}_AM]$). Compare also Theorem 3.9.

This kind of studies were also pursued by M. Sato in [70] where some properties of tilting and static modules are anticipated (e.g. [89], [90]).

For non-associative R-algebras (A, m), the endofunctor $A \otimes_R -$ is no monad and to study the relevance of separability in this case, one looks at the structure of closely related associative unital algebras.

3.16**. Multiplication algebra.** For any R-algebra A, not necessarily associative nor with unit, the left and right multiplications with elements $a \in A$ yield R-linear endomorphisms of A,

$$\lambda_a : A \to A,\ x \mapsto ax, \quad \rho_a : A \to A,\ x \mapsto xa,$$

Denote by $M(A)$ the subalgebra of $\mathrm{End}_R(A)$ generated by all $\{\lambda_a, \rho_a \,|a \in A\}$ and the identity map of A. $M(A)$ is called the *multiplication algebra* of A and since A is a module over $\mathrm{End}_R(A)$, it is also a module over $M(A)$, in fact an $(M(A), R)$-bimodule. The $M(A)$-submodules are just the two-sided ideals of A and hence the $M(A)$-module structure of A reflects its ring theoretic properties.

In case A is an associative unital algebra, there is a surjective algebra homomorphism

$$A^e = A \otimes A^o \to M(A), \quad a \otimes b \mapsto \lambda_a \rho_b,$$

and hence the A^e-module structure and the $M(A)$-module structure of A coincide. In fact, the attached categories $\sigma_{A^e}[A]$ and $\sigma_{M(A)}[A]$ can be identified and large parts of our results for separability and generating properties as A^e-modules can be formulated in terms of $M(A)$-modules. For example, A is an Azumaya algebra if and only if it is a generator in ${}_{M(A)}\mathbb{M}$ since, in this case, $A^e \simeq M(A) \simeq \mathrm{End}_R(A)$.

3.17. **Non-associative algebras.** The description of properties of non-associative algebras A by their multiplication algebras $M(A)$ was already considered by A.A. Albert and N. Jacobson (e.g. [36]). They observed that a finite dimensional algebra A over a field K is separable if and only if $M(A)$ is a separable (associative) K-algebra.

Considering the category $\sigma_{M(A)}[A]$ introduced in 3.16, one obtains a rich theory for non-associative algebras over rings without a priori finiteness restrictions. For example, characterisations of nonassociative Azumaya rings are obtained by simply replacing in 3.9 the algebra A^e by $M(A)$. This is elaborated in [87]. In this context, the algebra A^e, known as *enveloping algebra* in the associative case, is not relevant. Instead, for certain classes of non-associative algebras, e.g. alternative, Jordan or Lie algebras, the corresponding (associative) *enveloping algebras* can enter the picture (e.g. [87, 29.10], [36], [28], [10], [11]). For the study of regularity and radicals for non-associative algebras we refer to [85] and [82].

In 3.5, strong separability of associative algebras is characterised by a non-degenerate associative linear form $A \otimes_R A \to R$. This can also be achieved for some non-associative algebras, for example, certain *composition algebras* (e.g. [83]). For Lie algebras, separability is given by nondegeneracy of the *Killing form* (e.g. [20], [65]). In [80], a *Killing form* is also defined for Hopf algebras.

For associative algebras over a commutative ring R, Definition 2.7 yields:

3.18. **Definition.** An R-algebra (A, m, ι) is said to be a *Frobenius algebra* provided the forgetful functor $U_A : {}_A\mathbb{M} \to \mathbb{M}_R$ is Frobenius.

The categorical characterisations of this notion in 2.9 read now:

3.19. **Frobenius algebras.** *For an R-algebra (A, m, ι), there are equivalent:*

(a) *(A, m, ι) is a Frobenius algebra;*

(b) *the free functor $\phi_A : \mathbb{M}_R \to {}_A\mathbb{M}$ is Frobenius;*

(c) *A admits a comonad structure (A, δ, ε) and - equivalently -*

  (i) *(A, m, δ) satisfies the Frobenius conditions, that is,*

$$Am \cdot \delta A = \delta \cdot m = mA \cdot A\delta; \ \text{or}$$

  (ii) *$A \otimes_R - : \mathbb{M}_R \to \mathbb{M}_R$ is adjoint to itself with unit and counit,*

$$1_{\mathbb{A}} \xrightarrow{\iota} A \xrightarrow{\delta} A \otimes_R A, \quad A \otimes_R A \xrightarrow{m} A \xrightarrow{\varepsilon} 1_{\mathbb{A}};$$

(d) *$A \otimes_R - \simeq \mathrm{Hom}_R(A, -)$ as left A-module functors on $\mathbb{M}_R$;*

(e) *A is finitely generated and projective as an R-module and $A \simeq A^*$ as left (or right) A-modules.*

The isomorphism in (d) means that $A\otimes_R-$ preserves direct products and monomorphisms (as right adjoints do), that is, A_R has to be flat and finitely presented, hence finitely generated and projective as an R-module. In this case $\mathrm{Hom}_R(A,-)\simeq A^*\otimes_R-$.

For an R-module A with an algebra and a coalgebra structure, one may attach the categories of left A-modules ${}_A\mathbb{M}$ and left comodules ${}^A\mathbb{M}$. Then A yields a Frobenius algebra if and only if there is an isomorphism of categories (see Theorem 2.9, [54, Theorem 3.13])

$$Q : {}_A\mathbb{M} \to {}^A\mathbb{M} \quad \text{with} \quad U^A \cdot Q = U_A \quad \text{and} \quad Q\cdot\phi_A \simeq \phi^A.$$

By Proposition 1.10, finite separable field extensions $L\!:\!K$ allow for an L-isomorphism $L\simeq L^*$. This may have been one of the motivations for F. Frobenius to investigate (in [30], 1903) finite dimensional algebras A with the corresponding property. In [26] (1955), Eilenberg and Nakayama observed that the notion makes sense for algebras A over commutative rings R, provided A is finitely generated and projective as an R-module. These turned out to be of considerable interest in various areas of mathematics and theoretical physics.

