Algebra and Discrete Mathematics RESEARCH ARTICLE
Number ?. **(????).** pp. 1–13

Correct classes of modules

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Abstract. For a ring R, call a class $\mathcal{C}$ of R-modules *(pure-) mono-correct* if for any $M, N \in \mathcal{C}$ the existence of (pure) monomorphisms $M \to N$ and $N \to M$ implies $M \simeq N$. Extending results and ideas of Rososhek from rings to modules, it is shown that, for an R-module M, the class $\sigma[M]$ of all M-subgenerated modules is mono-correct if and only if M is semisimple, and the class of all weakly M-injective modules is mono-correct if and only if M is locally noetherian. Applying this to the functor ring of R-Mod provides a new proof that R is left pure semisimple if and only if R-Mod is pure-mono-correct. Furthermore, the class of pure-injective R-modules is always pure-mono-correct, and it is mono-correct if and only if R is von Neumann regular. The dual notion *epi-correctness* is also considered and it is shown that a ring R is left perfect if and only if the class of all flat R-modules is *epi-correct.* At the end some open problems are stated.

1. Introduction

Consider the following definitions in any category $\mathbf{A}$.

1.1. Mono- and epi-equivalent objects. Two objects A, B in $\mathbf{A}$ are called

mono-equivalent if there are monomorphisms $A \to B$ and $B \to A$,
epi-equivalent if there are epimorphisms $A \to B$ and $B \to A$.

We denote the first case by $A \overset{\text{m}}{\simeq} B$ and the second case by $A \overset{\text{e}}{\simeq} B$.

1.2. Correct objects and classes. An object A in $\mathbf{A}$ is said to be

mono-correct if, for every object $B \in \mathbf{A}$, $A \overset{\text{m}}{\simeq} B$ implies $A \simeq B$,
epi-correct if, for every $B \in \mathbf{A}$, $A \overset{\text{e}}{\simeq} B$ implies $A \simeq B$.

2001 Mathematics Subject Classification: *16D70, 16P40, 16D60.*
Key words and phrases: *Cantor-Bernstein Theorem, correct classes, homological classification of rings.*

We call a class $\mathcal{C}$ of objects in **A**

mono-correct if, for any objects $A, B \in \mathcal{C}$, $A \overset{\mathrm{m}}{\simeq} B$ implies $A \simeq B$,

epi-correct if, for any objects $A, B \in \mathcal{C}$, $A \overset{\mathrm{e}}{\simeq} B$ implies $A \simeq B$.

Notice that any subclass of a mono-correct (epi-correct) class of objects trivially is again a mono-correct (epi-correct) class. The reader should be aware of the difference between a *correct class of objects* and a *class of correct objects*.

The motivation for our investigation is the well-known property of the category **Set** of sets with maps as morphisms.

1.3. Cantor-Bernstein Theorem.
The class of all objects in **Set** *is mono-correct and epi-correct.*

Proof. The classical Cantor-Bernstein (or Schröder-Bernstein) Theorem says: if for two sets A, B there are injective maps $A \to B$ and $B \to A$, then there exists a bijection between A and B. In our terminology this means that every object in **Set** is mono-correct.

To prove epi-correctness assume that, for any sets A, B, there exist surjective maps $f : A \to B$ and $g : B \to A$. Then, by the Axiom of Choice, f and g are retractions, that is, there exist maps $f' : B \to A$ and $g' : A \to B$ such that $f \circ f' = \mathrm{id}_B$ and $g' \circ g = \mathrm{id}_B$ (e.g., [18, Satz 3.8]). Clearly f' and g' are injective and hence $A \simeq B$. □

For a discussion and history of the Cantor-Bernstein Theorem the reader is referred to [1], [2] and [9]. The fact that the class of all objects in **Set** is epi-correct is also called the *dual Cantor-Bernstein Theorem.* As stated in [1, Corollary 5.3], in Zermelo-Frankel set theory this is equivalent to the Axiom of Choice. There are several papers studying Cantor-Bernstein theorems for various algebraic situations, e.g., [4], [5] and [16].

