KASCH MODULES

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Abstract

An associative ring R is a *left Kasch ring* if it contains a copy of every simple left R-module. Transferring this notion to modules we call a left R-module M a *Kasch module* if it contains a copy of every simple module in $\sigma[M]$. The aim of this paper is to characterize and investigate this class of modules.

INTRODUCTION

Let M be a left R-module over an associative unital ring R, and denote by $\sigma[M]$ the full subcategory of R-Mod consisting of all M-subgenerated R-modules.

In section 1 we collect some basic facts about $\sigma[M]$, torsion theories, and modules of quotients in $\sigma[M]$. In section 2 we introduce the concept of a Kasch module. M is a Kasch module it its M-injective hull $\widehat{M}$ is an (injective) cogenerator in $\sigma[M]$. For ${}_RM = {}_RR$ we regain the classical concept of left Kasch ring. Various characterizations of Kasch modules are provided. In section 3 we present some properties of Kasch modules.

Note that the notion of Kasch module in [10] and [16] is different from ours. Also the notion of Kasch ring used in these papers (R is a Kasch ring if R_R and ${}_RR$ are injective cogenerators in Mod-R and R-Mod respectively) is different from the usual one.

1 PRELIMINARIES

Throughout this paper R will denote an associative ring with nonzero identity, R-Mod the category of all unital left R-modules and M a fixed left R-module. The notation ${}_RN$ will be used to emphasize that N is a left R-module. Module morphisms will be written as acting on the side opposite to scalar multiplication. All other maps will be written as acting on the left.

Any unexplained terminology or notation can be found in [7], [13], [14] and [15].

1.1 M-(co-)generated modules. A left R-module X is said to be *M-generated* (resp. *M-cogenerated*) if there exists a set I and an epimorphism $M^{(I)} \longrightarrow X$ (resp. a monomorphism $X \longrightarrow M^I$). The full subcategory of R-Mod consisting of all M-generated (resp. M-cogenerated) R-modules is denoted by $\mathrm{Gen}(M)$ (resp. $\mathrm{Cog}(M)$).

1.2 The category $\sigma[M]$. A left R-module X is called *M-subgenerated* if X is isomorphic to a submodule of an M-generated module, and the full subcategory of R-Mod consisting of all M-subgenerated R-modules is denoted by $\sigma[M]$. This is a Grothendieck category (see [14]) and it determines a filter of left ideals

$$F_M = \{\, I \leq {}_R R \mid R/I \in \sigma[M] \,\}\,,$$

which is precisely the set of all open left ideals of R in the so called *M-adic topology* on R (see [6]).

For any $X \in \sigma[M]$ we shall denote by $\widehat{X}$ the injective hull of X in $\sigma[M]$, called also the M-injective hull of X. With this terminology, the injective hull of X in R-Mod is the R-injective hull, denoted in the sequel by $E(X)$. It is known (see e.g. [14, 17.9]) that $\widehat{X} = Tr(M, X) = Tr(\sigma[M], X)$, where $Tr(M, X)$ (resp. $Tr(\sigma[M], X)$) denotes the trace of M (resp. $\sigma[M]$) in X.

1.3 Hereditary torsion theories in $\sigma[M]$. The concept of a torsion theory can be defined in any Grothendieck category (cf. [8]), so in particular in $\sigma[M]$. A *hereditary torsion theory* in $\sigma[M]$ is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of nonempty classes of modules in $\sigma[M]$ such that $\mathcal{T}$ is a *hereditary torsion class* or a *localizing subcategory* of $\sigma[M]$ (this means that it is closed under subobjects, factor objects, extensions, and direct sums) and

$$\mathcal{F} = \{\, X \in \sigma[M] \mid \mathrm{Hom}_R(T, X) = 0\,,\ \forall\, T \in \mathcal{T} \,\}\,.$$

The objects in $\mathcal{T}$ are called *τ-torsion modules*, and the object in $\mathcal{F}$ are called *τ-torsionfree modules*.

For any $X \in \sigma[M]$ we denote by $\tau(X)$ the *τ-torsion submodule* of X, which is the sum of all submodules of X belonging to $\mathcal{T}$. Clearly, one has $X \in \mathcal{T} \;\Leftrightarrow\; \tau(X) = X$, and $X \in \mathcal{F} \;\Leftrightarrow\; \tau(X) = 0$.

Note that any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ in $\sigma[M]$ if completely determined by its first component $\mathcal{T}$, and so usually the hereditary torsion theories are identified with hereditary torsion classes.

Any injective object $Q \in \sigma[M]$, i.e., any M-injective module belonging to $\sigma[M]$, determines a hereditary torsion theory $\tau_Q = (\mathcal{T}_Q, \mathcal{F}_Q)$, called the *hereditary torsion theory in* $\sigma[M]$ *cogenerated by* Q:

$$\mathcal{T}_Q = \{\, X \in \sigma[M] \mid \mathrm{Hom}_R(X, Q) = 0 \,\} \text{ and } \mathcal{F}_Q = \mathrm{Cog}(Q) \cap \sigma[M] \,.$$

Note that for any $N \in \sigma[M]$, $\mathrm{Cog}(N) \cap \sigma[M]$ is precisely the class $\mathrm{Cog}_M(N)$ of all objects in $\sigma[M]$ which are cogenerated by N in the category $\sigma[M]$ (i.e., are embeddable in direct products in $\sigma[M]$ of copies of N).

According to [15, 9.4, 9.5], any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ in $\sigma[M]$ has this form, i.e., for any such τ there exists an M-injective module Q in $\sigma[M]$ with $\tau = \tau_Q$.