3.20**. Corollary.** *Let A be a finitely generated and projective R-module allowing for an algebra structure (A,m,ι) and a coassociative coproduct $\delta : A\to A\otimes_R A$ such that (A,m,δ) satisfies the Frobenius conditions. Then:*

(1) *A is a separable algebra if and only if $m\cdot\delta = 1_A$.*

(2) *A is a Frobenius algebra if and only if (A,δ) allows for a counit $\varepsilon : A\to R$.*

3.21**. Frobenius extensions.** For ring extensions $B\to A$ with non-commutative rings A, B, the functor $A\otimes_B - : {}_B\mathbb{M}\to {}_B\mathbb{M}$ allows for a monad structure (see 3.11) and if this is Frobenius, the extension is called *Frobenius extension*, that is, the forgetful functor $U_A : {}_A\mathbb{M}\to {}_B\mathbb{M}$ is Frobenius.

The theory of such extensions was initiated by F. Kasch [38], T. Nakayama and T. Tsuku [62], K. Morita [60] and the literature around it is abundant.

4. Coseparable and Frobenius coalgebras

The categorical setting readily provides properties of coseparable comonads. In classical algebra, mainly comonads based on a tensor functor are considered and then are called *coalgebras*. We sketch the resulting framework for the category $\mathbb{M}_R$ of modules over a commutative ring R.

4.1**. Coalgebras.** A triple (C,δ,ε) is called an *R-coalgebra* provided C is an R-module and there are R-linear maps

$$\delta : C\to C\otimes_R C, \quad \varepsilon : C\to R,$$

satisfying the conditions to make the functor $C\otimes_R - : \mathbb{M}_R\to\mathbb{M}_R$ a comonad on $\mathbb{M}_R$ (see 7.6).

Notice that every free R-module V, with basis x_i, $i\in I$ any set, has a comodule structure by defining

$$\delta : x_i\mapsto x_i\otimes x_i, \quad \varepsilon : x_i\mapsto 1, \qquad \text{for } i\in I,$$

and extending this linearly to all of V.

From 7.6 we derive the notion of *C-comodules* obtaining the category of left comodules ${}^C\mathbb{M}$ (in which monomorphisms need not be injective and morphism need not have kernels). The morphism sets in ${}^C\mathbb{M}$ are denoted by Hom^C. We have the adjoint pair of forgetful and free functors,

$$\begin{aligned} U^C : {}^C\mathbb{M} \to \mathbb{M}_R, &\quad (M,\omega) \mapsto M, \\ C \otimes_R - : \mathbb{M}_R \to {}^C\mathbb{M}, &\quad X \mapsto (C \otimes_R X, \delta \otimes X), \end{aligned}$$

by the bijection

$$\begin{aligned} \mathrm{Hom}^C(M, C \otimes_R X) \to &\quad \mathrm{Hom}_R(M, X), \\ M \xrightarrow{h} C \otimes_R X \ \mapsto &\quad M \xrightarrow{h} C \otimes_R X \xrightarrow{\varepsilon \otimes X} X, \end{aligned}$$

leading, in particular, to the isomorphisms

$$\begin{aligned} \mathrm{End}^C(C) &\simeq \mathrm{Hom}_R(C, R) = C^*, \\ \mathrm{Hom}^C(M, C) &\simeq \mathrm{Hom}_R(M, R) = M^*. \end{aligned}$$

Of course, for a coalgebra (C, δ, ε), the setting of 7.6 also applies to the endofunctor $- \otimes_R C : \mathbb{M}_R \to \mathbb{M}_R$ leading to the category of *right C-comodules* $\mathbb{M}^C$. Clearly, C itself is a left as well as a right C-comodule.

The functor $C \otimes_R -$ has as right adjoint the functor $\mathrm{Hom}_R(C, -) : \mathbb{M}_R \to \mathbb{M}_R$ and the coalgebra structure on $C \otimes_R -$ induces a monad structure on $\mathrm{Hom}_R(C, -)$. The modules for this monad are also called *C-contramodules* (e.g,[27], [12], [93]). Following Definition 2.3 we have:

4.2. **Definition.** An R-coalgebra (C, δ, ε) is said to be *coseparable* if the forgetful functor $U^C : {}^C\mathbb{M} \to \mathbb{M}_R$ is separable.

From 2.4 we derive (see also [15, 3.29]):

4.3. **Coseparable coalgebras.** *For an R-coalgebra (C, δ, ε), the following are equivalent:*

(a) *C is a coseparable R-coalgebra;*
(b) *$\delta : C \to C \otimes C$ splits as a left and right C-comodule map (by some $m : C \otimes_R C \to C$);*
(c) *every C-monomorphism which splits as an R-module map splits as a C-comodule map (C is (C, R)-semisimple).*

As an application of 2.5 we obtain:

4.4. **Proposition.** *An R-coalgebra (C, δ, ε) is coseparable if and only if the induced monad on* $\mathrm{Hom}_R(C, -)$ *is separable.*

If C is finitely generated and projective as an R-module, this means that C^* (with the convolution product) is a separable R-algebra.

