WEAK FROBENIUS MONADS AND FROBENIUS BIMODULES

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Abstract. As observed by Eilenberg and Moore (1965), for a monad F with right adjoint comonad G on any category $\mathbb{A}$, the category of unital F-modules $\mathbb{A}_F$ is isomorphic to the category of counital G-comodules $\mathbb{A}^G$. The monad F is Frobenius provided we have $F = G$ and then $\mathbb{A}_F \simeq \mathbb{A}^F$. Here we investigate which kind of isomorphisms can be obtained for non-unital monads and non-counital comonads. For this we observe that the mentioned isomorphism is in fact an isomorphisms between $\mathbb{A}_F$ and the category of bimodules $\mathbb{A}_F^F$ subject to certain compatibility conditions (Frobenius bimodules). Eventually we obtain that for a weak monad (F, m, η) and a weak comonad (F, δ, ε) satisfying $Fm \cdot \delta F = \delta \cdot m = mF \cdot F\delta$ and $m \cdot F\eta = F\varepsilon \cdot \delta$, the category of compatible F-modules is isomorphic to the category of compatible Frobenius bimodules and the category of compatible F-comodules.

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Introduction

A monad (F, m, η) on a category $\mathbb{A}$ is called a *Frobenius monad* provided the functor F is (right) adjoint to itself (e.g. Street [6]). Then F also allows for a comonad structure (F, δ, ε) and the (Eilenberg-Moore) category $\mathbb{A}_F$ of F-modules is isomorphic to the category $\mathbb{A}^F$ of F-comodules. As shown in [5, Theorem 3.13], this isomorphism characterises a functor

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with monad and comonad structure as Frobenius monad. It is not difficult to see that the categories $\mathbb{A}_F$ and $\mathbb{A}^F$ are in fact isomorphic to the category $\mathbb{A}_F^F$ of (unital and counital) *Frobenius bimodules.* In this setting units and counits play a crucial role.

Here we are concerned with the question what is left from these correspondences when the conditions on units and counits are weakened. An elementary approach to this setting is offered in [7] and [8] where adjunctions between functors are replaced by *regular pairings* (L, R) of functors $L : \mathbb{A} \to \mathbb{B}$, $R : \mathbb{B} \to \mathbb{A}$ (see 1.4). The composition LR (resp. RL) yields endofunctors on $\mathbb{A}$ (resp. $\mathbb{B}$) and these are closely related to *weak (co)monads* as considered by Böhm et al. in [1, 3] (see Remark 1.12). In Section 1 we recall the definitions and collect basic results needed for our investigations.

Given a non-unital monad (F, m) on any category $\mathbb{A}$, a non-unital module $\varrho : F(A) \to A$ is called *firm* (see [2]) if the defining fork

$$FF(A) \underset{F(\varrho)}{\overset{m_A}{\rightrightarrows}} F(A) \xrightarrow{\varrho} A$$

is a coequaliser in the category of non-unital F-modules. This notion is generalised in Section 2 by restricting the coequaliser requirement to certain classes $\mathbb{K}$ of morphisms of F-modules. It turns out that compatible modules of a weak monad (F, m, η) satisfy the resulting conditions for a suitable class $\mathbb{K}$ (Proposition 2.10). Similar results hold for weak comonads.

In Section 3, we return to parings of the functors L and R. Given natural transformations $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{B}}$, one obtains a non-unital monad $(LR, L\widetilde{\varepsilon}R)$ and a non-counital comonad $(LR, L\eta R)$ on $\mathbb{B}$ for which the Frobenius condition is satisfied and this motivates the definition of Frobenius bimodules (see 3.1). Given a non-counital LR-comodule $\omega : B \to LR(B)$, the question arises when it can be extended to a Frobenius bimodule by some $\varrho : LR(B) \to B$. As sufficient condition it turns out that the defining cofork for ϱ is a coequaliser in the category of non-counital comodules (see Proposition 3.6). Further situations are investigated, in particular for regular pairings (Theorems 3.9, 3.10).

In Section 4, the results about the pairings (L, R) from Section 3 are reformulated for the (co)monad LR, that is, we consider an endofunctor F on $\mathbb{B}$ endowed with a weak monad structure (F, m, η), a weak comonad structure (F, δ, ε), and the compatibility between m and δ is postulated as the Frobenius property (see 4.1). (For $L\eta R$ and $L\widetilde{\varepsilon}R$ the latter follows by naturality, see (3.1)). The constructions lead to various functors between (compatible) module, comodule and bimodule categories (see 4.2, 4.3, 4.6). For proper (co)monads we get some results obtained by Böhm and Gómez-Torrecillas in [2] as Corollaries 4.7, 4.8.

1. Regular pairings

Throughout $\mathbb{A}$ and $\mathbb{B}$ will denote any categories. The symbols I_A, A, or just I will stand for the identity morphism on an object A, I_F or F denote the identity transformation on the functor F, and $I_{\mathbb{A}}$ means the identity functor on $\mathbb{A}$.

Given an endofunctor T on $\mathbb{A}$, an idempotent natural transformation $e : T \to T$ is said to *split* if there are an endofunctor $\underline{T}$ on $\mathbb{A}$ and natural transformations $p : T \to \underline{T}$ and $i : \underline{T} \to T$ such that $e = i \cdot p$ and $p \cdot i = I_{\underline{T}}$.

We recall some notions from [7], [8], [3].

1.1. Non-counital comodules. Let (G, δ) be a pair with an endofunctor $G : \mathbb{A} \to \mathbb{A}$ and a coassociative natural transformation (coproduct) $\delta : G \to GG$. Then (non-counital) G-comodules are defined as objects $A \in \mathbb{A}$ with a morphism $\upsilon : A \to G(A)$ satisfying $G(\upsilon) \cdot \upsilon = \delta_A \cdot \upsilon$ and the category of these G-comodules is denoted by $\underrightarrow{\mathbb{A}}^G$.

Consider a triple (G, δ, ε), with (G, δ) a pair as above and $\varepsilon : G \to I_{\mathbb{A}}$ any natural transformation (quasi-counit). Then a G-comodule (A, υ) is said to be *compatible* provided $\upsilon = G\varepsilon_A \cdot \delta_A \cdot \upsilon$. The full subcategory of $\underrightarrow{\mathbb{A}}^G$ consisting of compatible comodules is denoted by $\underline{\mathbb{A}}^G$.

(G, δ, ε) is called a *weak comonad* if

$$\varepsilon = \varepsilon \cdot G\varepsilon \cdot \delta, \;\; \delta = G\varepsilon G \cdot G\delta \cdot \delta, \quad \text{and} \quad G\varepsilon \cdot \delta = \varepsilon G \cdot \delta.$$

Then a G-comodule (A, υ) is compatible if $\varepsilon G_A \cdot \delta_A \cdot \upsilon = \upsilon = \upsilon \cdot \varepsilon_A \cdot \upsilon$. Furthermore, $G\varepsilon \cdot \delta : G \to G$ is idempotent and in case this is split by $G \xrightarrow{p} \underline{G} \xrightarrow{i} G$, one obtains a comonad $(\underline{G}, \underline{\delta}, \underline{\varepsilon})$ by putting

$$\underline{\delta} : \underline{G} \xrightarrow{i} G \xrightarrow{\delta} GG \xrightarrow{pp} \underline{GG}, \quad \underline{\varepsilon} : \underline{G} \xrightarrow{i} G \xrightarrow{\varepsilon} I_{\mathbb{A}}.$$

1.2. Non-unital modules. Let (F, m) be a pair with an endofunctor $F : \mathbb{A} \to \mathbb{A}$ and an associative natural transformation (product) $m : FF \to F$. Then (non-unital) F-modules are defined as objects $A \in \mathbb{A}$ with a morphism $\varrho : F(A) \to A$ satisfying $\varrho \cdot F\varrho = \varrho \cdot m_A$ and the category of these F-modules is denoted by $\underrightarrow{\mathbb{A}}_F$.

Consider a triple (F, m, η), with (F, m) a pair as above and any natural transformation $\eta : I_{\mathbb{A}} \to F$ (quasi-unit). An F-module (A, ϱ) is said to be *compatible* provided $\varrho = \varrho \cdot m_A \cdot F\eta_A$ and the full subcategory of $\underrightarrow{\mathbb{A}}_F$ consisting of compatible modules is denoted by $\underline{\mathbb{A}}_F$.