The purpose of this note is to consider such properties for classes of modules. In particular we will see that the class of all modules subgenerated by a module M is mono- (or epi-)correct if and only if M is semisimple. Further results are: the class of weakly M-injective modules is mono-correct if M is locally noetherian; the class of all flat R-modules is epi-correct if and only if R is left perfect; the class of all pure-injective modules is mono-correct if and only if R is von Neumann regular.

Restricting to the classes of pure morphisms, we obtain that the class of pure-injective modules is pure-mono-correct, and the class of all left R-modules is pure-mono-correct (or pure-epi-correct) if and only if R is left pure semisimple. This reproves and extends results in Rososhek [13, 14].

Restricting further to the class of splitting morphisms the resulting notions still make sense and some thoughts on this are mentioned at the end of the paper. Notice that the Cantor-Bernstein Theorem could be stated as the class of all sets being split-mono-correct and split-epi-correct.

2. Preliminaries

For convenience we recall some basic notions from module theory which will be used in the sequel. Let R be any associative ring with identity and let R-Mod denote the category of left R-modules.

2.1. The category $\sigma[M]$. For any R-module M, by $\sigma[M]$ we denote the full subcategory of R-Mod whose objects are M-subgenerated modules, that is, modules that are submodules of M-generated modules. $\sigma[M]$ is the smallest full Grothendieck subcategory of R-Mod containing M (see [17]). For any family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M]$ the coproduct in $\sigma[M]$ is the same as the coproduct in R-Mod, and the product in $\sigma[M]$ is the trace of $\sigma[M]$ in the product formed in R-Mod, i.e., $\mathrm{Tr}(\sigma[M], \prod_\Lambda N_\lambda)$.

Any module $N \in \sigma[M]$ has an injective cover in $\sigma[M]$, the M-injective cover of N, which is usually denoted by $\widehat{N}$. In particular, $\widehat{M}$ is the self-injective cover of M.

A module N is called *weakly M-injective* if $\mathrm{Hom}(-, N)$ transforms any monomorphism $K \to M^n$ into an epimorphism, whenever K is finitely generated and $n \in \mathbb{N}$. Clearly the direct sum of weakly M-injective modules is again weakly M-injective. Furthermore, every weakly M-injective module is M-injective if and only if M is locally noetherian (e.g., [17, 16.9,27.3]). Notice that for $M = R$ the weakly R-injective modules are just the FP-injective modules.

2.2. Purity in R-Mod. An exact sequence of left R-modules

$$(*) \qquad 0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} N \longrightarrow 0$$

is said to be *pure* provided $\mathrm{Hom}(P, -)$ transforms $(*)$ to an exact sequence of abelian groups, for any finitely presented R-module P. In this case f is called a *pure monomorphism* and g is a *pure epimorphism.* Notice that the purity of $(*)$ can also be characterized by the fact that it remains exact under the functors $F \otimes_R -$, for any (finitely presented) right R-module F.

A module $Q \in \sigma[M]$ is called *pure injective* if $\mathrm{Hom}(-, Q)$ is exact on all pure exact sequences $(*)$, and a module P is *pure projective* if $\mathrm{Hom}(P, -)$ is exact on all such sequences. Pure projective modules are precisely direct summands of direct sums of finitely presented modules. For any right R-module X, $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$ is a pure injective left R-module and $(*)$ is pure if and only if $\mathrm{Hom}(-, Q)$ is exact on $(*)$ for any pure injective left R-module Q.