For the investigation of R-coalgebras we have heavily used that the tensor product of two R-modules is again an R-module, that is, $\mathbb{M}_R$ allows for a tensor product (monoidal category). For a non-commutative ring A, endofunctors for the category of left A-modules can be provided by (A, A)-bimodules M, that is,

$$M \otimes_A - : {}_A\mathbb{M} \to {}_A\mathbb{M}, \quad X \mapsto M \otimes_A X.$$

This is the basis for our next definition.

4.5. **Corings.** Let A be a ring. A triple (C, δ, ε) is called an *A-coring* provided C is an (A, A)-bimodule and there are (A, A)-linear maps

$$\delta : C \to C \otimes_A C, \quad \varepsilon : C \to A,$$

satisfying the conditions to make the functor $C \otimes_A - : {}_A\mathbb{M} \to {}_A\mathbb{M}$ a comonad on ${}_A\mathbb{M}$ (see 7.6).

Left comodules over (C, δ, ε), the category of left comodules ${}^C_A\mathbb{M}$, the free and forgetful functors $\phi^C : {}_A\mathbb{M} \to {}^C_A\mathbb{M}$, $U^C : {}^C_A\mathbb{M} \to {}_A\mathbb{M}$ are defined by the corresponding definitions for comonads. In particular, *coseparable corings* are defined as coseparable comonads (see Definition 2.3). The results for coalgebras - with obvious modifications - apply widely to corings and this is outlined in detail in [15].

An interesting class of corings is obtained by tensoring an R-algebra A with an R-coalgebra C, R a commutative ring:

4.6**. Entwining algebras and coalgebras.** Consider an R-algebra (A, m, ι), an R-coalgebra (C, δ, ε) and a mixed entwining between the functors $A \otimes_R -$ and $C \otimes_R -$, that is, an R-linear map

$$\lambda : A \otimes_R C \to C \otimes_R A$$

with commutative diagrams as in 7.7. This makes $C \otimes_R A$ a left A-module by

$$A \otimes_R C \otimes_R A \xrightarrow{\lambda \otimes C} C \otimes_R A \otimes_R A \xrightarrow{C \otimes m} C \otimes_R A.$$

Together with the canonical right A-module structure, $C \otimes_R A$ is an (A, A)-bimodule and the coproduct and counit,

$$C \otimes_R A \xrightarrow{\delta \otimes A} C \otimes_R C \otimes_R A \simeq (C \otimes_R A) \otimes_A (C \otimes_R A), \quad C \otimes_R A \xrightarrow{\varepsilon \otimes A} A,$$

create a comonad on ${}_A\mathbb{M}$, that is, an A-coring (see 4.6).

We end this section with a further view on group actions. Recall that in 1.22, a $K[G]^*$-comodule structure on L is considered. This is a special case of a Hopf algebra H coacting on an algebra, namely an

4.7**. H-comodule algebra.** Let H be an R-Hopf algebra and (A, m, ι) an R-algebra with coaction $\omega : A \to A \otimes_R H$. A is said to be an *H-comodule algebra* provided m and ι are H-colinear. Then the *coinvariants of H in A*, defined as

$$A^{coH} = \{a \in A \,|\, \omega(a) = a \otimes 1_H\},$$

form a subalgebra of A. The extension $A^{coH} \subset A$ is called (right) *H-Galois* whenever the map

$$\beta : A \otimes_{A^{coH}} A \to A \otimes_R H, \quad (a \otimes 1_H)\, \omega(b),$$

is bijective. These structures are addressed, for example, in [57], [77, Chapter 4], and reconsidered for corings, e.g., in [15], [91]. A comprehensive survey is given in [58].

5. Application to semirings

In the last decades, semirings have turned out to be of considerable interest in various fields of pure and applied mathematics. For example, bialgebras over semirings offer an advantageous framework for automata and formal language theory (see [96]). For a survey and an introduction to this field of research the reader is referred to the book of J.S. Golan [31]. Aspects of linear algebra over semirings are nicely presented in D. Wilding's thesis [81]. We recall the basic notions in a form suitable for our setting.

A commutative monoid $(M, +)$ is a set M together with an associative and commutative composition (addition)

$$+ : M \times M \to M, \ (m, m') \mapsto m + m',$$

and a neutral element $0 \in M$, that is, $0 + m = m$, for all $m \in M$. The prototype for this structure are the non-negative integers, denoted by $\mathbb{N}_0$.

Morphisms between two (commutative) monoids are maps $f : M \to N$ respecting the addition and the neutral element. These data define the category $\textsc{Mon}$ of commutative monoids.

A *tensor product* between two objects $M, N \in \textsc{Mon}$ is defined as a bilinear map $\tau : M \times N \to M \otimes N$, for some $M \otimes N \in \textsc{Mon}$, with the universal property: For any $L \in \textsc{Mon}$ and bilinear map $\beta : M \times N \to L$, there is a unique monoid map $\widetilde{\beta} : M \otimes N \to L$ with commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[d, "\tau"'] & L \\ M \otimes N \arrow[ur, "\widetilde{\beta}"'] & \end{tikzcd} .$$

The existence of such a tensor product is shown by taking the free commutative monoid F generated by the set $M \times N$, and then the quotient monoid by the congruence relation on F generated by all pairs of the form

$$((m + m', n); (m, n) + (m', n)); \ ((m, n + n'); (m, n) + (m, n')),$$

with $m, m' \in M$, $n.n' \in N$. (e.g. [40], [67], [2], [8]).

Any object $R \in \textsc{Mon}$ defines a functor $R \otimes - : \textsc{Mon} \to \textsc{Mon}$, $X \mapsto R \otimes X$, and it is called a *semiring* if this functor allows for a monad structure, that is, an (associative) product $m : R \otimes R \to R$ and a unit $\iota : \mathbb{N}_0 \to R$. A *morphism* $f : R \to S$ of semirings is a map respecting the defining operations.

