Closure operations in module categories

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Abstract

Singular left modules over an associative ring A are characterized by the fact that the annihilator of every element is an essential left ideal. These modules play an important part in torsion theory. What can be said about A-bimodules whose elements are annihilated by (two-sided) essential ideals? It is shown that for semiprime rings A this property characterizes bimodules which are singular in the category $\sigma[A]$ of bimodules subgenerated by A. Based on this observation the closure operations on bimodules studied by M. Ferrero for prime and semiprime rings A are related to the singular torsion theory in $\sigma[A]$. In this context we give some characterizations of strongly prime rings. Our methods also apply to non-associative rings.

Introduction

In our first sections A will denote an associative ring with unit and A-*Mod* will stand for the category of all unital left A-modules.

Our first objective is to study singularity and non-singularity conditions in the category $\sigma[M]$ whose objects are submodules of M-generated modules.

In section 2 we give a characterization of strongly prime modules by properties of non-singular modules in $\sigma[M]$.

In section 3 we are concerned with the investigation of the singular closure of a submodule in any module in $\sigma[M]$. Our main observation is that for non-singular modules $K \subset N$ in $\sigma[M]$ the singular closure of K in N coincides with the essential closure of K in N. If K is essential in N this yields a bijective correspondence between closed submodules of K and N.

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In any self-injective module N the closed submodules are direct summands. Hence for any module M which is non-singular in $\sigma[M]$ we have a bijection between the closed submodules of M and the direct summands of its M-injective hull $\widehat{M}$.

In general, a direct sum of copies of $\widehat{M}$ need no longer be self-injective and the closed submodules of $\widehat{M}^{(\Lambda)}$ need no longer be direct summands. Nevertheless we still obtain nice relationships between closed A-submodules of $M^{(\Lambda)}$ and $\widehat{M}^{(\Lambda)}$ and the closed T-submodules of $T^{(\Lambda)}$, where $T := \mathit{End}_A(\widehat{M})$. In particular, putting $M = A$ we obtain, for left non-singular rings A, a bijection between closed A-submodules of module $A^{(\Lambda)}$ and $Q(A)^{(\Lambda)}$, and closed $Q(A)$-submodules of $Q(A)^{(\Lambda)}$, where $Q(A)$ denotes the maximal left quotient ring of A.

The last two sections are devoted to bimodule properties of a ring. For this, let A be any ring and $M(A)$ its multiplication ring, i.e. the subring of $\mathit{End}_{\mathbb{Z}}(A)$ generated by left and right multiplications with elements of A and the identity map of A. Then A can be considered as left $M(A)$-module and we form $\sigma[A]$ as a subcategory of $M(A)$-*Mod* (see [6]).

In section 4 we observe that for semiprime rings A the singular modules in $\sigma[A]$ are characterized by the fact that every cyclic $M(A)$-submodule is annihilated by an essential ideal of A. For an associative ring A this is equivalent to say that every element of the module is annihilated by an essential ideal of A.

We call a ring A *strongly prime* if it is strongly prime as an $M(A)$-module. The characterization of strongly prime modules mentioned above immediately yields a characterization of these rings. For an associative ring A we also have the notions *left strongly prime* and *right strongly prime* and we will point out some interactions between these properties. Notice that some authors call a ring 'strongly prime' if it is left and right strongly prime; this differs from our definition.

The final section is concerned with the closure operations for modules in $\sigma[A]$ for semiprime rings A. Based on the fact that these rings are non-singular in $\sigma[A]$ our previous results on non-singular modules apply. Moreover, the A-injective hull $\widehat{A}$ of A is the central closure of A and we obtain bijective correspondences between closed $M(A)$-submodules of $A^{(\Lambda)}$, closed $M(\widehat{A})$-submodules of $\widehat{A}^{(\Lambda)}$ and closed submodules of $T^{(\Lambda)}$, where $T := \mathit{End}_{M(A)}(\widehat{A})$ is the *extended centroid* of A.

These final results extend correspondence theorems for centred bimodules over prime and semiprime rings proved in [3, 5]. We point out that the methods used here are completely different from the methods used in those papers.

1 Basic notions

Let A be an associative ring with unit and let M be a left A-module. An A-module is said to be *subgenerated by* M if it is isomorphic to a submodule of an M-generated module. By $\sigma[M]$ we denote the full subcategory of A-*Mod* consisting of all modules which are subgenerated by M (see [8]).

A module $N \in \sigma[M]$ is said to be M-injective if the functor $Hom_A(-, N)$ is exact on short exact sequences with central object M. It is well-known that this is equivalent to the property that $Hom_A(-, N)$ is an exact functor on $\sigma[M]$.

A submodule $K \subset N$ is said to be *essential* in N, we write $K \trianglelefteq N$, if $K \cap L \neq 0$ for any non-zero submodule $L \subset N$. Every $N \in \sigma[M]$ is an essential submodule of some M-injective module $\widehat{N}$ in $\sigma[M]$ which is called the *M-injective hull* of N. Notice that every M-injective module in $\sigma[M]$ is M-generated.

For our investigations the following notion of singularity - which extends the well-known singularity in A-*Mod* - will be most important:

Definition. Let M and N be A-modules. N is called *singular in* $\sigma[M]$ or *M-singular* if $N \simeq L/K$ for some $L \in \sigma[M]$ and $K \trianglelefteq L$ (see [6, 2]).

In case $M = A$, we have the usual singularity in A-*Mod* and instead of *A-singular* we will just say *singular*. It is well-known that a module is singular in A-*Mod* if and only if the annihilator of each of its elements is an essential left ideal in A.