2.3. Left functor ring. Let $U = \bigoplus_A U_\alpha$ be the direct sum of a representing set $\{U_\alpha\}_A$ of the finitely presented left R-modules. Then the functor $\widehat{\mathrm{Hom}}(U, -)$ is defined, for any left R-module N, by

$$\widehat{\mathrm{Hom}}(U, N) = \{f \in \mathrm{Hom}(U, N) \mid (U_\alpha)f = 0 \text{ for almost all } \alpha \in A\}$$

and $T = \widehat{\mathrm{Hom}}(U, U)$ is called the *left functor ring* (of R-Mod). T is a ring with enough idempotents and by T-Mod we denote the category of all T-modules X with $TX = X$. The functor

$$\widehat{\mathrm{Hom}}(U, -) : R\text{-Mod} \to T\text{-Mod}$$

is fully faithful and, for any module N, $\widehat{\mathrm{Hom}}(U, N)$ is a flat T-module and every flat module in T-Mod is of this form ([17, Section 52]).

2.4. Right functor ring. Let $V = \bigoplus_A V_\alpha$ be the direct sum of a representing set $\{V_\alpha\}_A$ of the finitely presented right R-modules. Then the functor $\widehat{\mathrm{Hom}}(V, -)$ is defined as above and $S = \widehat{\mathrm{Hom}}(V, V)$ is the *right functor ring* (of Mod-R). Of course, S also has enough idempotents and by S-Mod we denote the category of all left S-modules Y with $SY = Y$. The functor

$$V \otimes_R - : R\text{-Mod} \to S\text{-Mod}$$

is fully faithful and, for any left module N, $V \otimes_R N$ is an FP-injective S-module and every FP-injective module in S-Mod is of this form ([17, 52.3]).

2.5. Remarks. The notion of purity can also be defined in the category $\sigma[M]$, for any module M, based on the finitely presented modules in $\sigma[M]$. However, there may be such categories that do not contain any non-zero finitely presented objects as shown by Example 1.7 in [12]. To get the expected results around purity in such categories one has to require that $\sigma[M]$ is *locally finitely presented*, that is, there is a generating set of finitely presented modules in $\sigma[M]$ (e.g., M is locally noetherian). In this case also the functor ring of $\sigma[M]$ can be defined and many results from R-Mod hold in this context (see [17]). For purity in more general Grothendieck categories and pure semisimplicity of these categories we refer to Simson [15].

3. Correct classes of modules

The definitions from Section 1 take the following form in module categories.

3.1. Mono- and epi-equivalent modules. Two modules M and N are called

mono-equivalent if there are monomorphisms $M \to N$ and $N \to M$,
epi-equivalent if there are epimorphisms $M \to N$ and $N \to M$.

We denote the first case by $M \overset{\mathrm{m}}{\simeq} N$ and the second case by $M \overset{\mathrm{e}}{\simeq} N$.

These relationships between two modules generalize the notion of isomorphisms. In the terminology of Facchini (e.g. [7, 8]), $M \overset{\mathrm{m}}{\simeq} N$ if the modules M, N belong to the same *monogeny class* and $M \overset{\mathrm{e}}{\simeq} N$ if they belong to the same *epigeny class*. In his work these notions play an important part in the study of uniqueness of decompositions.

Notice that a module M is compressible if it is mono-equivalent to each of its submodules. If M and N are mono-equivalent then the class of M-cogenerated modules is equal to the class of N-cogenerated modules. Dually, for epi-equivalent modules M,N, the class of M-generated modules coincides with the class of N-generated modules. In both cases the categories subgenerated by M and N are the same, that is, $\sigma[M]$ is equal to $\sigma[N]$. As a consequence, any self-injective (self-projective) module M is also N-injective (N-projective) for any $N \overset{\mathrm{m}}{\simeq} M$ or $N \overset{\mathrm{e}}{\simeq} M$.

3.2. Correct modules. An R-module M is said to be

mono-correct if for any module N, $M \overset{\mathrm{m}}{\simeq} N$ implies $M \simeq N$,
epi-correct if for any module N, $M \overset{\mathrm{e}}{\simeq} N$ implies $M \simeq N$.