A *left R-semimodule* is a module for the functor $R \otimes - : \textsc{Mon} \to \textsc{Mon}$, that is, an $M \in \textsc{Mon}$ together with a morphism $\rho : R \otimes M \to M$ in $\textsc{Mon}$, and *morphism of left R-semimodules* $M \to N$ are the module morphisms for $R \otimes -$, that is, monoid morphisms respecting the scalar multiplication, we denote them by $\mathrm{Hom}_R(M, N)$. These data define the category ${}_R\textsc{Mon}$ of left R-semimodules.

Right R-semimodules and their category $\textsc{Mon}_R$ are derived from the functor $- \otimes R : \textsc{Mon} \to \textsc{Mon}$.

Given two semirings R and S, a commutative monoid U with a left R-semimodule and a right S-module structure is said to be an (R, S)*-bisemimodule* provided these actions commute.

The tensor product between a right R-semimodule $\varrho : M \otimes R \to M$ and a left R-semimodule $\varrho' : R \otimes N \to N$ is defined by the coequaliser in $\textsc{Mon}$,

$$\begin{tikzcd} M \otimes R \otimes N \arrow[r, shift left, "\varrho \otimes N"] \arrow[r, shift right, "M \otimes \varrho'"'] & M \otimes N \arrow[r] & M \otimes_R N. \end{tikzcd}$$

If R is a commutative semiring, left R-semimodules M may be considered as right R-modules by putting $rm = mr$, for $m \in M$, $r \in R$. Then $M \otimes_R N$ obtains a canonical structure as an R-semimodule.

For an (R, S)-bisemimodule U, the functors

$$U \otimes_S - : {}_S\textsc{Mon} \to {}_R\textsc{Mon} \quad \text{and} \quad \mathrm{Hom}_R(U, -) : {}_R\textsc{Mon} \to {}_S\textsc{Mon}$$

are adjoint by the (canonical) bijection, for $M \in {}_S\textsc{Mon}$, $N \in {}_R\textsc{Mon}$,

$$\mathrm{Hom}_R(U \otimes_S M, N) \to \mathrm{Hom}_S(M, \mathrm{Hom}_R(U, N)), \ h \mapsto [m \mapsto h(- \otimes m)].$$

Let R be any semiring. An (R, R)-bisemimodule A yields an endofunctor $A \otimes_R - : {}_R\textsc{Mon} \to {}_R\textsc{Mon}$, and A is said to be an *R-semiring* provided this functor allows for a monad structure. Similarly, an (R, R)-bisemimodule C is called an *R-semicoring* if this functor allows for a comonad structure.

In case R is a commutative semiring, left (right) R-semimodules are considered as R-bimodules (as mentioned above), and the R-semirings are called *R-semialgebras* and R-semicorings are named *R-coalgebras*.

The categorical background leads to the definitions and properties of the corresponding categories of modules and comodules, respectively, and the appropriate free and forgetful functors.

5.1. **Definitions.** Let R be any semiring. An R-semiring A is called *separable* (*Frobenius)* if the forgetful functor $U_A : {}_A\text{Mon} \to {}_R\text{Mon}$ is separable (Frobenius). An R-semicoring C is called *coseparable* (*Frobenius*) if the forgetful functor $U^C : {}^C\text{Mon} \to {}_R\text{Mon}$ is separable (Frobenius).

Properties of theses structures can be derived from the results in Section 2. Further investigation on these notions may be of interest. Separable and central separable cancellative R-semialgebras (R a commutative semiring) are considered by R.P. Deore e.a. in [23], [24].

In the setting considered above, the definition of bimonads and Hopf monads in [50] provides the definition of *bisemialgebras* and *Hopf semialgebras*. This is worked out by J. Abuhlail and N. Al-Sulaiman in [3] where also separability for Hopf semialgebras is discussed. Semialgebras, semi-coalgebras and bi-semialgebras are used by J. Worthington as tools for automata theory in [96]. The Sweedler dual of a bialgebra over semirings is considered by G.H.E. Duchamp and C. Tollu in [25]. An extension of the Myhill-Nerode theorem to base semirings is subject of [49].

5.2. **Near-semirings.** For a set S, a quadruple $(S, +, \cdot, 0)$ is called a *(right) near-semiring* (also *semi-nearring*) if

(i) $(S, +, 0)$ is a monoid, $(S, \cdot\,)$ is a semigroup,

(ii) $0 \cdot a = 0$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.

A near-semiring $(S, +, \cdot 0)$ is a *nearring* if $(S, +, 0)$ is a (not necessarily commutative) group, and it is a *semiring*, if $(S, +, 0)$ is commutative and, in addition, $a \cdot 0 = 0$ and $c(a + b) = ca + cb$ for all $a, b, c \in S$.

The interest in these structures comes from the following observation: for any monoid $(N, +, 0)$, the maps from N to N, write $\text{Map}(N, N)$, form a near-semiring with respect to pointwise addition and composition of mappings. If $(N, +, 0)$ is a group, then $\text{Map}(N, N)$ is a nearring. Note that the terminology may differ in the literature.

These notions are also of interest in automata theory and computer science. To get an adjount pair of functors between categories of interest observe that for an object A in any category $\mathbb{A}$, there is a functor $\text{Mor}_{\mathbb{A}}(A, -) : \mathbb{A} \to \text{Set}$. A functor $T : \text{Set} \to \mathbb{A}$ that is left adjoint to it exists, provided for every family of copies of A there is a coproduct in $\mathbb{A}$. Based on this observation, a generalisation of nearrings and related tensor products are considered in [29]. Representations of near-semirings are investigated in [43].