(F, m, η) is called a *weak monad* if

$$\eta = m \cdot F\eta \cdot \eta, \;\; m = m \cdot mF \cdot F\eta F, \quad \text{and} \quad m \cdot F\eta = m \cdot \eta F.$$

Then an F-module (A, ϱ) is compatible if $\varrho \cdot m_A \cdot \eta F_A = \varrho = \varrho \cdot \eta_A \cdot \varrho$. Furthermore, $m \cdot F\eta : F \to F$ is idempotent and in case this is split by

$F \xrightarrow{p} \underline{F} \xrightarrow{i} F$, one obtains a monad $(\underline{F}, \underline{m}, \underline{\eta})$ by putting

$$\underline{m} : \underline{FF} \xrightarrow{ii} FF \xrightarrow{m} F \xrightarrow{p} \underline{F}, \quad \underline{\eta} : I_{\mathbb{A}} \xrightarrow{\eta} F \xrightarrow{p} \underline{F}.$$

1.3. **Pairings of functors.** For functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$, *pairings* are defined as maps, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\mathrm{Mor}_{\mathbb{B}}(L(A), B) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Mor}_{\mathbb{A}}(A, R(B)),$$

$$\mathrm{Mor}_{\mathbb{A}}(R(B), A) \underset{\widetilde{\beta}}{\overset{\widetilde{\alpha}}{\rightleftarrows}} \mathrm{Mor}_{\mathbb{A}}(B, L(A)) \, .$$

These - and their compositions - are determined by natural transformations obtained as images of the corresponding identity morphisms,

map	natural transformation	map	natural transformation
α	$\eta : I_{\mathbb{A}} \to RL,$	$\widetilde{\alpha}$	$\widetilde{\eta} : I_{\mathbb{B}} \to LR,$
β	$\varepsilon : LR \to I_{\mathbb{B}},$	$\widetilde{\beta}$	$\widetilde{\varepsilon} : RL \to I_{\mathbb{A}},$
$\beta \cdot \alpha$	$\ell : L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L$	$\widetilde{\beta} \cdot \widetilde{\alpha}$	$\widetilde{r} : R \xrightarrow{R\widetilde{\eta}} RLR \xrightarrow{\widetilde{\varepsilon}R} R$
$\alpha \cdot \beta$	$r : R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R$	$\widetilde{\alpha} \cdot \widetilde{\beta}$	$\widetilde{\ell} : L \xrightarrow{\widetilde{\eta}L} LRL \xrightarrow{L\widetilde{\varepsilon}} L$.

β (resp. α) is said to be *symmetric* if $Lr = \ell R$ (resp. $R\ell = rL$) (see [8, §3]). Under the given conditions (see [8]),

(i) $(LR, L\eta R, \varepsilon)$ is a non-counital comonad on $\mathbb{B}$ with quasi-counit ε;
(ii) $(RL, R\varepsilon L, \eta)$ is a non-unital monad on $\mathbb{A}$ with quasi-unit η;
(iii) $(LR, L\widetilde{\varepsilon}R, \widetilde{\eta})$ is a non-unital monad on $\mathbb{B}$ with quasi-unit $\widetilde{\eta}$;
(iv) $(RL, R\widetilde{\eta}L, \widetilde{\varepsilon})$ is a non-counital comonad on $\mathbb{A}$ with quasi-counit $\widetilde{\varepsilon}$.

Clearly, if α is a bijection, then (L, R) is an adjoint pair, if $\widetilde{\alpha}$ is a bijection, then (R, L) is an adjoint pair, and if α and $\widetilde{\alpha}$ are bijections, then LR and RL are Frobenius functors.

1.4. **Regular pairings.** A pairing (L, R, α, β) is said to be *regular* if

$$\alpha \cdot \beta \cdot \alpha = \alpha \text{ and } \beta \cdot \alpha \cdot \beta = \beta.$$

In this case, $\ell : L \to L$ and $r : R \to R$ (see 1.3) are idempotents and

$$\varepsilon = \varepsilon \cdot \ell r = \varepsilon \cdot \ell R = \varepsilon \cdot Lr,$$
$$\eta = r\ell \cdot \eta = R\ell \cdot \eta = rL \cdot \eta.$$

If β is symmetric, $\ell r = Lr = \ell R$; if α is symmetric, $r\ell = R\ell = rL$.

Assume the idempotents ℓ, r to be splitting, that is,

$$L \xrightarrow{\ell} L = L \xrightarrow{p} \underline{L} \xrightarrow{i} L, \quad R \xrightarrow{r} R = R \xrightarrow{p'} \underline{R} \xrightarrow{i'} R.$$

Then, for the natural morphisms

$$\underline{\eta} : \; I_{\mathbb{A}} \xrightarrow{\eta} RL \xrightarrow{p'p} \underline{RL}, \quad \underline{\varepsilon} : \; \underline{LR} \xrightarrow{ii'} LR \xrightarrow{\varepsilon} I_{\mathbb{B}},$$

one gets $\underline{\varepsilon L} \cdot \underline{L\eta} = I_{\underline{L}}$ and $\underline{R\varepsilon} \cdot \underline{\eta R} = I_{\underline{R}}$, hence yielding an adjunction $(\underline{L}, \underline{R}, \underline{\alpha}, \underline{\beta})$.

1.5. Proposition. *For functors* $\mathbb{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{B}$ *, there are equivalent:*

(a) (L, R) *allows for a regular pairing* (L, R, α, β) *with splitting idempotents* ℓ, r*;*

(b) *there are retractions* $\underline{L} \xrightarrow{i} L \xrightarrow{p} \underline{L}$ *and* $\underline{R} \xrightarrow{i'} R \xrightarrow{p'} \underline{R}$ *such that* $(\underline{L}, \underline{R})$ *allows for an adjunction.*

Proof. (a)⇒(b) The data from 1.4 yield an adjunction $(\underline{L}, \underline{R}, \underline{\alpha}, \underline{\beta})$ and the commutative diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{B}}(L(A), B) \arrow[r, "\alpha"] \arrow[d, "\mathrm{Mor}(i_A, B)"'] & \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \arrow[r, "\beta"] \arrow[d, "\mathrm{Mor}(A, p'_B)"] & \mathrm{Mor}_{\mathbb{B}}(L(A), B) \arrow[d, "\mathrm{Mor}(i_A, B)"] \\
\mathrm{Mor}_{\mathbb{B}}(\underline{L}(A), B) \arrow[r, "\underline{\alpha}"] & \mathrm{Mor}_{\mathbb{A}}(A, \underline{R}(B)) \arrow[r, "\underline{\beta}"] & \mathrm{Mor}_{\mathbb{B}}(\underline{L}(A), B).
\end{tikzcd}$$

(b)⇒(a) Given an adjunction $(\underline{L}, \underline{R}, \underline{\alpha}, \underline{\beta})$ and retracts $\underline{L} \xrightarrow{i} L \xrightarrow{p} \underline{L}$ and $\underline{R} \xrightarrow{i'} R \xrightarrow{p'} \underline{R}$, the above diagram tells us how to define (new) α and β to get commutativity. Then it is routine to check that (L, R, α, β) is a regular pairing and the resulting idempotents are split by (p, i) and (p', i'), respectively. □

Now assume that (L, R, α, β) and $(R, L, \widetilde{\alpha}, \widetilde{\beta})$ are regular pairings. Then ℓ, $\widetilde{\ell}$ are two natural transformations on L and r, $\widetilde{r}$ are two natural transformations on R. We are interested in the case when they coincide. Applying 1.5 and its dual yields:

1.6. Proposition. *For functors* $\mathbb{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{B}$ *, there are equivalent:*

(a) (L, R) *allows for regular pairings* (L, R, α, β) *and* $(R, L, \widetilde{\alpha}, \widetilde{\beta})$ *with splitting idempotents* $\ell = \widetilde{\ell}$, $r = \widetilde{r}$*;*

(b) *there are retractions* $\underline{L} \xrightarrow{i} L \xrightarrow{p} \underline{L}$ *and* $\underline{R} \xrightarrow{i'} R \xrightarrow{p'} \underline{R}$ *such that* $(\underline{L}, \underline{R})$ *and* $(\underline{R}, \underline{L})$ *allow for adjunctions, that is,* $(\underline{L}, \underline{R})$ *is a Frobenius pair of functors.*

1.7. Remark. Let (G, δ, ε) be a non-counital comonad on the category $\mathbb{A}$ with quasi-unit ε. For the Eilenberg-Moore category $\underrightarrow{\mathbb{A}}^G$ of non-counital G-comodules there are the free and the forgetful functors

$$\phi^G : \mathbb{A} \to \underrightarrow{\mathbb{A}}^G,\ A \mapsto (G(A), \delta_A), \qquad U^G : \underrightarrow{\mathbb{A}}^G \to \mathbb{A},\ (A, \omega) \mapsto A.$$

There is a pairing $(\phi^G, U^G, \alpha^G, \beta^G)$ with the maps, for $X \in \mathbb{A}$, $(A, \omega) \in \underrightarrow{\mathbb{A}}^G$,

$$\alpha^G : \mathrm{Mor}_{\mathbb{A}}(U^G(A), X) \to \mathrm{Mor}^G(A, \phi^G(X)), \quad f \mapsto G(f) \cdot \omega,$$
$$\beta^G : \mathrm{Mor}^G(A, \phi^G(X)) \to \mathrm{Mor}_{\mathbb{A}}(U^G(A), X), \quad g \mapsto \varepsilon_X \cdot g.$$

Compatible G-comodules $\upsilon : A \to G(A)$ are those with $\alpha^G \beta^G(\upsilon) = \upsilon$.