In these definitions the choice of N can be restricted to modules in $\sigma[M]$ since modules which are mono-equivalent or epi-equivalent to M always belong to $\sigma[M]$.

3.3. Correct modules in R-Mod.

(1) *M is mono-correct in case*

(i) *M is artinian;*

(ii) *M is uniserial and injective endomorphisms are epimorph.*

(2) *M is epi-correct in case*

(i) *M is noetherian;*

(ii) *M is uniserial and surjective endomorphisms are monomorph.*

(3) *M is mono- and epi-correct in case*

(i) *M is noetherian and self-injective;*

(ii) *M is artinian and self-projective;*

(iii) *M has finite length;*

(iv) *M is semisimple.*

Proof. (compare [13, Proposition 1])

(1)(i) Let $f : M \to N$ and $g : N \to M$ be monomorphisms. Then fg is an injective endomorphism of M and hence is an automorphism (e.g., [17, 31.13]). Then g is surjective and hence an isomorphism.

(ii) Consider f, g as in (1). Then fg is an isomorphism. Since N is isomorphic to a submodule of M it is hollow and this implies that f is an isomorphism.

(2)(i) This is shown dually to (1)(i) since surjective endomorphisms of noetherian modules are automorphisms (e.g., [17, 31.13]).

(ii) The proof is dual to that of (1)(ii).

(3)(i) Let M be self-injective and noetherian. Since M has finite uniform dimension every injective endomorphism is an automorphism and hence the assertion is obvious.

(ii) This can be shown with a proof dual to (i).

(iii) just combines (1)(i) and (2)(i).

(iv) If M is finitely generated and semisimple then it has finite length and (iii) applies. For an arbitrary semisimple M, any module N with $N \overset{\mathrm{m}}{\simeq} M$ is semisimple and hence the assertion will be a byproduct of 3.5 □

3.4. Correct classes of modules. A class $\mathcal{C}$ of R-modules is said to be

mono-correct if for any $N, M \in \mathcal{C}$, $M \overset{\mathrm{m}}{\simeq} N$ implies $M \simeq N$,

epi-correct if for any $N, M \in \mathcal{C}$, $M \overset{\mathrm{e}}{\simeq} N$ implies $M \simeq N$.

To describe correct classes of modules we recall some definitions. A module M is said to be π*-injective* if, for any submodules $K, L \subset M$ with $K \cap L = 0$, the canonical monomorphism $M \to M/K \oplus M/L$ splits (e.g., [17, 41.20]). Furthermore, M is called *direct injective* if for any direct summand $K \subset M$, any monomorphism $K \to M$ splits. The module M is *continuous* if it is π-injective and direct injective. Notice that any self-injective module M has these properties (e.g., [6], [11]).

The module M is called Σ*-self-projective* provided it is $M^{(\Lambda)}$-projective for any index set Λ. This is equivalent to M being projective in the category $\sigma[M]$ (e.g., [17, 18.3]). Notice that it is equivalent to self-projectivity of M provided M is a direct sum of finitely generated modules.

3.5. Correct module classes in R-Mod.

(1) *The following classes are mono-correct:*

  (i) *the class of artinian modules;*
  (ii) *the class of self-injective modules;*
  (iii) *the class of continuous modules;*
  (iv) *the class of semisimple modules.*

(2) *The following classes are epi-correct:*

  (i) *the class of noetherian modules;*
  (ii) *the class of Σ-self-projective supplemented modules;*
  (iii) *the class of semisimple modules.*

Proof. (1)(i) and (2)(i) are clear by 3.3(1),(2).

(1)(ii) is shown in [3] and it was observed in [11, Corollary 3.18] that the assertion can be extended to (iii).

(iv) Since semisimple modules are self-injective the assertion follows from (iii).

(2)(ii) Let M, N be Σ-self-projective supplemented modules. Then $M/\operatorname{Rad}(M)$ and $N/\operatorname{Rad}(N)$ are semisimple and epi-equivalent. Since epimorphisms of semisimple modules split they are also mono-equivalent and hence isomorphic by (1)(iv).