6. Application to S-acts

In the preceding section we considered semiring actions on commutative monoids. More generally, one may also study the action of monoids on any set. This is the appropriate setting for the theory of *automata* and we recall the basic definitions (e.g. [41]).

Let S be any monoid, that is, a set S with associative product $m : S \times S \to S$ and unit element 1_S. Then the endofunctor

$$S \times - : \text{SET} \to \text{SET}, \quad X \mapsto S \times X,$$

is a monad on SET. The (Eilenberg-Moore) modules of this functor (see 7.5) are called *(left) S-acts*. These are sets A with an associative and unital S-action

$$\varrho : S \times A \to A, \ (s, a) \mapsto sa.$$

We denote the category of S-acts by ${}_S\text{SET}$.

Induced by $S \times -$, we get the free and forgetful functors

$$\begin{aligned} \phi_S : \text{SET} \to {}_S\text{SET}, &\quad X \mapsto (S \times X, m_X), \\ U_S : {}_S\text{SET} \to \text{SET}, &\quad (X, \varrho) \mapsto X. \end{aligned}$$

Right S-acts are determined by the endofunctor $- \times S : \text{SET} \to \text{SET}$ and yield the category SET_S. For monoids S, T, (S, T)*-biacts* are given by commuting left S- and right T-actions.

In the literature, S-acts also show up under the name *S-sets, S-polygons, transition system, S-automata*, indicating the area where they are of interest.

Given any $(A, \varrho) \in \text{SET}_S$, $(B, \varrho') \in {}_S\text{SET}$, their *tensor product* is defined as the coequaliser in SET,

$$A \times S \times B \underset{A\times\varrho'}{\overset{\varrho\times B}{\rightrightarrows}} A \times B \xrightarrow{\ \tau\ } A \otimes_S B,$$

and thus is characterised by the universal property: for any set Y and S-balanced map $\beta : A \times B \to Y$ (i.e., $\beta(as, b) = \beta(a, sb)$, for $a \in A$, $b \in B$, $s \in S$), there is a unique map $\widetilde{\beta} : A \otimes_S B \to Y$ with commutative diagram

$$\begin{tikzcd} A \times B \arrow[r, "\beta"] \arrow[d, "\tau"'] & Y \\ A \otimes_S B \arrow[ur, "\widetilde{\beta}"'] & \end{tikzcd} .$$

Evidently, the formalism and the basic properties for this tensor product are the same as for the tensor product for modules and semimodules.

Given two monoids S and T, any (S, T)-biact A induces the adjoint pair of functors

$$\begin{aligned} A \otimes_T - : {}_T\text{SET} \to {}_S\text{SET}, &\quad X \mapsto A \otimes_T X, \\ \text{Hom}_S(A, -) : S_{\text{SET}} \to {}_T\text{SET}, &\quad Y \mapsto \text{Hom}_S(A, Y), \end{aligned}$$

with the canonical bijection

$$\text{Hom}_S(A \otimes_T X, Y) \to \text{Hom}_T(X, \text{Hom}_S(A, Y)), \ h \mapsto [x \mapsto h(- \otimes x)].$$

6.1. **Remarks.** For any monoid $(S, m, 1_S)$, $S \times -$ is a monad with a coproduct given by $d : S \to S \times S$, $s \mapsto (s, s)$. An easy argument shows that (S, m, d) does not satisfy the Frobenius conditions unless S consists only of one element. Furthermore, $m \cdot d = 1_S$ if and only if all elements of S are idempotent, i.e., S is a Boolean monoid.

On the other hand, $d : S \to S \times S$ respects the product and thus m and d are compatible in the sense of 7.8. Also, the map $\varepsilon : S \to 1_{\text{SET}}$, $s \mapsto [\omega]$, where $[\omega]$ is a singleton, makes (S, d, ε) a comonad such that $(S, m, 1_S, d, \varepsilon)$ becomes a bimonad. This is a Hopf monad if and only if the monoid S is a group (e.g. [92, 5.19]).

For the use of distributive laws in the theory of automata we refer to [17].

7. Appendix: Categorical background

For the convenience of the reader and to fix notation we recall some basic notions from category theory.

7.1. **Categories**. A *category* $\mathbb{A}$ consists of a class of objects $\mathrm{Obj}(\mathbb{A})$, and for any objects A, B, C, there are *morphism sets* $\mathrm{Mor}_{\mathbb{A}}(A, B)$ and $\mathrm{Mor}_{\mathbb{A}}(B, C)$ with associative *composition*

$$\mathrm{Mor}_{\mathbb{A}}(A, B) \times \mathrm{Mor}_{\mathbb{A}}(B, C) \to \mathrm{Mor}_{\mathbb{A}}(A, C), \ (f, g) \mapsto gf.$$

Sometimes the composition gf is also denoted by $g \cdot f$ or $g \circ f$.

A set $\mathrm{Mor}_{\mathbb{A}}(A, B)$ may be empty, except for $A = B$, since $\mathrm{Mor}_{\mathbb{A}}(B, B)$ always should contain an *identity morphism* 1_B satisfying $g1_B = g$ and $1_B f = f$, for $g \in \mathrm{Mor}_{\mathbb{A}}(B, C)$, $f \in \mathrm{Mor}_{\mathbb{A}}(A, B)$.

It is customary to write $A \in \mathbb{A}$ instead of $A \in \mathrm{Obj}(\mathbb{A})$, and $\mathrm{Mor}(B, C)$ instead of $\mathrm{Mor}_{\mathbb{A}}(B, C)$ if no uncertainty can arise.