The class $\mathcal{S}_M$ of all M-singular modules is closed under direct sums, submodules, and factor modules, i.e. it is a hereditary pretorsion class in $\sigma[M]$. Hence every module $N \in \sigma[M]$ has a largest M-singular submodule which we denote by $\mathcal{S}_M(N)$. If $\mathcal{S}_M(N) = 0$ then N is said to be *non-M-singular*.

If the module M itself is non-M-singular it is also called *polyform* (see [2]). Such modules can be characterized in the following way:

1.1 Polyform modules. Characterization.

For a module M with M-injective hull $\widehat{M}$, the following are equivalent:

(a) M is polyform (i.e. $\mathcal{S}_M(M) = 0$)

(b) for any submodule $K \subset M$ and $0 \neq f : K \to M$, $Ke\,f$ is not essential in K;

(c) for any $K \in \sigma[M]$ and $0 \neq f : K \to M$, $Ke\,f$ is not essential in K;

(d) $End_A(\widehat{M})$ is regular.

2 Strongly prime modules

In this section A again will denote an associative ring with unit, and M will be a left A-module.

An A-module M is said to be *strongly prime* if for every submodule $N \subset M$, $M \in \sigma[N]$ (see [7, 1]). The following equivalences are fairly obvious:

2.1 Strongly prime modules.

For an A-module M with M-injective hull $\widehat{M}$, the following are equivalent:

(a) M is strongly prime;

(b) $\widehat{M}$ is strongly prime;

(c) $\widehat{M}$ is generated by each of its nonzero submodules.

To get some more characterizations we introduce a new

Definition. A module $N \in \sigma[M]$ is called an *absolute subgenerator in $\sigma[M]$* if every non-zero submodule $K \subset N$ is a subgenerator in $\sigma[M]$ (i.e. $M \in \sigma[K]$).

It is obvious that every absolute subgenerator is a strongly prime module, and M itself is an absolute subgenerator in $\sigma[M]$ if and only if it is strongly prime. Moreover, if N is an absolute subgenerator in $\sigma[M]$ with $\mathcal{S}_M(N) \neq 0$, then all modules in $\sigma[M]$ are M-singular.

With this notions we have the following

2.2 Characterization of strongly prime modules.

For a polyform A-module M, the following assertions are equivalent;

(a) M is strongly prime;

(b) every module $K \in \sigma[M]$ with $\mathcal{S}_M(K) \neq K$ is a subgenerator;

(c) every non-zero module $K \in \sigma[M]$ with $\mathcal{S}_M(K) = 0$ is an absolute subgenerator;

(d) for every non-zero $N \in \sigma[M]$,

$$\mathcal{S}_M(N) = \bigcap \{K \subset N \mid N/K \text{ is an absolute subgenerator in } \sigma[M]\};$$

(e) there exists an absolute subgenerator in $\sigma[M]$.

In this case, every projective module in $P \in \sigma[M]$ is an absolute subgenerator and hence $\mathcal{S}_M(P) = 0$.

Proof. $(a) \Rightarrow (b)$ Since M is polyform the M-singular modules $X \in \sigma[M]$ are characterized by the property $Hom_A(X, \widehat{M}) = 0$ (see [6]). Hence for any $K \in \sigma[M]$ with $\mathcal{S}_M(K) \neq K$, there exists a non-zero homomorphism $f : K \to \widehat{M}$. Since M (and $\widehat{M}$) is strongly prime the image of f is a subgenerator in $\sigma[M]$ and so is K.

$(b) \Rightarrow (c)$ If $\mathcal{S}_M(N) = 0$, every non-zero submodule of N is non-M-singular.

$(c) \Rightarrow (d)$ Let $K \subset N$ be such that N/K is an absolute subgenerator in $\sigma[M]$. Assume $\mathcal{S}_M(N) \not\subset K$. Then $(K + \mathcal{S}_M(N))/K$ is an M-singular submodule of N/K.

This is impossible since not all modules in $\sigma[M]$ are M-singular. So $\mathcal{S}_M(N) \subset K$ and $\mathcal{S}_M(N)$ is contained in the given intersection.

Since M is polyform, $N/\mathcal{S}_M(N)$ is non-M-singular and hence an absolute subgenerator in $\sigma[M]$. This proves our assertion.

$(d) \Rightarrow (e)$ Since $\mathcal{S}_M(M) = 0$ the equality in (d) implies the existence of an absolute subgenerator in $\sigma[M]$.

$(e) \Rightarrow (a)$ Let N be an absolute subgenerator in $\sigma[M]$. Then clearly $\mathcal{S}_M(N) = 0$ and N is a strongly prime module. Now the proof $(a) \Rightarrow (c)$ applies and so M is an absolute subgenerator in $\sigma[M]$.

Every projective module $P \in \sigma[M]$ is isomorphic to a submodule of some $M^{(\Lambda)}$ which is an absolute subgenerator (by (c)). Hence P is also an absolute subgenerator. □

Applying 2.2 to $M = A$, we immediately have the following

2.3 Characterization of left strongly prime rings.

For the ring A the following properties are equivalent:

(a) A is a left strongly prime ring;

(b) every left A-module which is not singular is a subgenerator in A-Mod;

(c) every non-singular left A-module is an absolute subgenerator in A-Mod;

(d) for every non-zero left A-module N,

$$Z(N) = \bigcap \{K \subset N \mid N/K \text{ is an absolute subgenerator in } A\text{-Mod}\};$$

(e) there exists an absolute subgenerator in A-Mod.

3 Closure operations on modules in $\sigma[M]$

Let A be an associative ring with unit and let M be any left A-module.