Moreover, the projectivity condition implies that M and N have small radicals (see [17, 42.3]). Therefore the isomorphism $M/\operatorname{Rad}(M) \simeq N/\operatorname{Rad}(N)$ can be lifted to an isomorphism $M \simeq N$.

(iii) follows from (ii) since any semisimple module is Σ-self-projective. □

3.6. Class $\sigma[M]$ mono-correct. *For a module M the following are equivalent:*

(a) *The class of all modules in $\sigma[M]$ is mono-correct (epi-correct);*

(b) *every (injective) module in $\sigma[M]$ is mono-correct;*

(c) *every module in $\sigma[M]$ is epi-correct;*

(d) *M is semisimple.*

Proof. (a)⇒(b) resp. (c) is trivial.

(b)⇒(d) Assume M not to be semisimple. Then there exists some module $N \in \sigma[M]$ which is not M-injective. Denote by $\widehat{N}$ the M-injective hull of N and put $L = \operatorname{Tr}(\sigma[M], \widehat{N}^{\mathbb{N}})$, the countable product of $\widehat{N}$ in $\sigma[M]$ (see 2.1). Then L is M-injective and it is mono-equivalent to $N \oplus L$ which is not M-injective. So L and $N \oplus L$ are not isomorphic contradicting condition (a).

(d)⇒(a),(c) If M is semisimple then all modules in $\sigma[M]$ are semisimple and the assertion follows by 3.5.

(c)⇒(d) Assume M not to be semisimple. Then there exists a simple module $E \in \sigma[M]$ which is not M-projective, that is, there exists an epimorphism $p : N \to E$ in $\sigma[M]$ which does not split. Assume N has a semisimple direct summand K. If $(K)p = E$, then the restriction $p : K \to E$ splits, contradicting the choice of p. So we may assume that N has no simple direct summand. Put $L = N^{(\mathbb{N})}$. Then L and $E \oplus L$ are epi-equivalent. By an exchange property (e.g., [10, 18.17]), this implies that E is isomorphic to a direct summand of N, contradicting the choice of N. This shows that every simple module in $\sigma[M]$ is M-projective, that is, M is semisimple (e.g., [17, 20.3]). □

As a special case we have a corollary which is partly proved in [13, Theorem 1].

3.7. All R-modules mono-correct. *For R the following are equivalent:*

(a) *The class of all left R-modules is mono-correct (epi-correct);*

(b) *every (injective) left R-module is mono-correct;*

(c) *every (projective) left R-module is epi-correct;*

(d) *R is left semisimple (= right semisimple).*

Now we consider correctness of some smaller classes of modules.

3.8. Class of weakly M-injectives mono-correct. *For M the following are equivalent:*

(a) *The class of weakly M-injective modules in $\sigma[M]$ is mono-correct;*

(b) *M is locally noetherian.*

Proof. (b)⇒(a) If M is locally noetherian, then each weakly M-injective module is M-injective and the class of these modules in $\sigma[M]$ is mono-correct (see 3.5).

(a)⇒(b) We follow the pattern of the proof of 3.6(b)⇒(d). Assume M not to be locally noetherian. Then there exists some weakly M-injective module $N \in \sigma[M]$ which is not M-injective. Denote by $\widehat{N}$ the M-injective hull of N and put $L = \mathrm{Tr}(\sigma[M], \widehat{N}^{\mathbb{N}})$. Then L is M-injective and it is mono-equivalent to $N \oplus L$ which is not M-injective. So L and $N \oplus L$ are not isomorphic contradicting condition (a). □

Recalling that weakly R-injective is the same as FP-injective we have:

3.9. Class of FP-injectives mono-correct. *For a ring R the following are equivalent:*

(a) *The class of FP-injective left R-modules is mono-correct;*

(b) *R is left noetherian.*

Dually we get a characterization of left perfect rings. For this we do not need an identity element in the ring - it suffices to have a ring T with enough idempotents. Recall that such a ring T is left perfect if and only if all flat left T-modules are projective (e.g., [17, 49.9]).