(G, δ, ε) is a weak comonad if and only if $(\phi^G, U^G, \alpha^G, \beta^G)$ is a regular pairing with β^G symmetric (see [8, Proposition 4.4]).

Similar characterisations hold for weak monads ([8, Proposition 3.4]).

1.8. **Related comonads.** *Let (L, R, α, β) be a regular pairing (see 1.4).*

(1) *For the coproduct*

$$\delta : LR \xrightarrow{L\eta R} LRLR \xrightarrow{\ell RLr} LRLR,$$

$(LR, \delta, \varepsilon)$ is a weak comonad on $\mathbb{B}$. If β is symmetric, $\delta = L\eta R$.

(2) *$\ell r : LR \to LR$ induces morphisms of non-counital comonads respecting the quasi-counits,*

$$(LR, L\eta R, \varepsilon) \to (LR, L\eta R, \varepsilon) \text{ and } (LR, L\eta R, \varepsilon) \to (LR, \delta, \varepsilon),$$

and an endomorphism of weak comonads $(LR, \delta, \varepsilon) \to (LR, \delta, \varepsilon)$.

Proof. Direct verification shows $\varepsilon LR \cdot \delta = \ell r = LR\varepsilon \cdot \delta$, the conditions for a weak comonad. For the next claims, consider the commutative diagram

$$\begin{tikzcd}
LR \arrow[r, "\ell r"] \arrow[d, "L\eta R"'] & LR \arrow[d, "L\eta R"'] \arrow[dr, "\delta"] & \\
LRLR \arrow[r, "\ell RLr"] \arrow[dr, "\ell r\ell r"'] & LRLR \arrow[r, "\ell RLr"] \arrow[d, "Lr\ell R"] & LRLR \arrow[d, "Lr\ell R"] \\
 & LRLR \arrow[r, "\ell RLr"'] & LRLR\,;
\end{tikzcd}$$

the left hand part proves the assertion about the first morphism and the outer paths show the properties of the second and third morphisms. □

1.9. **Related monads.** *Let (L, R, α, β) be a regular pairing (see 1.4).*

(1) *For the product*

$$m : RLRL \xrightarrow{rLR\ell} RLRL \xrightarrow{R\varepsilon L} RL,$$

(RL, m, η) is a weak monad on $\mathbb{A}$. If α is symmetric, $m = R\varepsilon L$.

(2) *$r\ell : RL \to RL$ yields morphisms of non-unital monads respecting the quasi-units,*

$$(RL, R\varepsilon L, \eta) \to (RL, R\varepsilon L, \eta) \text{ and } (RL, R\varepsilon L, \eta) \to (RL, m, \eta).$$

and an endomorphism of weak monads $(RL, m, \eta) \to (RL, m, \eta)$.

Proof. One easily verifies $m \cdot \eta RL = r\ell = m \cdot RL\eta$, the condition for a weak monad. The other claims are shown similarly to 1.8 $\square$

Combining the preceding observations we have shown:

1.10. **Proposition.** *Let* (L, R, α, β) *be a regular pairing and assume the idempotents* ℓ *and* r *to split. With the notation from* 1.4, $(\underline{LR}, \underline{L}\eta\underline{R}, \underline{\varepsilon})$ *is a comonad on* $\mathbb{B}$ *and* $(\underline{RL}, \underline{R}\varepsilon\underline{L}, \underline{\eta})$ *is monad on* $\mathbb{A}$. *Then,*

(1) *the natural transformation* $pp' : LR \to \underline{LR}$ *induces morphisms of non-counital comonads* $(LR, L\eta R, \varepsilon) \to (\underline{LR}, \underline{L}\eta\underline{R}, \underline{\varepsilon})$, *and morphisms of weak comonads* $(LR, \delta, \varepsilon) \to (\underline{LR}, \underline{L}\eta\underline{R}, \underline{\varepsilon})$;

(2) *the natural transformation* $p'p : RL \to \underline{RL}$ *induces morphisms of non-unital monads* $(RL, R\varepsilon L, \eta) \to (\underline{RL}, \underline{R}\varepsilon\underline{L}, \underline{\eta})$ *and morphisms of weak monads* $(RL, m, \eta) \to (\underline{RL}, \underline{R}\varepsilon\underline{L}, \underline{\eta})$.

1.11. **Regular pairings and comodules.** Let (L, R, α, β) be a regular pairing and consider the weak comonad $(LR, \delta, \varepsilon)$ defined in 1.8. Then a non-counital $(LR, \delta, \varepsilon)$-comodule (B, υ) is compatible (see 1.1) if $\upsilon = \varepsilon LR_B \cdot \delta_B \cdot \upsilon = r\ell_B \cdot \upsilon$.

Write $\overline{\mathbb{B}}^{LR,\delta}$ for the full subcategory of $\underrightarrow{\mathbb{B}}^{LR,\delta}$ formed by the compatible $(LR, \delta, \varepsilon)$-comodules. For any $B \in \mathbb{B}$, $(LR(B), \delta_B)$ is a compatible $(LR, \delta, \varepsilon)$-comodule, and thus we have a functor

$$\phi^{LR,\delta} : \mathbb{B} \to \overline{\mathbb{B}}^{LR,\delta}, \quad B \mapsto (LR(B), \delta_B).$$

The obvious forgetful functor $U^{LR,\delta} : \overline{\mathbb{B}}^{LR,\delta} \to \mathbb{B}$ need not be (left) adjoint to ϕ^{LR} but $(\phi^{LR}, U^{LR,\delta})$ allows for a regular pairing (see 1.7).

Denoting by $\underrightarrow{\mathbb{B}}^{LR,\eta}$ the non-counital comodules for $(LR, L\eta R, \varepsilon)$, the natural transformation $(LR, L\eta R, \varepsilon) \to (LR, \delta, \varepsilon)$ induced by ℓr (see 1.8) defines a functor $t_{\ell r} : \underrightarrow{\mathbb{B}}^{LR,\eta} \to \underrightarrow{\mathbb{B}}^{LR,\delta}$. It is easy to see that hereby the image of any comodule in $\underrightarrow{\mathbb{B}}^{LR,\eta}$ is a compatible comodule in $\underrightarrow{\mathbb{B}}^{LR,\delta}$ leading to a commutative diagram

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "\phi^{LR,\eta}"] \arrow[dr, "\phi^{LR,\delta}"'] & \underrightarrow{\mathbb{B}}^{LR,\eta} \arrow[d, "t_{\ell r}"] \\
& \overline{\mathbb{B}}^{LR,\delta}.
\end{tikzcd}$$

In case the idempotents ℓ and r are splitting, we get the splitting natural transformation $pp' : LR \to \underline{LR}$ (from 1.4) which induces functors $\overline{\mathbb{B}}^{LR,\delta} \to \overline{\mathbb{B}}^{\underline{LR}}$ and $\overline{\mathbb{B}}^{LR,\eta} \to \overline{\mathbb{B}}^{\underline{LR}}$, also denoted by pp', with commutative diagram

$$\begin{tikzcd}
\underrightarrow{\mathbb{B}}^{LR,\eta} \arrow[r, "t_{\ell r}"] \arrow[dr, "pp'"] & \overline{\mathbb{B}}^{LR,\delta} \arrow[d, "pp'"] \\
\mathbb{B} \arrow[u, "\phi^{LR,\eta}"] \arrow[r, "\phi^{\underline{LR}}"'] & \overline{\mathbb{B}}^{\underline{LR}}.
\end{tikzcd}$$

Since $\underline{LR}$ is a comonad, every non-counital $\underline{LR}$-comodule is compatible, that is $\overline{\mathbb{B}}^{\underline{LR}} = \underrightarrow{\mathbb{B}}^{\underline{LR}}$, but need not be counital.

1.12**. Remark.** As pointed out by an anonymous referee, a regular pairing (L, R, α, β) defined in 1.4 is in fact the same as an adjunction in the local idempotent closure $\overline{\mathrm{Cat}}$ of the 2-category Cat of categories and hence corresponds to a comonad in $\overline{\mathrm{Cat}}$. This lives on the 1-cell $(LR, \ell r)$ with coproduct $\ell RLr \cdot L\eta R$ and counit ε (see [3]). In this approach, similar to Proposition 1.6, the properties of the weak comonad LR are described by properties of a related comonad $\underline{LR}$.

We are also interested in the modules and comodules induced directly by RL and LR, respectively.

2. (Co)firm (co)modules

To develop further constructions for pairings of functors, symmetry conditions are needed and so we consider weak (co)monads.

The notion of (co-)equalisers in categories may be modified in the following way.