Definitions. Let $K \subset N$ be modules in $\sigma[M]$. A maximal essential extension of K in N will be called an *essential closure* of K in N. K is said to be *closed in* N if it has no proper essential extension in N.

We define the *M-singular closure* $[K]_N$ of K in N by

$$[K]_N/K = \mathcal{S}_M(N/K).$$

K is said to be *M-singular closed* in N if $K = [K]_N$, i.e. if N/K is non-M-singular.

Notice that forming $[K]_N$ depends on the category $\sigma[M]$ we are working in. Usually it should be clear from the context which closure is meant.

In particular, for $M = A$ the notion A-singular defines precisely those A-modules X for which the annihilator of each element is an essential left ideal in A. Hence we have the following characterization of

3.1 Singular closure in ***A-Mod*****.**

Let $K \subset N$ be left A-modules. Then in A-Mod the singular closure of K in N is

$$[K]_N = \{x \in N \mid Ix \subset K \ \textit{ for some essential left ideal } I \subset A\}.$$

It should be observed that, in general, forming the singular closure need not be an idempotent operation on the submodules of N. However, it will be idempotent in the cases we are interested in, for example, if M is polyform or in the following situation:

3.2 Essential closure in non-M-singular modules.

Let $K \subset N$ be non-M-singular modules in $\sigma[M]$. Then for every essential closure $\overline{K}$ in N,

$$\overline{K}/K = \mathcal{S}_M(N/K) = [K]_N/K.$$

This implies that K has a unique essential closure in N.

Proof. Clearly $\overline{K} \subset [K]_N$. Since $[K]_N$ is non-M-singular, any map from $[K]_N$ to an M-singular module has essential kernel. Hence $K \trianglelefteq [K]_N$ and $\overline{K} = [K]_N$. □

The preceding observation implies a close relationship between closed submodules of essential extensions of non-M-singular modules.

Denote by $\mathcal{L}(X)$ the lattice of submodules of any module X.

3.3 Correspondence of closed submodules.

Let $K \trianglelefteq N$ be non-M-singular modules in $\sigma[M]$. Then the mappings

$$\mathcal{L}(K) \to \mathcal{L}(N), \quad U \mapsto [U]_N,$$
$$\mathcal{L}(N) \to \mathcal{L}(K), \quad V \mapsto V \cap K,$$

provide a bijection between closed submodules of K and closed submodules of N.

Proof. For a closed submodule $U \subset K$, $U = K \cap [U]_N$. If $V \subset N$ is a closed submodule, then $V \cap K \trianglelefteq V$ and hence $[V \cap K]_N = V$. Therefore the composition of the two maps yields the identity on closed submodules. □

For polyform modules M, we can relate singular closed submodules of any module $N \in \sigma[M]$ to closed submodules of the non-M-singular module $N/\mathcal{S}_M(N)$:

3.4 Correspondence of singular closed submodules.

Let M be a polyform A-module and $N \in \sigma[M]$. Then the canonical projection

$$p : N \to N/\mathcal{S}_M(N)$$

provides a bijection between singular closed submodules of N and (singular-) closed submodules of $N/\mathcal{S}_M(N)$.

Proof. Since $N/\mathcal{S}_M(N)$ is non-M-singular, its closed submodules coincide with the M-singular closed submodules (by 3.2).

Let $U \subset N$ be singular closed and assume $\mathcal{S}_M(N) \not\subset U$. Then $(U + \mathcal{S}_M(N))/U$ is a non-zero M-singular submodule of N/U, a contradiction.

Now the bijection suggested follows from the canonical isomorphism

$$N/U \simeq (N/\mathcal{S}_M(N))/(U/\mathcal{S}_M(N)).$$

□

Because of the above correspondence we will concentrate our investigation on non-M-singular modules.

The correspondence described in 3.3 has remarkable consequences. They are based on the well-known fact (e.g. [2, Proposition 7.2]):

3.5 Closed submodules in self-injective modules.

In a self-injective module, every closed submodule is a direct summand.

3.6 Lemma. *Let $K \subset N$ be non-M-singular modules in $\sigma[M]$. Let $U \subset K$ be a closed submodule and $\overline{U} \subset N$ its essential closure in N.*

(1) The canonical map $K/U \to N/\overline{U}$ is an essential monomorphism.

(2) If N is M-injective then $N/\overline{U}$ is an M-injective hull on K/U.

(3) If K/U is M-injective then $K/U \simeq N/\overline{U}$.

Proof. (1) The composition of the canonical homomorphisms

$$K/U \to N/U \to N/\overline{U}$$

is a monomorphism since $K \cap \overline{U} = U$. Assume its image is not essential in $N/\overline{U}$. Then there exists a submodule $\overline{U} \subset V \subset N$ such that

$$((K + \overline{U})/\overline{U}) \cap (V/\overline{U}) = (K \cap V + \overline{U})/\overline{U} = 0,$$

and hence $K \cap V \subset K \cap \overline{U} = U$ implying $V \subset \overline{U}$.

(2) If N is M-injective the closed submodule $\overline{U} \subset N$ is a direct summand (by 3.5) and hence $N/\overline{U}$ is M-injective. Now the assertion is clear by (1).

(3) is obvious by (1). □

For the module M itself we obtain the following properties:

3.7 Relations with the injective hull.
Let M be a polyform module with M-injective hull $\widehat{M}$ and $T := End_A(\widehat{M})$.

(1) There exist bijections between

(i) the closed submodules of M,

(ii) the direct summands of $\widehat{M}$,

(iii) the left ideals which are direct summands of T.

(2) (i) For any essential left ideal $I \trianglelefteq T$, $\widehat{M}I \trianglelefteq \widehat{M}$.