The connection between two categories is given by

7.2. **Functors.** A *covariant functor* $F : \mathbb{A} \to \mathbb{B}$ between two categories consists of assignments of $\mathrm{Obj}(\mathbb{A}) \to \mathrm{Obj}(\mathbb{B})$, $A \mapsto F(A)$, and of morphisms $\mathrm{Mor}(A, B) \to \mathrm{Mor}(A, B)$, $f \mapsto F(f)$, such that

$$F(1_A) = 1_{F(A)} \quad \text{and} \quad F(fg) = F(f)F(g).$$

By definition, one has a map of sets,

$$\Phi^F_{A,A'} : \mathrm{Mor}_{\mathbb{A}}(A, A') \to \mathrm{Mor}_{\mathbb{B}}(F(A), F(A')),$$

and the functor F is called *faithful* if $\Phi^F_{A,A'}$ is injective, *full* if $\Phi^F_{A,A'}$ is surjective, and *fully faithful* if $\Phi^F_{A,A'}$ is bijective.

Contravariant functors F reverse the composition of morphisms, that is, $F(fg) = F(g)F(f)$

The relation between two functors is described by

7.3. **Natural transformations.** Let $F, F' : \mathbb{A} \to \mathbb{B}$ be covariant functors. A *natural transformation* $\alpha : F \to F'$ is given by a family of morphisms

$$\alpha_A : F(A) \to F'(A) \text{ in } \mathbb{B},\ A \in \mathrm{Obj}(\mathbb{A}),$$

such that any $f : A \to B$ in $\mathbb{A}$ induces commutativity of the diagram in $\mathbb{B}$,

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ F'(A) \arrow[r, "F'(f)"] & F'(B)\,. \end{tikzcd}$$

7.4. **Adjoint pairs of functors.** A pair (L, R) of functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ between categories $\mathbb{A}$ and $\mathbb{B}$ is called *adjoint* if there are bijections, natural in $A \in \mathrm{Obj}(\mathbb{A})$ and $B \in \mathrm{Obj}(\mathbb{B})$,

$$\vartheta_{A,B} : \mathrm{Mor}_{\mathbb{B}}(L(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, R(B)).$$

Such a pair can be characterised by the associated natural transformations

$$\textit{unit} \ \ \eta : 1_{\mathbb{A}} \to RL \quad \text{and} \quad \textit{counit} \ \ \varepsilon : LR \to 1_{\mathbb{B}}.$$

with the *trianglular identities*

$$R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R = 1_R, \quad L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L = 1_L.$$

Vector spaces with products, that is, algebras, lead to the consideration of functors with products. The products here are given by natural transformations and the rules for them are taken from the relevant properties for algebras.

7.5. Monads and their modules. A *monad* on $\mathbb{A}$ is a triple $\mathbb{F} = (F, m, \iota)$, where $F : \mathbb{A} \to \mathbb{A}$ is a functor and

$$m : FF \to F, \quad \iota : 1_{\mathbb{A}} \to F,$$

are natural transformations with commutative diagrams

$$\begin{tikzcd} FFF \arrow[r, "m_F"] \arrow[d, "Fm"'] & FF \arrow[d, "m"] \\ FF \arrow[r, "m"'] & F\,, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\iota F"] \arrow[d, "F\iota"'] \arrow[dr, "="] & FF \arrow[d, "m"] \\ FF \arrow[r, "m"'] & F\,. \end{tikzcd}$$

An $\mathbb{F}$-*module* is an object A in $\mathbb{A}$ with a morphism $\varrho_A : F(A) \to A$ inducing commutative diagrams

$$\begin{tikzcd} FF(A) \arrow[r, "m_A"] \arrow[d, "F\varrho_A"'] & F(A) \arrow[d, "\varrho_A"] \\ F(A) \arrow[r, "\varrho_A"'] & A\,, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[dr, "1_A"'] & F(A) \arrow[d, "\varrho_A"] \\ & A\,. \end{tikzcd}$$

Morphisms between two F-modules (A, ϱ) and (A', ϱ') are given by a morphism $h : A \to A'$ in $\mathbb{A}$ implying commutativity of the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(h)"] \arrow[d, "\varrho"'] & F(A') \arrow[d, "\varrho'"] \\ A \arrow[r, "h"] & A'. \end{tikzcd}$$

With this morphisms, the F-modules form a category, denoted by $\mathbb{A}_F$.

Clearly, for any object A, $F(A)$ has a an F-module structure given by $m_A : FF(A) \to F(A)$ and for any morphism h in $\mathbb{A}$, $F(h)$ is an F-module morphism. This leads to the *free functor* and the *forgetful functor*,

$$\phi_F : \mathbb{A} \to \mathbb{A}_{\mathbb{F}},\ A \ \mapsto\ (F(A), m_A), \quad U_F : \mathbb{A}_{\mathbb{F}} \to \mathbb{A},\ (A, \varrho) \mapsto A,$$

and (ϕ_F, U_F) form an adjoint pair by the bijections for $A \in \mathrm{Obj}(\mathbb{A})$ and $B \in \mathrm{Obj}(\mathbb{A}_{\mathbb{F}})$,

$$\mathrm{Mor}_{\mathbb{A}_F}(F(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, U_F(B)), \quad f \mapsto f \circ \iota_A.$$

Notice that $U_F \phi_F = F$.