2.1**. Definitions.** Let $\mathbb{K}$ be a class of morphisms in a category $\mathbb{A}$ closed under composition. A cofork

$$B \xrightarrow{k} C \underset{f}{\overset{g}{\rightrightarrows}} D$$

is said to be a $\mathbb{K}$-*equaliser* provided $k \in \mathbb{K}$ and, for any $h : Q \to C$ in $\mathbb{K}$ with $f \cdot h = g \cdot h$, there exists a unique $q : Q \to B$ in $\mathbb{K}$ such that $h = k \cdot q$. If this holds, then, for morphisms $r, s : X \to B$ in $\mathbb{K}$, $k \cdot r = k \cdot s$ implies $r = s$.

Similarly, a fork

$$B \underset{f}{\overset{g}{\rightrightarrows}} C \xrightarrow{s} D$$

is said to be a $\mathbb{K}$-*coequaliser* provided $s \in \mathbb{K}$ and, for any $h : C \to Q$ in $\mathbb{K}$ with $h \cdot f = h \cdot g$, there is a unique $q : D \to Q$ in $\mathbb{K}$ such that $h = q \cdot s$. In this case, for morphisms $t, u : D \to Y$ in $\mathbb{K}$, $t \cdot s = u \cdot s$ implies $t = u$.

A class $\mathbb{K}$ of morphisms in $\mathbb{A}$ is called an *ideal class* if for any morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{A}$, f or g in $\mathbb{K}$ implies that $g \cdot f$ is in $\mathbb{K}$.

Taking for $\mathbb{K}$ the class of all morphisms in $\mathbb{A}$, the notions defined above yield the usual equalisers and coequalisers in the category $\mathbb{A}$.

2.2. **$\mathbb{K}$-cofirm comodules.** Let (G,δ) be a non-counital comonad. Given an ideal class $\mathbb{K}$ of morphisms in the category $\underset{\to}{\mathbb{B}}^{LR}$ of non-counital G-comodules, a comodule (B,ω) is called $\mathbb{K}$-*cofirm* provided the defining cofork

$$B \xrightarrow{\omega} G(B) \underset{G(\omega)}{\overset{\delta_B}{\rightrightarrows}} GG(B)$$

is a $\mathbb{K}$-equaliser. If we choose for $\mathbb{K}$ all morphisms in $\underset{\to}{\mathbb{B}}^{LR}$, a $\mathbb{K}$-cofirm comodule is just called *cofirm.*

2.3. **Compatible comodule morphisms.** Now let (G,δ,ε) be a weak comonad and $\gamma := G\varepsilon\cdot\delta : G\to G$ the idempotent comonad morphism. We call a morphism h between G-comodules (B,ω) and (B',ω') γ-*compatible*, provided it induces commutativity of the triangles in the diagram

$$\begin{tikzcd} B \arrow[r, "\omega"] \arrow[d, "h"'] \arrow[drr, "h"] & G(B) \arrow[r, "\varepsilon_B"] & B \arrow[d, "h"] \\ B' \arrow[r, "\omega'"'] & G(B') \arrow[r, "\varepsilon_{B'}"'] & B'. \end{tikzcd}$$

Clearly, since the outer diagram is always commutative for comodule morphisms, it is enough to require commutativity for one of the triangles. Thus one readily obtains:

(1) *The class $\mathbb{K}_\gamma$ of all γ-compatible morphisms in $\overline{\mathbb{B}}^G$ is an ideal class.*
(2) *A morphism $h: Q\to G(B)$ of G-comodules is in $\mathbb{K}_\gamma$ if and only if $\gamma_B\cdot h = h$.*
(3) *A morphism $h: G(B)\to Q$ of G-comodules is in $\mathbb{K}_\gamma$ if and only if $h\cdot\gamma_B = h$.*

Evidently, a G-comodule (B,ω) is compatible (as in 1.1) if and only if $\omega\in\mathbb{K}_\gamma$, that is, $\omega = \gamma_B\cdot\omega$.

Notice that $\gamma = I_G$ implies that every non-counital G-comodule is γ-compatible, that is, $\underset{\to}{\mathbb{B}}^G = \overline{\mathbb{B}}^G$; in this case, however, not every G-comodule morphism need to be γ-compatible and a G-comodule (B,ω) need not be counital but only satisfies $\omega = \omega\cdot\varepsilon_B\cdot\omega$.

2.4. **Proposition.** *If (G,δ,ε) is a weak comonad, then any compatible G-comodule (B,ω) is $\mathbb{K}_\gamma$-cofirm.*

Proof. We have to show that the cofork

$$B \xrightarrow{\omega} G(B) \underset{G(\omega)}{\overset{\delta_B}{\rightrightarrows}} GG(B)$$

is a $\mathbb{K}_\gamma$-equaliser. Let (Q, κ) be a G-comodule and $h : Q \to G(B)$ a morphism in $\mathbb{K}_\gamma$ with $G(\omega) \cdot h = \delta_B \cdot h$. In the diagram

$$\text{(2.1)} \qquad \begin{tikzcd}
Q \arrow[rr, "h"] \arrow[dr, "h"] \arrow[dd, "\kappa"'] & & G(B) \arrow[r, "\varepsilon_B"] \arrow[d, "G(\omega)"] & B \arrow[dd, "\omega"] \\
& G(B) \arrow[r, "\delta_B"] \arrow[d, "\delta_B"] & GG(B) \arrow[dr, "\varepsilon_{G(B)}"] & \\
G(Q) \arrow[r, "G(h)"'] & GG(B) \arrow[rr, "G\varepsilon_B"'] & & G(B),
\end{tikzcd}$$

all inner diagrams are commutative. This shows that $\widetilde{h} := \varepsilon_B \cdot h : Q \to B$ is a G-comodule morphism with

$$\begin{aligned}
\omega \cdot \widetilde{h} \;&=\; \omega \cdot \varepsilon_B \cdot h \;=\; \varepsilon_{G(B)} \cdot G(\omega) \cdot h \\
&=\; \varepsilon_{G(B)} \cdot \delta_B \cdot h \;=\; \gamma_B \cdot h \;=\; h, \\
\varepsilon_B \cdot \omega \cdot \widetilde{h} \;&=\; \varepsilon_B \cdot \gamma_B \cdot h \;=\; \varepsilon_B \cdot h \;=\; \widetilde{h},
\end{aligned}$$

thus $\widetilde{h} \in \mathbb{K}_\gamma$. Moreover, for any $q : Q \to B$ in $\mathbb{K}_\gamma$ with $\omega \cdot q = h$, we have $\varepsilon_B \cdot h = \varepsilon_B \cdot \omega \cdot q = q$, showing uniqueness of q. □

Replacing (Q, h) in diagram (2.1) by (B, ω), we see that $\varepsilon_B \cdot \omega$ is a comodule morphism and this leads to the following observation.

2.5. **Proposition.** *If (G, δ, ε) is a (proper) comonad, then any non-counital G-comodule (B, ω) is cofirm if and only if it is counital.*

Proof. Since we have a comonad, $\gamma = I_G$, every G-comodule (B, ω) is γ-compatible, and $\omega = \omega \cdot \varepsilon_B \cdot \omega$ (see 2.3).

If (B, ω) is cofirm, then ω is monomorph in $\underrightarrow{\mathbb{B}}^{LR}$; since $\varepsilon_B \cdot \omega$ and I_B are morphisms in $\underrightarrow{\mathbb{B}}^G$ we conclude $\varepsilon_B \cdot \omega = I_B$, that is, (B, ω) is counital.

It is folklore that any counital G-comodule is cofirm. □

2.6. **$\mathbb{K}$-firm modules.** Let (F, m) be a non-unital monad on $\mathbb{B}$. Given an ideal class of morphisms in the category $\underrightarrow{\mathbb{B}}_F$ of non-unital F-modules (see [8]), a module (B, ϱ) is called $\mathbb{K}$-*firm* provided the defining fork

$$\begin{tikzcd}
FF(B) \arrow[r, shift left, "m_B"] \arrow[r, shift right, "F(\varrho)"'] & F(B) \arrow[r, "\varrho"] & B
\end{tikzcd}$$

is a $\mathbb{K}$-coequaliser (Definitions 2.1).

2.7. **Remark.** Following [2, 2.3], a non-unital F-module (B, ϱ) is called *firm* provided it is $\mathbb{K}$-firm for the class $\mathbb{K}$ of all morphisms in $\underrightarrow{\mathbb{B}}_F$ and ϱ is an epimorphism in $\mathbb{B}$. The term *firm* was coined by Quillen for non-unital algebras A over a commutative ring k with the property that the map

$A \otimes_A A \to A$, $a \otimes b \mapsto ab$, is an isomorphism. Then, an A-module is firm provided it is firm for the monad $A \otimes_k -$ on the category of k-modules. In the category of non-unital A-modules, coequalisers are induced by coequalisers of k-modules and hence are epimorph (in fact surjective) as k-module morphisms (e.g. [2, 6.1]).