(ii) For every $V \subset \widehat{M}$ with $Tr(\widehat{M}, V) \trianglelefteq \widehat{M}$, $Hom_A(\widehat{M}, V) \trianglelefteq T$.

Proof. (1) This follows from 3.3 and 3.5 and the fact that direct summands in $\widehat{M}$ correspond to left ideals which are direct summands in T.

(2) Assume that $\widehat{M}I$ is not essential in $\widehat{M}$. Then the essential closure $\overline{\widehat{M}I}$ of $\widehat{M}I$ is a proper direct summand in $\widehat{M}$ and $I \subset Hom_A(\widehat{M}, \widehat{M}I) \subset Hom_A(\widehat{M}, \overline{\widehat{M}I})$, which is a proper direct summand in T. This is a contradiction to $I \trianglelefteq T$.

Now let $V \subset \widehat{M}$ such that $Tr(\widehat{M}, V) \trianglelefteq \widehat{M}$. If $Hom_A(\widehat{M}, V)$ is not essential in T, then it is contained in a proper direct summand Te, $e^2 = e \in T$. Then

$$Tr(\widehat{M}, V) = \widehat{M} Hom_A(\widehat{M}, V) \subset \widehat{M}e,$$

contradicting $Tr(\widehat{M}, V) \trianglelefteq \widehat{M}$. □

For an infinite index set Λ, the direct sum $\widehat{M}^{(\Lambda)}$ need not be M-injective. Nevertheless we have nice characterizations of its closed submodules.

First we make some technical observations.

3.8 Lemma. *Let M be a polyform A-module with M-injective hull $\widehat{M}$. Then every closed submodule of $\widehat{M}^{(\Lambda)}$ is $\widehat{M}$-generated.*

Proof. Let $U \subset \widehat{M}^{(\Lambda)}$ be a closed submodule, and denote by $\{U_\gamma\}_\Gamma$ the (directed) set of finitely generated submodules of U. Then each U_γ is contained in some finite partial sum of $\widehat{M}^{(\Lambda)}$, which is M-injective and contains the unique essential closure $\overline{U}_\gamma$ (of U_γ in $\widehat{M}^{(\Lambda)}$) as a direct summand. Clearly $\overline{U}_\gamma \subset U$ and $\varinjlim \overline{U}_\gamma = U$. This implies that U is $\widehat{M}$-generated. □

3.9 Lemma. *Let M be a finitely generated polyform A-module with M-injective hull $\widehat{M}$ and $T := End_A(\widehat{M})$. Then:*

(1) For every $f : \widehat{M} \to \widehat{M}^{(\Lambda)}$, $\widehat{M}f$ is contained in a finite partial sum of $\widehat{M}^{(\Lambda)}$ and hence we may identify

$$T^{(\Lambda)} = Hom_A(\widehat{M}, \widehat{M}^{(\Lambda)}).$$

(2) *for any* $f_1, \dots, f_n \in Hom_A(\widehat{M}, \widehat{M}^{(\Lambda)})$, $\sum_{i=1}^n \widehat{M} f_i$ *is a direct summand in* $\widehat{M}^{(\Lambda)}$ *and the exact sequence determined by the* f_i *splits:*

$$\widehat{M}^n \to \sum_{i=1}^{n} \widehat{M} f_i \to 0;$$

(3) *for every left* T*-submodule* $X \subset T^{(\Lambda)}$, $Hom_A(\widehat{M}, \widehat{M}X) = X$.

Proof. (1) Since M is finitely generated, for every $f : \widehat{M} \to \widehat{M}^{(\Lambda)}$, we have the following diagram, where $k \in I\!N$,

$$\begin{array}{cccc} 0 \to & M & \to & \widehat{M} \\ & \downarrow f\mid_M & & \downarrow f \\ & \widehat{M}^k & \to & \widehat{M}^{(\Lambda)}. \end{array}$$

Since $\widehat{M}^k$ is M-injective we can extend $f\mid_M$ to some $g : \widehat{M} \to \widehat{M}^k$. However, since M is polyform there is a unique extension of $f\mid_M$ from M to $\widehat{M}$. This means $f = g$ and $\widehat{M} f \subset \widehat{M}^k$.

As a consequence, for every $f \in Hom_A(\widehat{M}, \widehat{M}^{(\Lambda)})$ we have in fact, for some $k \in I\!N$,

$$f \in Hom_A(\widehat{M}, \widehat{M}^k) = T^k,$$

which implies our assertion.

(2) By (1), $\sum\limits_{i=1}^{n} \widehat{M} f_i$ is contained in some finite partial sum $\widehat{M}^k$, $k \in I\!N$. Then we have

$$\widehat{M}^n \xrightarrow{f} \sum_{i=1}^{n} \widehat{M} f_i \subset \widehat{M}^k,$$

where f is determined by the f_i.

Now f may be considered as an endomorphism of $\widehat{M}^{n+k}$. Since $End_A(\widehat{M}^{n+k})$ is regular the image and the kernel of f are direct summands (see [8, 37.7]) proving our assertion.

(3) Let $g \in Hom_A(\widehat{M}, \widehat{M}X)$. Then $Mg \subset \sum\limits_{i=1}^{k} \widehat{M} x_i$, for some $x_i \in X$. By (2), $\sum\limits_{i=1}^{k} \widehat{M} x_i$ is M-injective and $g|_M$ can be uniquely extended from M to $\widehat{M}$. Hence we may assume $\widehat{M} g \subset \sum\limits_{i=1}^{k} \widehat{M} x_i$. We describe the situation in the diagram

$$\begin{array}{ccccc} & & \widehat{M} & & \\ & & \downarrow g & & \\ \widehat{M}^k & \to & \sum_{i=1}^k \widehat{M} x_i & \to & 0\,. \end{array}$$

By (2), the lower row splits. Hence we have a map $\widehat{M} \to \widehat{M}^k$ which yields a commutative diagram and is determined by some $t_1, \dots, t_k \in T$ satisfying $g = \sum_{i \le k} t_i x_i \in X$.