7.6. Comonads and their comodules. A *comonad* is a triple $\mathbb{G} = (G, \delta, \varepsilon)$, where $G : \mathbb{A} \to \mathbb{A}$ is a functor with *coproduct* and *counit*, that is, natural transformations

$$\delta : G \to GG, \quad \varepsilon : G \to 1_{\mathbb{A}},$$

with commuting diagrams

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & GG \arrow[d, "G\delta"] \\ GG \arrow[r, "\delta_G"] & GGG\,, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, "="] & GG \arrow[d, "\varepsilon_G"] \\ GG \arrow[r, "G\varepsilon"'] & G. \end{tikzcd}$$

A $\mathbb{G}$-*comodule* is an object $A \in \mathrm{Obj}(\mathbb{A})$ with a morphism $\rho^A : A \to G(A)$ in $\mathbb{A}$, inducing commutativity of the diagrams

$$\begin{tikzcd} A \arrow[r, "\rho^A"] \arrow[d, "\rho^A"'] & G(A) \arrow[d, "\delta_A"] \\ G(A) \arrow[r, "G\rho^A"] & GG(A)\,, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\rho^A"] \arrow[dr, "1_A"'] & G(A) \arrow[d, "\varepsilon_A"] \\ & A. \end{tikzcd}$$

Morphisms between comodules (A, ρ), (A', ρ') are morphisms $h : A \to A'$ in $\mathbb{A}$ with commutative diagrams

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\rho"'] & A' \arrow[d, "\rho'"] \\ G(A) \arrow[r, "G(h)"] & G(A'). \end{tikzcd}$$

The resulting category of G-comodules is denoted by $\mathbb{A}^G$.

Similar to the module case, for any $A \in \mathbb{A}$, $(G(A), \delta_A)$ is a G-comodule and one has the *(cofree) functor* and the *forgetful functor*,

$$\phi^G : \mathbb{A} \to \mathbb{A}^G,\ A \ \mapsto\ (G(A),\ \delta_A), \quad U^G : \mathbb{A}^G \to \mathbb{A},\ (A, \rho) \mapsto A,$$

and (U^G, ϕ^G) form an adjoint pair of functors by the bijections

$$\mathrm{Mor}_{\mathbb{A}^G}(B, G(A)) \to \mathrm{Mor}_{\mathbb{A}}(U^G(B), A), \quad f \mapsto \varepsilon_A \circ f,$$

for any $A \in \mathrm{Obj}(\mathbb{A})$ and $B \in \mathrm{Obj}(\mathbb{A}^G)$. Notice that $U^G \phi^G = G$.

Composing two monads one expects to get - under certain conditions - again a monad. New structures arise when a monad is combined with a comonad. For this one considers

7.7. Mixed distributive laws. Let (F, m, ι) be a monad and (G, δ, ε) a comonad on the category $\mathbb{A}$. Then a natural transformation

$$\lambda : FG \to GF$$

is said to be *mixed distributive law* or *entwining* provided it induces commutativity of the diagrams

$$\begin{tikzcd} FFG \arrow[rr, "m_G"] \arrow[d, "F\lambda"'] & & FG \arrow[d, "\lambda"] \\ FGF \arrow[r, "\lambda_F"] & GFF \arrow[r, "Gm"] & GF, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\iota_G"] \arrow[dr, "G\iota"'] & FG \arrow[d, "\lambda"] \\ & GF, \end{tikzcd}$$

$$\begin{tikzcd} FG \arrow[r, "F\delta"] \arrow[d, "\lambda"'] & FGG \arrow[r, "\lambda_G"] & GFG \arrow[d, "G\lambda"] \\ GF \arrow[rr, "\delta_F"] & & GGF, \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "F\varepsilon"] \arrow[d, "\lambda"'] & F \\ GF \arrow[ur, "\varepsilon_F"'] & \end{tikzcd} .$$

In this setting, it is of interest to consider *mixed bimodules*, that is, objects $A \in \mathbb{A}$ with an F-module structure $\varrho : F(A) \to A$ and a G-comodule structure $\omega : A \to G(A)$ with commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "\varrho"] \arrow[d, "F(\omega)"'] & A \arrow[r, "\omega"] & G(A) \\ FG(A) \arrow[rr, "\lambda_A"] & & GF(A). \arrow[u, "G(\varrho)"'] \end{tikzcd}$$

Morphisms between two mixed bimodules are required to be F-module and G-comodule morphisms.

Formally one can also consider entwinings $GF \to FG$. These, however, do not relate in the same way to mixed bimodules (involves Kleisli categories, e.g. [12]).

Of particular interest are endofunctors which are both a monad as well as a comonad. In this case the question for the compatibility of the two structure arises.

7.8. **Bimonads.** Let $B : \mathbb{A} \to \mathbb{A}$ be a functor allowing for a monad (B, m, ι), a comonad (B, δ, ε), and a mixed distributive law $\lambda : BB \to BB$. These data are called a *bimonad* provided they induce commutativity of the diagram

$$\begin{array}{ccccc} BB & \xrightarrow{m} & B & \xrightarrow{\delta} & BB \\ {\scriptstyle B\delta}\downarrow & & & & \uparrow{\scriptstyle Bm} \\ BBB & & \xrightarrow{\quad\lambda_B\quad} & & BBB. \end{array} \tag{7.1}$$

It is customary to call the mixed (B, B)-bimodules *Hopf modules* and we denote them by $\mathbb{A}_B^B$. Commutativity of (7.1) implies that, for every $A \in \mathbb{A}$, $B(A)$ is a Hopf module with

$$\text{coaction } \delta_A : B(A) \to BB(A) \text{ and action } m_A : BB(A) \to B(A).$$

Thus one obtains a functor

$$\phi_B^B : \mathbb{A} \to \mathbb{A}_B^B, \quad A \mapsto (B(A), m_A, \delta_A).$$

A natural transformation $S : B \to B$ is called an *antipode* if

$$m \cdot SB \cdot \delta = 1_B \cdot \varepsilon = m \cdot BS \cdot \delta,$$

and the bimonad (B, m, δ, λ) is called a *Hopf monad* provided such an antipode exists. Under mild restrictions, this is the case if and only if ϕ_B^B is an equivalence of categories (e.g. [92], [50, 5.6]).