2.8. **Compatible module morphisms.** Let (F, m, η) be a weak monad with idempotent monad morphism $\vartheta := m \cdot \eta F : F \to F$. A morphism h between F-modules (B, ϱ) and (B', ϱ') is called *ϑ-compatible*, provided it induces commutativity of the triangles in the diagram

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "h"'] \arrow[rrd, "h"] & F(B) \arrow[r, "\varrho"] & B \arrow[d, "h"] \\ B' \arrow[r, "\eta_{B'}"'] & F(B') \arrow[r, "\varrho'"'] & B'. \end{tikzcd}$$

Similar to 2.3 one obtains:

(1) *The class $\mathbb{K}_\vartheta$ of all ϑ-compatible morphisms in $\underline{\mathbb{B}}_F$ is an ideal class.*

(2) *A morphism $h : Q \to F(B)$ of F-modules is in $\mathbb{K}_\vartheta$ if and only if $\vartheta_B \cdot h = h$.*

(3) *A morphism $h : LR(B) \to Q$ of F-modules is in $\mathbb{K}_\vartheta$ if and only if $h \cdot \vartheta_B = h$.*

Clearly, an F-module (B, ϱ) is compatible (see 1.2) if and only if $\varrho \in \mathbb{K}_\vartheta$, that is, $\varrho \cdot \vartheta_B = \varrho$.

2.9. **Remark.** Given the assumptions in 2.8, one may consider the subcategory of $\underline{\mathbb{B}}_F$ consisting of the same objects and as morphisms the ϑ-compatible morphisms. Then the identity morphism on a ϑ-compatible module (B, ϱ) is $\varrho \cdot \eta_B : B \to B$ and equalisers in this category are essentially the $\mathbb{K}_\vartheta$-equalisers. This situation is also addressed in [3, Remark 2.5] (with different terminology).

Dual to the Propositions 2.4 and 2.5 we now have:

2.10. **Proposition.** *If (F, m, η) is a weak monad, then any ϑ-compatible F-module (B, ϱ) is $\mathbb{K}_\vartheta$-firm.*

2.11. **Proposition.** *If (F, m, η) is a (proper) monad, then a non-unital F-module (B, ϱ) is firm if and only if it is unital.*

3. Frobenius property and Frobenius bimodules

In the setting of 1.3, assume α and $\widetilde{\beta}$ to be given, that is, there are natural transformations $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$. Then $(LR, L\eta R)$ is a non-counital comonad, and $(LR, L\widetilde{\varepsilon}R)$ is a non-unital monad on $\mathbb{B}$ (see

[8]). This section is for studying the interplay between the corresponding module and comodule structures.

Let $\underrightarrow{\mathbb{B}}{}^{LR}_{LR}$ denote the category of objects in $\mathbb{B}$ which have an LR-module as well as an LR-comodule structure (LR-bimodules) and with morphisms which are LR-module and LR-comodule morphisms.

By naturality, we have the commutative diagram (Frobenius property)

$$
\begin{tikzcd}
 & LRLRLR \arrow[dr, "LRL\widetilde{\varepsilon}R"] & \\
LRLR \arrow[ur, "L\eta RLR"] \arrow[r, "L\widetilde{\varepsilon}R"] \arrow[dr, "LRL\eta R"'] & LR \arrow[r, "L\eta R"] & LRLR \\
 & LRLRLR \arrow[ur, "L\widetilde{\varepsilon}RLR"'] &
\end{tikzcd} \qquad . \tag{3.1}
$$

We are interested in LR-modules and LR-comodules subject to a reasonable compatibility condition.

3.1. **Frobenius bimodules.** A triple (B, ϱ, ω) with an object $B \in \mathbb{B}$ and two morphisms $\varrho : LR(B) \to B$ and $\omega : B \to LR(B)$ is called a *Frobenius bimodule* provided the data induce commutativity of the diagram

$$
\begin{tikzcd}
LRLR(B) \arrow[r, "LR(\varrho)"] \arrow[d, "L\widetilde{\varepsilon}R"'] \arrow[dr, phantom, "\mathrm{(I)}"] & LR(B) \arrow[r, "LR(\omega)"] \arrow[d, "\varrho"] \arrow[dr, phantom, "\mathrm{(II)}"] & LRLR(B) \arrow[d, "L\widetilde{\varepsilon}R"] \\
LR(B) \arrow[r, "\varrho"] \arrow[d, "L\eta R"'] \arrow[dr, phantom, "\mathrm{(III)}"] & B \arrow[r, "\omega"] \arrow[d, "\omega"] \arrow[dr, phantom, "\mathrm{(IV)}"] & LR(B) \arrow[d, "L\eta R"] \\
LRLR(B) \arrow[r, "LR(\varrho)"'] & LR(B) \arrow[r, "LR(\omega)"'] & LRLR(B).
\end{tikzcd}
$$

This implies that $\varrho : LR(B) \to B$ defines a (non-unital) LR-module and $\omega : B \to LR(B)$ a (non-counital) LR-comodule; if that is already known, the conditions on Frobenius bimodules reduce to commutativity of the diagrams (II) and (III), that is commutativity of (Frobenius property for modules)

$$
\begin{tikzcd}
 & LRLR(B) \arrow[dr, "LR(\varrho)"] & \\
LR(B) \arrow[ur, "L\eta R"] \arrow[r, "\varrho"] \arrow[dr, "LR(\omega)"'] & B \arrow[r, "\omega"] & LR(B) \\
 & LRLR(B) \arrow[ur, "L\widetilde{\varepsilon}R"'] &
\end{tikzcd} \qquad . \tag{3.2}
$$

Denote by $\mathbb{B}^{LR}_{LR}$ the category with the Frobenius LR-bimodules as objects and morphisms which are LR-module as well as LR-comodule morphisms.

By the commutative diagram (3.1), for any $B \in \mathbb{B}$, $LR(B)$ is a Frobenius bimodule with the canonical structures, that is, there is a functor

$$K^{LR}_{LR} : \mathbb{B} \to \mathbb{B}^{LR}_{LR}, \quad B \mapsto (LR(B), L\widetilde{\varepsilon}_{R(B)}, L\eta_{R(B)}).$$

3.2. **Natural mappings.** Assume again $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ to be given (see 1.3). Then there are maps, natural in $A, A' \in \mathbb{A}$,

$$\begin{aligned} \Phi_{A,A'} &: \mathrm{Mor}_{\mathbb{B}}(L(A), L(A')) \to \mathrm{Mor}_{\mathbb{A}}(A, A'), \quad g \mapsto \widetilde{\varepsilon}_{A'} \cdot R(g) \cdot \eta_A, \\ L_{A,A'} &: \mathrm{Mor}_{\mathbb{A}}(A, A') \to \mathrm{Mor}_{\mathbb{B}}(L(A), L(A')), \quad f \mapsto L(f), \end{aligned}$$

$$\Phi_{A,A'} \cdot L_{A,A'} : \mathrm{Mor}_{\mathbb{A}}(A, A') \to \mathrm{Mor}_{\mathbb{A}}(A, A'),\ f \mapsto f \cdot \widetilde{\varepsilon}_A \cdot \eta_A = \widetilde{\varepsilon}_{A'} \cdot \eta_{A'} \cdot f.$$

- If $\widetilde{\varepsilon} \cdot \eta = I_{\mathbb{A}}$, then $\Phi \cdot L_{-,-}$ is the identity (L is separable).
- If $\eta \cdot \widetilde{\varepsilon} \cdot \eta = \eta$, then $\Phi \cdot L_{-,-} \cdot \Phi = \Phi$ ($\Phi \cdot L_{-,-}$ is idempotent).

The natural transformation

$$\theta : LR \xrightarrow{L\eta R} LRLR \xrightarrow{L\widetilde{\varepsilon}R} LR$$

is an LR-module as well as an LR-comodule morphism. From diagram (3.1) one immediately obtains the equalities

$$\begin{aligned} L\eta R \cdot \theta &= LR\,\theta \cdot L\eta R = \theta\, LR \cdot L\eta R, \\ \theta \cdot L\widetilde{\varepsilon}R &= L\widetilde{\varepsilon}R \cdot \theta\, LR = L\widetilde{\varepsilon}L \cdot LR\,\theta. \end{aligned}$$

Similar relations are obtained for Frobenius bimodules.

3.3. **Proposition.** *Given $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$, let (B, ϱ, ω) be a Frobenius LR-bimodule (see 3.1). Then*

$$\varrho \cdot \omega \cdot \varrho = \varrho \cdot \theta_B \quad \textit{and} \quad \omega \cdot \varrho \cdot \omega = \theta_B \cdot \omega.$$

(1) *If $\widetilde{\varepsilon} \cdot \eta = I_{\mathbb{A}}$, then $\varrho \cdot \omega \cdot \varrho = \varrho$ and $\omega \cdot \varrho \cdot \omega = \omega$.*
Then, if ϱ is an epimorphism in $\underrightarrow{\mathbb{B}}^{LR}$ or ω is a monomorphism in $\underrightarrow{\mathbb{B}}_{LR}$, one gets $\varrho \cdot \omega = I_B$.