This proves $Hom_A(\widehat{M}, \widehat{M}X) = X$. □

With these preparations we are able to prove correspondences for closed submodules of infinite direct sums.

3.10 Correspondences for closed submodules of $M^{(\Lambda)}$.

Let M be a finitely generated polyform A-module with M-injective hull $\widehat{M}$. We denote $T := End_A(\widehat{M})$ and identify $T^{(\Lambda)} = Hom_A(\widehat{M}, \widehat{M}^{(\Lambda)})$ (by 3.9).

There are bijective correspondences between

(i) the closed submodules of $M^{(\Lambda)}$,

(ii) the closed submodules of $\widehat{M}^{(\Lambda)}$,

(iii) the closed left T-submodules of $T^{(\Lambda)}$.

For closed submodules $V \subset M^{(\Lambda)}$ and $X \subset T^{(\Lambda)}$, these are given by

$$\begin{array}{ccccc} V & \rightarrow & [V]_{\widehat{M}^{(\Lambda)}} & \rightarrow & Hom_A(\widehat{M}, [V]_{\widehat{M}^{(\Lambda)}}), \\ M^{(\Lambda)} \cap \widehat{M}X & \leftarrow & \widehat{M}X & \leftarrow & X\,. \end{array}$$

Proof. The correspondence between (i) and (ii) is just a special case of the situation described in 3.3.

Let $U \subset \widehat{M}^{(\Lambda)}$ be a closed submodule. We want to show that $Hom_A(\widehat{M}, U)$ is a closed T-submodule in $T^{(\Lambda)}$. Since T is left non-singular it is to show that any $f \in T^{(\Lambda)}$, which satisfies $If \subset Hom_A(\widehat{M}, U)$ for an essential left ideal $I \subset T$, already belongs to $Hom_A(\widehat{M}, U)$. In fact, this condition implies $\widehat{M}If \subset U$, where $\widehat{M}I \trianglelefteq \widehat{M}$ (by 3.7). Assume that $\widehat{M}f \not\subset U$. Then the map $\widehat{M} \xrightarrow{f} \widehat{M}^{(\Lambda)} \rightarrow \widehat{M}^{(\Lambda)}/U$ is non-zero and has essential kernel. This is not possible since $\widehat{M}^{(\Lambda)}/U$ is non-M-singular and we conclude that $f \in Hom_A(\widehat{M}, U)$.

Moreover, $\widehat{M}Hom_A(\widehat{M}, U) = U$ (by 3.8).

Now let $X \subset T^{(\Lambda)}$ be a closed T-submodule and put $U := [\widehat{M}X]_{\widehat{M}^{(\Lambda)}}$. Denote by $\{X_\gamma\}_\Gamma$ the family of finitely generated submodules of X. Assume there is an

$$f \in Hom_A(\widehat{M}, U) \text{ such that } \widehat{M}f \not\subset \widehat{M}X.$$

Then $V := (\widehat{M}X)f^{-1} \trianglelefteq \widehat{M}$. Clearly $(\widehat{M}X)f^{-1} = \bigcup_\Gamma(\widehat{M}X_\gamma)f^{-1}$.

The $\widehat{M}X_\gamma$ are direct summands in $\widehat{M}^{(\Lambda)}$ (by 3.9). So any $(\widehat{M}X_\gamma)f^{-1} \subset \widehat{M}$ is closed and hence a direct summand in $\widehat{M}$. This shows that V is $\widehat{M}$-generated, and by 3.7, $Hom_A(\widehat{M}, V)$ is an essential left ideal in T. By construction and 3.9,

$$Hom_A(\widehat{M}, V)f \subset Hom_A(\widehat{M}, Vf) \subset Hom_A(\widehat{M}, \widehat{M}X) = X,$$

and this implies $f \in X$ (since X is closed in $T^{(\Lambda)}$). □

In particular the preceding result applies to $A = M$. In this case we detect some more interesting relationships for submodules of free modules. Recall that for a left non-singular ring A with injective hull $E(A)$, the maximal left quotient ring is $Q(A) = End_A(E(A))$ and 3.10 reads as follows:

3.11 Closed left submodules of $A^{(\Lambda)}$.

Let A be a left non-singular ring with maximal left quotient ring $Q(A)$. There is a bijective correspondence between

(i) the closed left A-submodules of $A^{(\Lambda)}$,

(ii) the closed left A-submodules of $Q(A)^{(\Lambda)}$,

(iii) the closed left $Q(A)$-submodules of $Q(A)^{(\Lambda)}$.

Combining 3.11 with 3.10 we recover some properties of non-singular left A-modules known from localization theory.

3.12 Corollary. *Let A be a left non-singular ring with maximal left ring of quotients $Q(A)$, and let L be any non-singular left A-module. Then:*

(1) L is an essential A-submodule of a $Q(A)$-module $\widetilde{L}$.

(2) If L is a finitely generated A-module, then $\widetilde{L}$ is a finitely generated $Q(A)$-module.

(3) If L is A-injective then it is a $Q(A)$-module.