Over a commutative ring R, an R-bialgebra H is a Hopf algebra provided the bimonad $H \otimes_R -$ is a Hopf monad. If the Picard group of the ring R is zero, then any Hopf algebra H with H finitely generated and projective as an R-module, has also the structure of a Frobenius algebra (see [66]).

7.9. **Remarks.** The notion of distributive laws (of mixed type) goes back to Beck [9] and we refer to [92, 12, 15] and the references given there for further information. In more general situations, (simple) distributive laws are used by B. Klin in [42] in his study of operational semantics. He shows how *stream systems* and *Mealy machines* can be described by comodules and distributive laws.

For any monoid S, the monoid algebra $K[S]$ over a field K is a bialgebra and provides the setting for the *Myhill-Nerode Theorem* in the theory of formal languages and (finite) automata (e.g., [64], [77]).

Monads and comonads are closely related to adjoint pairs of functors (e.g. [27]):

7.10. **Adjoint pairs and (co)monads.** Let $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ be an adjoint pair of functors (see 7.4) with

$$\textit{unit } \eta : 1_{\mathbb{A}} \to RL \quad \text{and} \quad \textit{counit } \varepsilon : LR \to 1_{\mathbb{B}},$$

Then $RL : \mathbb{A} \to \mathbb{A}$ has a monad structure with

$$\textit{product } \mu = R\varepsilon_L : RLRL \to RL \;\text{ and } \textit{unit } \eta : 1_{\mathbb{A}} \to RL,$$

and $LR : \mathbb{B} \to \mathbb{B}$ has a comonad structure with

$$\textit{coproduct } \delta = L\eta_R : LR \to LRLR \;\text{ and } \textit{counit } \varepsilon : LR \to 1_{\mathbb{B}}.$$

Our view on monads and comonads may suggest to consider *comonads* as *dual* to *monads* and vice versa. As a special case one has the dual A^* of a finite dimensional algebra A. However, this only works because of finite dimension since then the endofunctors $A^* \otimes_K -$ and $\mathrm{Hom}_K(A,-)$ are isomorphic. Without finite dimension we nevertheless get that the functor $\mathrm{Hom}_K(A,-)$ is a comonad. This can be formulated in full generality.

7.11. **Adjoint endofunctors.** Let (F,G) be an adjoint pair of endofunctors on a category $\mathbb{A}$ with bijection

$$\varphi_{X,Y} : \mathrm{Mor}_{\mathbb{A}}(F(X),Y) \to \mathrm{Mor}_{\mathbb{A}}(X,G(Y)),$$

and $\eta : 1_{\mathbb{A}} \to GF$, $\varepsilon : FG \to 1_{\mathbb{A}}$ as unit and counit.

Assume (F,m,ι) to be a monad. Then, for $X,Y \in \mathbb{A}$, there are diagrams

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{A}}(F(X),Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "\mathrm{Mor}(m_X,Y)"'] & \mathrm{Mor}_{\mathbb{A}}(X,G(Y)) \arrow[d, dotted, "\mathrm{Mor}(X,?)"] \\
\mathrm{Mor}_{\mathbb{A}}(FF(X),Y) \arrow[r, "\simeq"] & \mathrm{Mor}_{\mathbb{A}}(X,GG(Y)),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{A}}(F(X),Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "\mathrm{Mor}(\iota_X,Y)"'] & \mathrm{Mor}_{\mathbb{A}}(X,G(Y)) \arrow[dl, dotted, "\mathrm{Mor}(X,??)"] \\
\mathrm{Mor}_{\mathbb{A}}(X,Y) & ,
\end{tikzcd}$$

in which the dotted morphisms exist by composition of the other morphisms (φ is invertible). By the *Yoneda Lemma* it follows that they are induced by morphisms

$$\underline{\delta}_Y : G(Y) \to GG(Y) \quad \text{and} \quad \underline{\varepsilon}_Y : G(Y) \to Y,$$

and these are explicitly given by the natural transformations

$$\underline{\delta} : G \xrightarrow{\eta G} GFG \xrightarrow{G\eta FG} GGFFG \xrightarrow{GGmG} GGFG \xrightarrow{GG\varepsilon} GG,$$
$$\underline{\varepsilon} : G \xrightarrow{eG} FG \xrightarrow{\varepsilon} 1_{\mathbb{A}}\,,$$

yielding a comonad $(G,\underline{\delta},\underline{\varepsilon})$.

Based on this kind of arguments one obtains (e.g. [12, 2.8]):

7.12. **Theorem.** *Let (F,G) be an adjoint pair of endofunctors on $\mathbb{A}$ with unit $\eta : 1_{\mathbb{A}} \to GF$ and counit $\varepsilon : FG \to 1_{\mathbb{A}}$. Then*

(1) *F has a monad structure if and only if G allows for a comonad structure. In this case, the category of F-modules is isomorphic to the category of G-comodules by the functors*

$$Q : \mathbb{A}_F \to \mathbb{A}^G, \quad F(A) \xrightarrow{h} A \;\mapsto\; A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A),$$
$$Q^{-1} : \mathbb{A}^G \to \mathbb{A}_F, \quad A \xrightarrow{\rho} G(A) \;\mapsto\; F(A) \xrightarrow{F(\rho)} FG(A) \xrightarrow{\varepsilon_A} A.$$

(2) *F has a comonad structure if and only if G allows for a monad structure. In this case the corresponding Kleisli categories are isomorphic.*

More about these structures may be found, for example, in [12, 92]. For an explicit outline for the Hom-tensor functors in module categories in this context we refer to [94].

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