3.10. Class of flat modules epi-correct. *For a ring T with enough idempotents, the following are equivalent:*

(a) *The class of flat left T-modules is epi-correct;*

(b) *T is left perfect.*

Proof. (a)⇒(b) We slightly modify the proof of 3.6,(c)⇒(d). Assume there exists a flat T-module which is not projective and let $P \to F$ be an epimorphism where P is a projective T-module. Put $L = P^{(\mathbb{N})}$. Then L and $F \oplus L$ are epi-equivalent. Since one of them is projective and the other is not they cannot be isomorphic, contradicting (a). Hence all flat modules are projective and so T is left perfect.

(b)⇒(a) If T is left perfect, then every projective module is supplemented and hence the class of projectives is epi-correct by 3.5. □

The proof of the following observation is similar to the proof of 3.8.

3.11. Pure-injectives mono-correct. *For a ring R the following are equivalent:*

(a) *The class of pure-injective left R-modules is mono-correct;*

(b) *every pure-injective left R-module is injective;*

(c) *every short exact sequence of left R-modules is pure;*

(d) *R is a von Neumann regular ring.*

Proof. (c)⇔(d) A well-known characterization of von Neumann regular rings.

(a)⇒(b) Let N be a pure-injective left R-module and denote its injectve hull by $E(N)$. Then (as in the proof (b)⇒(d) in 3.6) $L = E(N)^{\mathbb{N}}$

is mono-equivalent to the pure-injective module $N \oplus L$ and so $L \simeq N \oplus L$. This implies that N is injective.

(b)⇒(d) This is obvious by the fact that an exact sequence is pure provided $\mathrm{Hom}(-, Q)$ is exact on it for any pure-injective module Q (see 2.2). □

4. Pure-correct classes of modules

Replacing in Section 3 the morphisms by pure morphisms leads to the notion of pure-correct classes of modules.

4.1. Pure-isomorphic modules. Two modules M and N are called

pure-mono-equivalent if there are pure monomorphisms $M \to N$ and $N \to M$,

pure-epi-equivalent if there are pure epimorphisms $M \to N$ and $N \to M$.

We denote the first case by $M \overset{\mathrm{m}}{\underset{\mathrm{p}}{\simeq}} N$ and the second case by $M \overset{\mathrm{e}}{\underset{\mathrm{p}}{\simeq}} N$.

4.2. Pure-correct modules. An R-module M is said to be

pure-mono-correct if for any module N, $M \overset{\mathrm{m}}{\underset{\mathrm{p}}{\simeq}} N$ implies $M \simeq N$,

pure-epi-correct if for any module N, $M \overset{\mathrm{e}}{\underset{\mathrm{p}}{\simeq}} N$ implies $M \simeq N$.

Notice that for classes of FP-injective (absolutely pure) modules the condition pure-mono-correct is equivalent to mono-correct. Similarly, for classes of flat modules the property pure-epi-correct is the same as epi-correct. In particular, over a von Neumann regular ring R, for any class of modules any correctness condition is equivalent to the corresponding pure version.

Applying the right functor ring we can show the following.

4.3. Pure-injectives are pure-mono-correct. *For any ring R the class of pure-injective modules is pure-mono-correct.*

Proof. We refer to the notions in 2.4. The functor $V \otimes_R - : R\text{-Mod} \to S\text{-Mod}$ takes pure injectives to injective S-modules (see [17, 52.3]) and takes pure-mono-equivalent pure injective R-modules $M \overset{\mathrm{m}}{\underset{\mathrm{p}}{\simeq}} N$ to mono-equivalent injective S-modules $V \otimes_R M \overset{\mathrm{m}}{\simeq} V \otimes_R N$. This implies that $V \otimes_R M \simeq V \otimes_R N$ (by 3.5) and we conclude $M \simeq N$. □

Recall that R is *left pure semisimple* if every left R-module is a direct sum of finitely generated modules. The equivalence (c)⇔(e) in the next theorem was proved in [13, Theorem 3].