(2) *If $\eta \cdot \widetilde{\varepsilon} \cdot \eta = \eta$ or $\widetilde{\varepsilon} \cdot \eta \cdot \widetilde{\varepsilon} = \widetilde{\varepsilon}$, then $\omega \cdot \varrho$ is an idempotent morphism.*

Proof. The equalities claimed and (1) can be derived from the commutative diagram

$$\begin{tikzcd}
 & & LRLR(B) \arrow[r, "L\widetilde{\varepsilon}_{R(B)}"] \arrow[dr, "LR(\varrho)"] & LR(B) \arrow[drr, "\varrho"] & & \\
B \arrow[r, "\omega"] \arrow[dr, "\omega"'] & LR(B) \arrow[ur, "L\eta_{R(B)}"] \arrow[r, "\varrho"] \arrow[dr, "LR(\omega)"] & B \arrow[r, "\omega"] & LR(B) \arrow[rr, "\varrho"'] & & B \\
 & LR(B) \arrow[r, "L\eta_{R(B)}"'] & LRLR(B) \arrow[ur, "L\widetilde{\varepsilon}_{R(B)}"'] & & &
\end{tikzcd}\quad .$$

(2) To show this, extend the above diagram by ω on the right or by ϱ on the left, respectively. $\square$

3.4. **Compatible bimodule morphisms.** Assume $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ to be given. A morphism h between Frobenius modules (B, ϱ, ω) and (B', ϱ', ω') is called *θ-compatible*, provided it induces commutativity of the diagram

$$\begin{tikzcd}
B \arrow[r, "\omega"] \arrow[d, "h"'] \arrow[rrd, "h"] & LR(B) \arrow[r, "\varrho"] & B \arrow[d, "h"] \\
B' \arrow[r, "\omega'"'] & LR(B') \arrow[r, "\varrho'"'] & B'.
\end{tikzcd}$$

One easily obtains the following.

(1) *The class $\mathbb{K}_\theta$ of all θ-compatible bimodule morphisms in $\underrightarrow{\mathbb{B}}^{LR}_{LR}$ is an ideal class.*

(2) *A morphism $h : Q \to LR(B)$ of LR-bimodules is in $\mathbb{K}_\theta$ if and only if $\theta_B \cdot h = h$.*

(3) *A morphism $h : LR(B) \to Q$ of LR-bimodules is in $\mathbb{K}_\theta$ if and only if $h \cdot \theta_B = h$.*

(4) *If $\widetilde{\varepsilon} \cdot \eta \cdot \widetilde{\varepsilon} = \widetilde{\varepsilon}$, then $L\widetilde{\varepsilon}R = \theta \cdot L\widetilde{\varepsilon}R$, that is, $L\widetilde{\varepsilon}R$ is θ-compatible.*

(5) *If $\eta \cdot \widetilde{\varepsilon} \cdot \eta = \eta$, then $L\eta R = L\eta R \cdot \theta$, that is, $L\eta R$ is θ-compatible.*

(6) *For a Frobenius bimodule (B, ω, ϱ), ω is θ-compatible if and only if $\omega = \omega \cdot \varrho \cdot \omega$, and ϱ is θ-compatible if and only if $\varrho = \varrho \cdot \omega \cdot \varrho$.*

The next result shows how (co)firm (co)modules enter the picture.

3.5. **Proposition.** *Let $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ be given and consider a Frobenius LR-bimodule (B, ϱ, ω).*

(1) *If ω is θ-compatible, then (B, ω) is $\mathbb{K}_\theta$-cofirm;*
if ϱ is θ-compatible, then (B, ϱ) is $\mathbb{K}_\theta$-firm.

(2) *If $\widetilde{\varepsilon} \cdot \eta \cdot \widetilde{\varepsilon} = \widetilde{\varepsilon}$, then $(LR(B), L\widetilde{\varepsilon}_{R(B)})$ is a $\mathbb{K}_\theta$-firm module;*
if $\eta \cdot \widetilde{\varepsilon} \cdot \eta = \eta$, $(LR(B), L\eta_{R(B)})$ is a $\mathbb{K}_\theta$-cofirm comodule.

Proof. (compare Proposition 2.4) (1) For a non-counital LR-comodule (Q, κ), let $h : Q \to LR(B)$ be a comodule morphism with $L\eta R \cdot h = LR(\omega) \cdot h$ and $h = \theta \cdot h$. For $\widetilde{h} := \varrho \cdot h$ we get

$$\omega \cdot \widetilde{h} = \omega \cdot \varrho \cdot h = L\widetilde{\varepsilon}R \cdot LR(\omega) \cdot h = L\widetilde{\varepsilon}R \cdot L\eta R \cdot h = h.$$

For any θ-compatible comodule morphism $q : Q \to B$ with $\omega \cdot q = h$, we have $g = \varrho \cdot \omega \cdot q = \varrho \cdot h = \widetilde{h}$, showing uniqueness of $\widetilde{h}$.

The second claim is shown similarly.

(2) In view of 3.4, (4) and (5), the assertions follow from (1). $\square$

3.6. **Proposition.** *Assume $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ to be given. Let $\mathbb{K}$ be an ideal class of LR-comodule morphisms and suppose $L\widetilde{\varepsilon}_{R(B)}$ in $\mathbb{K}$ for any $B \in \mathbb{B}$.*

(1) *If (B, ω) in $\mathbb{B}^{LR}_{\rightarrow}$ is a $\mathbb{K}$-cofirm comodule (see 2.2), there is a unique $\varrho : LR(B) \to B$ in $\mathbb{K}$ making (B, ϱ, ω) a Frobenius bimodule.*

(2) *With this module structure, LR-comodule morphisms between $\mathbb{K}$-cofirm LR-comodules (B, ω) and (B', ω') are morphisms of the Frobenius bimodules (B, ω, ϱ) and (B', ω', ϱ').*

Proof. (1) Consider the diagram (see 3.1)

$$\begin{tikzcd}
LRLR(B) \arrow[r, dotted, "LR(\varrho)"] \arrow[d, "L\widetilde{\varepsilon}R"'] \arrow[dr, phantom, "\text{(I)}"] & LR(B) \arrow[r, "LR(\omega)"] \arrow[d, dotted, "\varrho"] \arrow[dr, phantom, "\text{(II)}"] & LRLR(B) \arrow[d, "L\widetilde{\varepsilon}R"] \\
LR(B) \arrow[r, dotted, "\varrho"] \arrow[d, "L\eta R"'] \arrow[dr, phantom, "\text{(III)}"] & B \arrow[r, "\omega"] \arrow[d, "\omega"] \arrow[dr, phantom, "\text{(IV)}"] & LR(B) \arrow[d, "L\eta R"] \\
LRLR(B) \arrow[r, dotted, "LR(\varrho)"'] & LR(B) \arrow[r, "LR(\omega)"'] & LRLR(B),
\end{tikzcd}$$

where (IV) is assumed to be a $\mathbb{K}$-equaliser. Since

$$\begin{aligned}
L\eta_{R(B)} \cdot L\widetilde{\varepsilon}_{R(B)} \cdot LR(\omega) &= L\widetilde{\varepsilon}_{RLR(B)} \cdot LRL\eta_{R(B)} \cdot LR(\omega) \\
&= L\widetilde{\varepsilon}_{RLR(B)} \cdot LRLR(\omega) \cdot LR(\omega) \\
&= LR(\omega) \cdot L\widetilde{\varepsilon}_{R(B)} \cdot LR(\omega),
\end{aligned}$$

and $L\widetilde{\varepsilon}R_B \cdot LR(\omega)$ is in $\mathbb{K}$, there exists a unique $\varrho : LR(B) \to B$ in $\mathbb{K}$ leading to the commutative diagram (II), and (III) commutes since ϱ is required to be a comodule morphism. Moreover,

$$\begin{aligned}
\omega \cdot \varrho \cdot L\widetilde{\varepsilon}_{R(B)} &= LR(\varrho) \cdot L\eta_{R(B)} \cdot L\widetilde{\varepsilon}_{R(B)} \\
&= LR(\varrho) \cdot L\widetilde{\varepsilon}_{RLR(B)} \cdot LRL\eta_{R(B)} \\
&= L\widetilde{\varepsilon}_{R(B)} \cdot LRLR(\varrho) \cdot LRL\eta_{R(B)} \\
&= L\widetilde{\varepsilon}_{R(B)} \cdot LR(\omega) \cdot LR(\varrho) \;=\; \omega \cdot \varrho \cdot LR(\varrho),
\end{aligned}$$

and hence $\varrho \cdot L\widetilde{\varepsilon}_{R(B)} = \varrho \cdot LR\varrho$ since ω is a $\mathbb{K}$-equaliser. This means that the diagram (I) is also commutative.