Proof. L is A-generated and we have the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & U & \to & A^{(\Lambda)} & \to & L & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \overline{U} & \to & Q(A)^{(\Lambda)} & \to & \widetilde{L} & \to & 0\,, \end{array}$$

where U is a closed submodule of $A^{(\Lambda)}$ (since L is non-singular), and $\overline{U}$ denotes the essential closure of U in $Q(A)^{(\Lambda)}$. Now apply 3.10 and 3.11. □

4 Singular pretorsion theory in $\sigma[A]$

Now let A be any ring with multiplication algebra $M(A)$, and denote by $\sigma[A]$ the full subcategory of $M(A)$-*Mod* whose objects are submodules of A-generated modules.

For any $M(A)$-module X and $x \in X$, left (right) multiplication with an $a \in A$ is defined by $ax := L_a x$ ($xa := R_a x$).

Recall that a module $N \in \sigma[A]$ is said to be *singular in* $\sigma[A]$, or A-*singular*, if $N \simeq L/K$, for some $L \in \sigma[A]$ and $K \trianglelefteq L$.

We denote by $\mathcal{S}$ the pretorsion class of all A-singular modules in $\sigma[A]$, and for any $X \in \sigma[A]$ we write $\mathcal{S}(X)$ for the largest A-singular submodule of X.

Recall that for a semiprime ring A, $\mathcal{S}(A) = 0$ and $\mathcal{S}$ is a torsion class, i.e. it is closed under extensions in $\sigma[A]$ (see [6]).

Over an associative ring A with unit, a left module is singular if and only if the annihilator of each of its elements is an essential left ideal. This characterization is based on the fact that A is projective as a left A-module and hence any homomorphism from A to a singular left A-module has essential kernel.

In general, A is not projective as a bimodule (even if it is associative) and hence we do not have a corresponding characterization of A-singular modules in $\sigma[A]$. However, any homomorphism from a non-A-singular module to an A-singular module has essential kernel. So we may expect a similar characterization of A-singularity in case A is non-A-singular, in particular, if A is semiprime. This is what we show now.

4.1 A-singular modules for semiprime rings.

Let A be a semiprime ring. Then for any $N \in \sigma[A]$, the following are equivalent:

(a) N is A-singular;

(b) for every cyclic $M(A)$-submodule $X \subset N$, there exists an essential ideal $I \trianglelefteq A$ such that $IX = 0$ (or $XI = 0$).

If A is associative, (b) is equivalent to:

(c) For every $x \in N$, there exists an essential ideal $I \trianglelefteq A$ with $Ix = 0$ (or $xI = 0$).

Proof. $(a) \Rightarrow (b)$ Let $X \subset N$ be generated by one element. Then X is contained in a finitely A-generated, A-singular module $\widetilde{X}$. For this we have an epimorphism $f : A^n \to \widetilde{X}$, for some $n \in I\!N$. Since A is non-A-singular, for each inclusion $\varepsilon_i : A \to A^n$, $i = 1, \ldots, n$, we have $Ke\,\varepsilon_i f \trianglelefteq A$. Then $I := \bigcap_{i=1}^n Ke\,\varepsilon_i f$ is an essential ideal in A and

$$I\widetilde{X} = I(A^n)f \subset (I^n)f = 0.$$

Similarly we get $\widetilde{X}I = 0$. From this the assertion follows.

$(b) \Rightarrow (a)$ Let $X \subset N$ be a cyclic $M(A)$-submodule and let $H \trianglelefteq A$ be such that $HX = 0$. Take $\widetilde{X}$ to be a finitely A-generated essential extension of X. Then we have the exact diagram

$$\begin{array}{ccccc} & & A^n & & \\ & & \downarrow f & & \\ 0 \longrightarrow X \longrightarrow & & \widetilde{X} & \stackrel{p}{\longrightarrow} \widetilde{X}/X \longrightarrow 0\,, \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

and for every inclusion $\varepsilon_i : A \to A^n$, $i = 1, \ldots, n$,

$$I_i := (X)(\varepsilon_i f)^{-1} \trianglelefteq A, \text{ and } I := \bigcap_{i=1}^n I_i \trianglelefteq A.$$

For this we have

$$<HI> \widetilde{X} = <HI> (A^n)f \subset H(I^n)f \subset HX = 0\,,$$

where $<HI>$ denotes the ideal generated by HI. This shows that $< HI >^n \subset Ke\, f$. Also, $< HI >$ is an essential ideal in A since for every non-zero ideal $L \subset A$, $0 \neq (I \cap H \cap L)^2 \subset IH \cap L$. Hence $\widetilde{X}$ and X are A-singular.

$(b) \Leftrightarrow (c)$ For an associative algebra A, the ideal generated by $x \in N$ has the form $(\mathbb{Z} + A)x(\mathbb{Z} + A)$. Hence for any ideal $I \subset A$, $Ix = 0$ implies

$$I(\mathbb{Z} + A)x(\mathbb{Z} + A) = Ix(\mathbb{Z} + A) = 0.$$

□

For an associative left non-singular ring A with unit, we have a nice one sided characterization of the A-singular (bi)modules in $\sigma[A]$. Denoting the singular submodule of any left module X by $\mathcal{S}_l(X)$ we can generalize Theorem 4.6 of [3] from prime to semiprime rings:

4.2 A-singularity over left-nonsingular rings.
For an associative semiprime ring A with unit, the following are equivalent:

(a) $\mathcal{S}_l(A) = 0$, i.e. A is left non-singular;

(b) for every module $X \in \sigma[A]$, $\mathcal{S}(X) = \mathcal{S}_l(X)$.

Proof. $(a) \Rightarrow (b)$ By Proposition 4.1, the elements of $\mathcal{S}(X)$ are annihilated by essential ideals. Since A is semiprime, any essential ideal $I \subset A$ is also essential as left ideal. So $\mathcal{S}(X) \subset \mathcal{S}_l(X)$.