4.4. Left pure semisimple rings. *For a ring R the following are equivalent:*

(a) *The class of all left R-modules is pure-epi-correct;*

(b) *the left functor ring T (of R-*Mod*) is left perfect;*

(c) *the class of all left R-modules is pure-mono-correct;*

(d) *the right functor ring S (of* Mod-*R) is left locally noetherian;*

(e) *R is left pure semisimple.*

Proof. (b)⇔(d)⇔(e) are well known (e.g., [17, 53.6, 53.7]).

(a)⇔(b) Apply the notions from 2.3. Obviously $\widehat{\mathrm{Hom}}(U,-)$ respects pure epi-morphisms and hence any pair of pure-epi-equivalent modules $M \overset{\mathrm{e}}{\underset{\mathrm{p}}{\simeq}} N$ yields an epi-equivalent pair $\widehat{\mathrm{Hom}}(U,M) \overset{\mathrm{e}}{\simeq} \widehat{\mathrm{Hom}}(U,N)$ of flat T-modules. Also, every pair of flat T-modules can be obtained by a pair of pure-epi-equivalent left R-modules. Hence all R-modules are pure-epi-correct if and only if the class of flat T-modules is epi-correct. By 3.10 this is equivalent to T being left perfect.

(c)⇔(d) We refer to the notions from 2.4. The functor $V \otimes_R -$ respects pure monomorphisms and relates the pure-mono-equivalent left R-modules to the mono-equivalent FP-injective left S-modules. Hence all R-modules are pure-mono-correct if and only if the class of all FP-injective S-modules is mono-correct. By 3.8 this is equivalent to ${}_SS$ being locally noetherian. □

We finish the paper with some questions and suggestions.

4.5. π-injective modules mono-correct? As mentioned in 3.5, it is shown in [11, Corollary 3.18] that the class of continuous modules is mono-correct. It is also pointed out there - and an example is given - that this does not hold for the class of π-injective (quasi-continuous) modules in general: Take a commutative domain R which is not a principal domain and consider any ideal $I \subset R$ which is not principal. Then R and I are (trivially) π-injective and mono-equivalent but not isomorphic.

Under which conditions on the ring R (besides semisimplicity) is the class of π-injective modules mono-correct?

4.6. Discrete modules epi-correct? The mono-correctness of the class of continuous modules generalizes the mono-correctness of the class of self-injective modules. Dually, in 3.5(2) we have shown the epi-correctness of the class of Σ-self-projective supplemented modules. Can this be extended to the (some) class of *discrete modules*, i.e., supplemented modules which are direct projective and π-projective?

4.7. Projective modules epi-correct? We have seen that for left perfect rings the class of projectives (and flats) is epi-correct. Is there another condition which makes this happen?

It is easy to see that the epi-correctness of the class of pure-projectives implies that R is von Neumann regular. Is the class of projective modules epi-correct in this case?

4.8. Split-correctness. Splitting exact sequences are special cases of pure exact sequences. In an obvious way the notions of split-mono-equivalent and split-epi-equivalent modules can be introduced and applied to define *split-mono-correct* and *split-epi-correct classes of modules.* Clearly, for classes of injective modules split-mono-correctness is equivalent to mono-correctness, and for classes of projective modules split-epi-correctness is the same as epi-correctness. Notice that in the category **Set** every mono-morphism splits (is a coretraction) and (with the Axiom of Choice) every epimorphism splits. Hence the Cantor-Bernstein Theorem could be also phrased as: *The class of all objects in* **Set** *is split-mono-correct and split-epi-correct.*

Here the question arises, which rings can be characterized by the split-epi-correctness or the split-mono-correctness of some classes of modules?

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