(2) Now let $h : B \to B'$ be an LR-comodule morphism. Then

$$\begin{aligned}
\omega' \cdot h \cdot \varrho &= LR(h) \cdot \omega \cdot \varrho \\
&= LR(h) \cdot L\widetilde{\varepsilon}_{R(B)} \cdot LR(\omega) \\
&= L\widetilde{\varepsilon}_{R(B')} \cdot LRLR(h) \cdot LR(\omega) \\
&= L\widetilde{\varepsilon}_{R(B')} \cdot LR(\omega') \cdot LR(h) \;=\; \omega' \cdot \varrho' \cdot LR(h)
\end{aligned}$$

and, since both $h \cdot \varrho$ and $\varrho' \cdot LR(h)$ are in $\mathbb{K}$, this implies that they are equal (see Definition 2.1), that is, h is also an LR-module morphism. $\square$

Symmetric to Proposition 3.6 we get:

3.7**. Proposition.** *Assume $\eta : I_{\mathbb{A}} \to RL$ and $\widehat{\varepsilon} : RL \to I_{\mathbb{A}}$ to be given. Let $\mathbb{K}'$ be an ideal class of LR-module morphisms and suppose $L\eta_{R(B)}$ belongs to $\mathbb{K}'$ for any $B \in \mathbb{B}$.*

(1) *If (B, ϱ) in $\mathbb{B}_{\underrightarrow{LR}}$ is a $\mathbb{K}'$-firm module (see* 2.6*), there is a unique $\omega : B \to LR(B)$ in $\mathbb{K}'$ making (B, ϱ, ω) a Frobenius bimodule.*

(2) *With this comodule structure, LR-module morphisms between $\mathbb{K}'$-firm LR-modules (B, ϱ) and (B', ϱ') are morphisms of the Frobenius bimodules (B, ω, ϱ) and (B', ω', ϱ').*

So far we have only considered the case when α and $\widetilde{\beta}$ (in 1.3) exist. Now we want to include more mappings in our assumptions.

3.8**. Lemma.** *Refer to the notation in* 1.3 *and* 3.2*.*

(1) *Let (L, R, α, β) be any pairing and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ a natural transformation satisfying $\eta \cdot \widetilde{\varepsilon} \cdot \eta = \eta$. Then $\ell r \cdot \theta = \ell r$.*

(2) *Let $(R, L, \widetilde{\alpha}, \widetilde{\beta})$ be any pairing and $\eta : I_{\mathbb{A}} \to RL$ a natural transformation satisfying $\widetilde{\varepsilon} \cdot \eta \cdot \widetilde{\varepsilon} = \widetilde{\varepsilon}$. Then $\theta \cdot \widetilde{\ell r} = \widetilde{\ell r}$.*

Proof. The assertions follow immediately from the definitions. □

3.9**. Theorem.** *Let (L, R, α, β) be a regular pairing with β symmetric and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ any natural transformation. Then,*

(1) *$\widehat{\varepsilon} := \widetilde{\varepsilon} \cdot r\ell : RL \to I_{\mathbb{A}}$ is a natural transformation with $\widehat{\varepsilon} = \widehat{\varepsilon} \cdot r\ell$. Furthermore, $\ell r \cdot L\widehat{\varepsilon}R = L\widehat{\varepsilon}R$, that is, $L\widehat{\varepsilon}R$ is ℓr-compatible as an LR-comodule morphism;*

(2) *$(LR, L\eta R, \varepsilon)$ is a weak comonad and if $\omega : B \to LR(B)$ is an ℓr-compatible LR-comodule, there is a unique $\varrho : LR(B) \to B$ in $\mathbb{K}_{\ell r}$ making (B, ϱ, ω) a Frobenius $(LR, \eta, \widehat{\varepsilon})$-module, given by*

$$\varrho : LR(B) \xrightarrow{LR(\omega)} LRLR(B) \xrightarrow{L\widehat{\varepsilon}_{R(B)}} LR(B) \xrightarrow{\varepsilon_B} B;$$

(3) *morphisms between ℓr-compatible LR-comodules (B, ω), (B', ω') are LR-bimodule morphisms between (B, ϱ, ω) and (B', ϱ', ω').*

Proof. (1) By our symmetry assumption, $\ell R = Lr$ and the diagram

$$\begin{tikzcd} LRL \arrow[r, "Lr\ell R", "\ell RLr"'] \arrow[d, "L\widehat{\varepsilon}R"'] \arrow[dr, "L\widehat{\varepsilon}R"] & LRLR \arrow[d, "L\widehat{\varepsilon}R"] \\ LR \arrow[r, "\ell r"'] & LR \end{tikzcd}$$

commutes, showing $L\widehat{\varepsilon}R = \ell r \cdot L\widehat{\varepsilon}R$.

(2) As shown in Proposition 2.4, (B, ω) is $\mathbb{K}_{\ell r}$-cofirm and hence the existence of ϱ follows by Proposition 3.6. For the Frobenius module

(B,ϱ,ω), we have the commutative diagram

$$\begin{tikzcd}
 & LRLR(B) \arrow[r, "\varepsilon LR"] \arrow[dr, "LR(\varrho)"] & LR(B) \arrow[dr, "\varrho"] & \\
LR(B) \arrow[ur, "L\eta R"] \arrow[r, "\varrho"'] \arrow[dr, "LR(\omega)"'] & B \arrow[r, "\omega"'] & LR(B) \arrow[r, "\varepsilon_B"'] & B \\
 & LRLR(B) \arrow[ur, "L\widehat{\varepsilon}R"'] & &
\end{tikzcd}.$$

Since ϱ is ℓR-compatible, the upper paths yields $\varrho \cdot \ell R = \varrho$. The lower path is the composite given for ϱ.

(3) Since $\mathbb{K}_{\ell r}$ is an ideal class, the assertion about the bimodule morphisms follows by Proposition 3.6. $\square$

Instead of (L,R,α,β), we may require $(R,L,\widetilde{\alpha},\widetilde{\beta})$ to be a regular pairing (see 1.3) and relate the bimodules for $(LR,\eta,\widetilde{\varepsilon})$ with modules for $(LR,L\widehat{\varepsilon}R)$. By symmetry we obtain:

3.10. **Theorem.** *Let $(R,L,\widetilde{\alpha},\widetilde{\beta})$ be a regular pairing of functors with $\widetilde{\alpha}$ symmetric and $\eta : I_{\mathbb{A}} \to RL$ any natural transformation. Then,*

(1) *$\widehat{\eta} := \widetilde{r}\widetilde{\ell}\cdot\eta : I_{\mathbb{A}} \to RL$ is a natural transformation with $\widehat{\eta} = \widetilde{r}\widetilde{\ell}\cdot\widehat{\eta}$ and $L\widehat{\eta}R = \widetilde{r}\widetilde{\ell}\cdot L\widehat{\eta}R$, that is, $L\widehat{\eta}R$ is $\widetilde{r}\widetilde{\ell}$-compatible as an LR-module morphism (see* 2.8*);*

(2) *$(LR, L\widehat{\varepsilon}R, \widehat{\eta})$ is a weak monad and if $\varrho : LR(B) \to B$ is an $\widetilde{\ell}\widetilde{r}$-compatible LR-module, there is a unique $\omega : B \to LR(B)$ in $\mathbb{K}_{\widetilde{\ell}\widetilde{r}}$ making (B,ϱ,ω) a Frobenius $(LR,\widehat{\eta},\widetilde{\varepsilon})$-bimodule given by*

$$\omega : B \xrightarrow{\widetilde{\eta}_B} LR(B) \xrightarrow{L\widehat{\eta}_{R(B)}} LRLR(B) \xrightarrow{LR(\varrho)} LR(B);$$

(3) *morphisms between $\widetilde{\ell}\widetilde{r}$-compatible LR-modules (B,ϱ), (B',ϱ') are $(LR,\widehat{\eta},\widetilde{\varepsilon})$-bimodule morphism between (B,ϱ,ω) and (B',ϱ',ω').*

4. Weak Frobenius monads

As we have seen in the previous section, for results on the interplay between (co)module and bimodule structures for Frobenius monads symmetry conditions on our pairings were needed, that is, the intrinsic non-(co)unital (co)monads became weak (co)monads. Hence we will concentrate in this section on this kind of (co)monads and also apply results from Section 2.