Denote $N := \mathcal{S}_l(X)$. This is obviously an A-bimodule and hence it is contained in an A-generated essential extension $\widetilde{N} \in \sigma[A]$. Consider the exact sequence of A-bimodules

$$0 \to N \to \widetilde{N} \to \widetilde{N}/N \to 0.$$

By construction, $\widetilde{N}/N$ is A-singular and hence left singular by the above argument.

Considering this sequence in A-*Mod*, we have $\widetilde{N}$ as an extension of the left singular A-modules N and $\widetilde{N}/N$. Since $\mathcal{S}_l(A) = 0$ we know that the class of singular left modules is closed under extensions and hence $\widetilde{N}$ is left singular.

By construction, there is a bimodule epimorphism $f : A^{(\Lambda)} \to \widetilde{N}$. For every inclusion $\varepsilon_\lambda : A \to A^{(\Lambda)}$, $Ke\,\varepsilon_\lambda f$ is an ideal, which is essential as a left ideal, since $\widetilde{N}$ is left singular. Then it is certainly essential as an ideal showing that $\widetilde{N}$ is A-singular.

$(b) \Rightarrow (a)$ Since $A \in \sigma[A]$, we have $\mathcal{S}_l(A) = \mathcal{S}(A) = 0$. □

Remark. In Proposition 4.2 we needed a unit in A to have the usual notion of left non-singularity. For rings A without unit one can work in the category $\sigma[_AA]$ of A-subgenerated left A-modules and use the singularity defined in this category. Since obviously $\sigma[_{M(A)}A] \subset \sigma[_AA]$ a result similar to 4.2 can be shown with this notion.

Since a semiprime ring A is non-A-singular we obtain from 2.2 the following

4.3 Characterization of strongly prime rings.

For a semiprime ring A, the following properties are equivalent:

(a) *A is a strongly prime ring (in $\sigma[A]$);*

(b) *any module in $\sigma[A]$ is A-singular or a subgenerator in $\sigma[A]$;*

(c) *every non-A-singular module in $\sigma[A]$ is an absolute subgenerator in $\sigma[A]$;*

(d) *for every non-zero $N \in \sigma[A]$,*

$$\mathcal{S}(N) = \bigcap \{K \subset N \mid N/K \text{ is an absolute subgenerator in } \sigma[A]\};$$

(e) *there exists an absolute subgenerator in $\sigma[A]$.*

An associative ring A is an object of $\sigma[A]$ and of A-*Mod*. Accordingly A can be strongly prime in each of these categories. Left strongly prime rings A (i.e. A strongly prime in A-*Mod*) were characterized in 2.3. Combining this with 4.3 we arrive at a description of left strongly prime associative rings by properties of two-sided modules which was already shown in [3, Theorem 4.13]:

4.4 More characterizations of left strongly prime rings.

For an associative semiprime ring A with unit the following are equivalent:

(a) *A is a left strongly prime ring;*

(b) *any module $N \in \sigma[A]$ is A-singular (in $\sigma[A]$) or is a subgenerator in A-Mod;*

(c) *every non-A-singular module in $\sigma[A]$ is an absolute subgenerator in A-Mod;*

(d) *for every non-zero $N \in \sigma[A]$,*

$$\mathcal{S}(N) = \bigcap \{K \subset N \mid \begin{array}{l} K \text{ is a sub-bimodule and} \\ N/K \text{ is an absolute subgenerator in } A\text{-Mod}\} ;\end{array}$$

(e) *there exists a module in $\sigma[A]$ which is an absolute subgenerator in A-Mod.*

Proof. $(a) \Rightarrow (b)$ Let A be left strongly prime. Then A is left non-singular and, by 4.2, modules which are not A-singular in $\sigma[A]$ are not singular left A-modules. Hence, by 2.3, they are subgenerators in A-*Mod*.

$(b) \Rightarrow (c)$ By the same argument this follows from 2.3.

$(c) \Rightarrow (d)$ By 4.3, $\mathcal{S}(N)$ is the intersection of those $K \subset N$ for which N/K is an absolute subgenerator in $\sigma[A]$. Then N/K is non-A-singular and hence an absolute subgenerator in A-*Mod*.

$(d) \Rightarrow (e) \Rightarrow (a)$ are obvious since $\mathcal{S}(A) = 0$. □

5 Closure operations in $\sigma[A]$

We are now going to outline the transfer of the closure operations for modules in $\sigma[M]$ to the category $\sigma[A]$, where A is any ring with multiplication algebra $M(A)$.

Recall that for modules $K \subset N$ in $\sigma[A]$ we have two types of 'closures' of K in N: the maximal essential extension of K in N (which is unique if N is non-A-singular), and the A-singular closure of K in N defined by $[K]_N/K := \mathcal{S}(N/K)$.

Using the characterization of A-singular modules given in 4.1 we obtain:

5.1 Singular closure in $\sigma[A]$.
Let A be a semiprime ring and $K \subset N$ in $\sigma[A]$. Then

$$[K]_N = \{x \in N \mid Ix \subset K \text{ (or } xI \subset K) \text{ for some essential ideal } I \trianglelefteq A\}.$$

Notice that for associative semiprime rings this property was used in [3, 5] to define the closure of K in N.

Over a semiprime ring A, for any $N \in \sigma[A]$, the A-singular closed submodules are in one-to-one correspondence with the closed submodules of $N/\mathcal{S}(N)$ (by 3.4). Hence in what follows we will focus on non-A-singular modules.