4.1. **Frobenius property.** Let (F,m) be a non-unital monad, (F,δ) a non-counital comonad, $(B,\varrho) \in \underrightarrow{\mathbb{B}}_F$ and $(B,\omega) \in \underrightarrow{\mathbb{B}}^F$. We say that

(F, m, δ) satisfies the *Frobenius property* and (B, ϱ, ω) is a *Frobenius bimodule*, provided they induce commutativity of the respective diagrams,

$$\begin{tikzcd}
 & FFF \arrow[dr, "Fm"] & \\
FF \arrow[ur, "\delta F"] \arrow[r, "m"] \arrow[dr, "F\delta"'] & F \arrow[r, "\delta"] & FF \\
 & FFF \arrow[ur, "mF"'] &
\end{tikzcd}, \qquad
\begin{tikzcd}
 & FF(B) \arrow[dr, "F(\varrho)"] & \\
F(B) \arrow[ur, "\delta_B"] \arrow[r, "\varrho"] \arrow[dr, "F(\omega)"'] & B \arrow[r, "\omega"] & F(B) \\
 & FF(B) \arrow[ur, "m_B"'] &
\end{tikzcd}.$$

The Frobenius bimodules as objects and the morphisms, which are F-module as well as F-comodule morphisms, form a category which we denote by $\underrightarrow{\mathbb{B}}{}_F^F$. Transferring the Propositions 3.6 and 3.7 yields:

4.2. **Theorem.** *Assume (F, m, δ) to satisfy the Frobenius property. Let (F, δ, ε) be a weak comonad, $\gamma := \varepsilon F \cdot \delta$, and assume $m = \gamma \cdot m$. Then,*

(1) *for any γ-compatible F-comodule (B, ω), there is a unique γ-compatible F-comodule morphism*

$$\varrho : F(B) \xrightarrow{F(\omega)} FF(B) \xrightarrow{m_B} F(B) \xrightarrow{\varepsilon_B} B$$

making (B, ϱ, ω) a Frobenius bimodule;

(2) *any F-comodule morphism between γ-compatible comodules (B, ω) and (B', ω') becomes a morphism between the Frobenius bimodules (B, ϱ, ω) and (B', ϱ', ω');*

(3) *there is an isomorphism of categories*

$$\Psi : \overline{\mathbb{B}}^F \to \overline{\mathbb{B}}{}^F_F, \quad (B, \omega) \mapsto (B, \varrho, \omega),$$

with the forgetful functor $U_F : \overline{\mathbb{B}}{}^F_F \to \overline{\mathbb{B}}^F$ as inverse, where $\overline{\mathbb{B}}{}^F_F$ denotes the category of Frobenius bimodules which are γ-compatible as F-comodules.

Proof. By our compatibility condition on m, we can apply Proposition 3.9 and the formula for ϱ given there. The assertions about the functors follow directly from the constructions. □

4.3. **Theorem.** *Assume (F, m, δ) to satisfy the Frobenius property. Let (F, m, η) be a weak monad, $\vartheta := m \cdot F\eta$, and assume $\delta = \delta \cdot \vartheta$. Then,*

(1) *for a ϑ-compatible F-module (B, ϱ), there is a unique ϑ-compatible module morphism*

$$\omega : B \xrightarrow{\eta_B} F(B) \xrightarrow{\delta_B} FF(B) \xrightarrow{F(\varrho)} F(B)$$

making (B, ϱ, ω) a Frobenius bimodule;

(2) *any F-morphism between ϑ-compatible modules (B, ϱ), (B', ϱ') becomes a morphism between the Frobenius bimodules (B, ϱ, ω) and (B', ϱ', ω');*

(3) *there is an isomorphism of categories*

$$\Phi : \underline{\mathbb{B}}_F \to \underline{\mathbb{B}}_F^F, \quad (B, \varrho) \mapsto (B, \varrho, \omega),$$

with the forgetful functor $U^F : \underline{\mathbb{B}}_F^F \to \underline{\mathbb{B}}_F$ as inverse, where $\underline{\mathbb{B}}_F^F$ denotes the category of Frobenius modules which are ϑ-compatible as F-modules.

Proof. By Proposition 3.10 and the formula for ω given there. □

4.4. **Definition.** We call $(F, m, \eta; \delta, \varepsilon)$ a *weak Frobenius monad* provided (F, m, η) is a weak monad, (F, δ, ε) is a weak comonad, (F, m, δ) has the Frobenius property (see (4.1)), and $m \cdot F\eta = F\varepsilon \cdot \delta$ (i.e. $\vartheta = \gamma$).

As a first property we observe:

4.5. **Proposition.** *Let $(F, m, \eta; \delta, \varepsilon)$ be a weak Frobenius monad and assume the idempotent $m \cdot F\eta = F\varepsilon \cdot \delta$ to be split by $F \to \underline{F} \to F$. Then $\underline{F}$ has a canonical monad and comonad structure $(\underline{F}, \underline{m}, \underline{\eta}; \underline{\delta}, \underline{\varepsilon})$ which makes it a Frobenius monad.*

Proof. The monad and comonad structures on $\underline{F}$ are obtained from 1.1 and 1.2 and a routine diagram chase shows that the Frobenius property (see 4.1) is satisfied. □

Summarising we obtain our main result for these structures.

4.6. **Theorem.** *Let $(F, m, \eta; \delta, \varepsilon)$ be a weak Frobenius monad. Then the constructions in* 4.2 *and* 4.3 *yield category isomorphisms*

$$\overline{\mathbb{B}}^F \xrightarrow{\Psi} \overline{\underline{\mathbb{B}}}_F^F \xrightarrow{U^F} \underline{\mathbb{B}}_F \;, \qquad \underline{\mathbb{B}}_F \xrightarrow{\Phi} \overline{\underline{\mathbb{B}}}_F^F \xrightarrow{U_F} \overline{\mathbb{B}}^F \;.$$

where $\overline{\underline{\mathbb{B}}}_F^F$ denotes the category of those Frobenius F-bimodules which are (γ-)compatible as F-comodules and (ϑ-)compatible as F-modules.

Proof. For a weak monad (F, m, η), m is ϑ-compatible and hence γ-compatible by our assumption $\gamma = \vartheta$. Similarly, δ is ϑ compatible and hence the conditions in the preceding propositions are satisfied. □

For (proper) monads and comonads the assertions simplify. For Proposition 4.2 this situation is considered in [2, Section 4] and our results for this case correspond essentially to [2, Lemma 2, Corollary 1].

4.7. **Corollary.** *Let (F, m, δ) satisfy the Frobenius property and assume (F, δ, ε) to be a comonad.*

(1) *For any counital F-comodule $\omega : B \to F(B)$, there is some F-module morphism $\varrho : F(B) \to B$ making (B, ϱ, ω) a Frobenius bimodule.*

(2) *If (F, m) allows for a unit, then (B, ϱ) is a unital F-module.*

(3) *If $m \cdot \delta = I_F$, then, for any Frobenius bimodule (B, ϱ, ω), (B, ϱ) is a firm F-module.*

Proof. (1), (2) hold by Theorem 4.2; (3) follows from Theorem 3.5. □

4.8. **Corollary.** *Let (F, m, δ) satisfy the Frobenius property and assume (F, m, η) to be a monad.*

(1) *For any unital F-module $\varrho : F(B) \to B$, there is some F-comodule morphism $\omega : B \to F(B)$ (given in 3.10) making (B, ϱ, ω) a Frobenius bimodule.*

(2) *If (F, δ) allows for a counit, then (B, ω) is a counital F-comodule.*

(3) *If $m \cdot \delta = I_F$, then, for any Frobenius bimodule (B, ϱ, ω), (B, ω) is a firm F-comodule.*

For proper monads and comonads F, all non-unital F-modules are compatible and all non-counital F-comodules are compatible, that is, $\underset{\rightarrow}{\mathbb{B}}_F = \underline{\mathbb{B}}_F$ and $\underset{\rightarrow}{\mathbb{B}}^F = \overline{\mathbb{B}}^F$. Thus we have:

4.9. **Corollary.** *Let $(F, m, \eta; \delta, \varepsilon)$ be a Frobenius monad. There are category isomorphisms*

$$\Psi : \underset{\rightarrow}{\mathbb{B}}^F \to \underset{\rightarrow}{\mathbb{B}}^F_F, \quad \Phi : \underset{\rightarrow}{\mathbb{B}}_F \to \underset{\rightarrow}{\mathbb{B}}^F_F,$$

where $\underset{\rightarrow}{\mathbb{B}}^F_F$ denotes the category of non-unital and non-counital Frobenius F-bimodules, and

$$\Psi' : \mathbb{B}^F \to \mathbb{B}^F_F, \quad \Phi' : \mathbb{B}_F \to \mathbb{B}^F_F,$$

where $\mathbb{B}^F_F$ is the category of unital and counital Frobenius F-bimodules.

It is easy to see that (by (co)restriction) these isomorphisms induce isomorphisms between the category of unital F-modules, counital F-comodules, and of unital and counital Frobenius bimodules, an observation following from Eilenberg-Moore [4], which may be considered as the starting point for the categorical treatment of Frobenius algebras.

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