Applying 3.7 to A, and recalling that the central closure $\widehat{A}$ is just the A-injective hull of A we have:

5.2 Relations with the central closure.
Let A be a semiprime ring with central closure $\widehat{A}$ and extended centroid $T := End_{M(A)}(\widehat{A})$.

(1) There exists bijections between

(i) the closed ideals in A,

(ii) the ideals which are direct summands in $\widehat{A}$,

(iii) the ideals which are direct summands in T.

(2) (i) For any essential ideal $I \trianglelefteq T$, $\widehat{A}I \subset \widehat{A}$ is an essential ideal.

(ii) For every $\widehat{A}$-generated essential ideal $V \subset \widehat{A}$, $Hom_{M(\widehat{A})}(\widehat{A}, V) \trianglelefteq T$.

Proof. (1) is immediately clear by 3.7(1).

(2) It is easy to see that the $M(A)$-submodule $\widehat{A}I$ is in fact an ideal in $\widehat{A}$. Moreover, since T is commutative, $Hom_{M(A)}(\widehat{A}, V)$ is a (two-sided) ideal in T and $Tr(\widehat{A}, V)$ (in $M(A)$-*Mod*) is an ideal in $\widehat{A}$. With this remark the assertions follow from 3.7 (2). □

Transferring the correspondece theorem for submodules of any $M^{(\Lambda)}$ and $\widehat{M}^{(\Lambda)}$ to $A^{(\Lambda)}$ and $\widehat{A}^{(\Lambda)}$ a new phenomenon occurs (similar to 5.2): In $\widehat{A}^{(\Lambda)}$ we have $M(A)$-submodules and $M(\widehat{A})$-submodules. It is a nice aspect of the theory that closed

$M(A)$-submodules and closed $M(\widehat{A})$-submodules coincide. We are going to prove this fact, which is similar to the observation (made in 3.11) that for associative rings A with unit the closed left A-submodules of $Q(A)^{(\Lambda)}$ are precisely the same as the closed left $Q(A)$-submodules.

5.3 $M(A)$-submodules of $A^{(\Lambda)}$.

Let A be a semiprime ring with central closure $\widehat{A}$ and $T := End_{M(A)}(\widehat{A})$. Then:

(1) $Hom_{M(A)}(\widehat{A}, \widehat{A}^{(\Lambda)}) = Hom_{M(\widehat{A})}(\widehat{A}, \widehat{A}^{(\Lambda)})$.

(2) Closed $M(A)$-submodules of $\widehat{A}^{(\Lambda)}$ are closed $M(\widehat{A})$-submodules, and conversely.

(3) Closed $M(A)$-submodules of $\widehat{A}^{(\Lambda)}$ are $\widehat{A}$-generated $M(\widehat{A})$-submodules.

(4) If A is a finitely generated $M(A)$-module, then, for every T-submodule $X \subset Hom_{M(\widehat{A})}(\widehat{A}, \widehat{A}^{(\Lambda)})$,

$$Hom_{M(\widehat{A})}(\widehat{A}, \widehat{A}X) = X.$$

Proof. (1) Recall that $T = End_{M(A)}(\widehat{A}) = End_{M(\widehat{A})}(\widehat{A})$. Now the assertion follows from the fact that every $f \in Hom_{M(A)}(\widehat{A}, \widehat{A}^{(\Lambda)})$ is determined by the $f\pi_\lambda \in T$, where $\pi_\lambda : \widehat{A}^{(\Lambda)} \to \widehat{A}$ denote the canonical projections.

(2) Since $M(\widehat{A}) = M(A)T$ we have to show that every closed $M(A)$-submodule $U \subset \widehat{A}^{(\Lambda)}$ is a T-submodule: Let $t \in T$ and $I \trianglelefteq A$ such that $It \subset A$. Then $Ut \cdot I = U(It) \subset U$. Since U is closed this implies $Ut \subset U$ (by 5.1).

Clearly every $M(A)$-closed submodule is $M(\widehat{A})$-closed.

Suppose that $V \subset \widehat{A}^{(\Lambda)}$ is a closed $M(\widehat{A})$-submodule. Let $u \in \widehat{A}^{(\Lambda)}$ be such that $uI \subset V$ for some $I \trianglelefteq A$. Then obviously $uIT \subset V$, where IT is an essential ideal in $\widehat{A}$. Since V is $M(\widehat{A})$-closed this implies $u \in V$ showing that V is $M(A)$-closed.

(3) By 3.8, a closed submodule $U \subset \widehat{A}^{(\Lambda)}$ is $\widehat{A}$-generated as $M(A)$-module. Now it is obvious by (1) that $\widehat{A}$ generates U as $M(\widehat{A})$-module.

(4) Applying (1) this follows from 3.9. □

We are now prepared to present the following

5.4 Correspondence of closed submodules of $A^{(\Lambda)}$.

Let A be a semiprime ring which is finitely generated as $M(A)$-module. We put $T := End_{(M(A)}(\widehat{A})$ and identify $T^{(\Lambda)} = Hom_{M(\widehat{A})}(\widehat{A}, \widehat{A}^{(\Lambda)})$. Then there are bijections between

(i) the closed $M(A)$-submodules of $A^{(\Lambda)}$,

(ii) the closed $M(\widehat{A})$-submodules of $\widehat{A}^{(\Lambda)}$,

(iii) the closed T-submodules of $T^{(\Lambda)}$.

Correspondences as above can also be obtained by the methods used in [5]. However, in [5, Corollary 3.20] not every closed submodule of $T^{(\Lambda)}$ was involved. Our result obtained here is more precise and from this we conclude that every T-closed submodule of $T^{(\Lambda)}$ is a 'special' closed submodule.

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