For any M-injective module Q in $\sigma[M]$ we can also consider the hereditary torsion theory $\tau_{E(Q)} = (\mathcal{T}_{E(Q)}, \mathcal{F}_{E(Q)})$ in R-Mod cogenerated by $E(Q)$:

$$\mathcal{T}_{E(Q)} = \{\, {}_R X \mid \mathrm{Hom}_R(X, E(Q)) = 0 \,\} \text{ and } \mathcal{F}_{E(Q)} = \mathrm{Cog}(E(Q)) \,.$$

Since for any $X \in \sigma[M]$ and $f \in \mathrm{Hom}_R(X, E(Q))$, one has $\mathrm{Im}(f) \in Tr(\sigma[M], E(Q)) = \widehat{Q} = Q$, we deduce that

$$\mathcal{T}_Q = \mathcal{T}_{E(Q)} \cap \sigma[M] \text{ and } \mathcal{F}_Q = \mathcal{F}_{E(Q)} \cap \sigma[M] \,,$$

that is, any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ in $\sigma[M]$ is the "trace" $\tau' \cap \sigma[M]$ of a certain hereditary torsion theory $\tau' = (\mathcal{T}', \mathcal{F}')$ in R-Mod: this means that

$$\mathcal{T} = \mathcal{T}' \cap \sigma[M] \text{ and } \mathcal{F} = \mathcal{F}' \cap \sigma[M] \,.$$

1.4 The Lambek torsion theory in $\sigma[M]$**.** The M-injective hull $\widehat{M}$ of the module ${}_R M$ cogenerates a hereditary torsion theory $\tau_{\widehat{M}} = (\mathcal{T}_{\widehat{M}}, \mathcal{F}_{\widehat{M}})$ in $\sigma[M]$, namely:

$$\mathcal{T}_{\widehat{M}} = \{\, X \in \sigma[M] \mid \mathrm{Hom}_R(X, \widehat{M}) = 0 \,\} \,,$$

$$\mathcal{F}_{\widehat{M}} = \mathrm{Cog}_M(\widehat{M}) = \sigma[M] \cap \mathrm{Cog}(\widehat{M}) \,,$$

called the *Lambek torsion theory* in $\sigma[M]$. Note that this torsion theory depends on the choice of the subgenerator of $\sigma[M]$. If $\sigma[M] = \sigma[N]$ for some ${}_R N$, then in general $\tau_{\widehat{M}} \neq \tau_{\widehat{N}}$.

If ${}_RM = {}_RR$ then we obtain the torsion theory $\tau_{E(R)}$ on R-Mod, which is precisely the well-known *Lambek* torsion theory in R-Mod. The corresponding Gabriel topology on R is the set

$$D_R = \{\, I \leq {}_RR \,|\, \mathrm{Hom}_R(R/I, E(R)) = 0 \,\}$$

of all *dense* left ideals of R.

In the sequel, we shall denote by D_M the Gabriel topology on R corresponding to the hereditary torsion theory in R-Mod cogenerated by $E(M)$,

$$D_M = \{\, I \leq {}_RR \,|\, \mathrm{Hom}_R(R/I, E(M)) = 0 \,\}\,.$$

1.5 Modules of quotients in $\sigma[M]$. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in $\sigma[M]$. For any module $X \in \sigma[M]$ one defines the τ-*injective hull* of X (see [15, 9.10]) as being the submodule $E_\tau(X)$ of the M-injective hull $\widehat{X}$ of X for which

$$E_\tau(X)/X := \tau(\widehat{X}/X)\,.$$

The *module of quotients* $Q_\tau(X)$ of X with respect to τ is defined (see [15, 9.14]) by

$$Q_\tau(X) := E_\tau(X/\tau(X))\,.$$

In particular one can consider for any $X \in \sigma[M]$ the module of quotients of X with respect to the Lambek torsion theory $\tau_{\widehat{M}}$ in $\sigma[M]$.

The module of quotients of a module ${}_RX$ with respect to the Lambek torsion theory $\tau_{E(R)}$ in R-Mod is denoted by $Q_{\max}(X)$ and is called the *maximal module of quotients* of X. For ${}_RX = {}_RR$ one obtains a ring denoted by $Q^\ell_{\max}(R)$ and called the *maximal left ring of quotients* of R.

2 Definition and Characterizations

The following result is well-known (see e.g. [13, Lemma 5.1, p. 235]):

2.1 Proposition. *The following assertions are equivalent for a ring R:*

(1) $D_R = \{R\}$;

(2) $E(R)$ *is an injective cogenerator of* R-Mod*;*

(3) Every simple left R–module is isomorphic to a (minimal) left ideal of R;

(4) $\mathrm{Hom}_R(C, R) \neq 0$ *for every nonzero cyclic left R-module C;*

(5) $\ell(I) \neq 0$ *for every left ideal I of R, where* $\ell(I) = \{\, r \in R \,|\, rI = 0 \,\}$.

A ring satisfying one of the equivalent conditions above is called a *left Kasch ring*.

The above proposition suggests the following:

2.2 Definition. *A module ${}_RM$ is called a* Kasch module *if $\widehat{M}$ is an (injective) cogenerator in $\sigma[M]$.*

So, the ring R is a left Kasch ring if and only if ${}_RR$ is a Kasch module.

2.3 Remarks. (1) Clearly, if M is a Kasch module, then so is $M \oplus N$ for any $N \in \sigma[M]$.

(2) For any ${}_RN$ there exists ${}_RK \in \sigma[N]$ such that $\sigma[N] = \sigma[K]$ and K is a Kasch module. Indeed, $\sigma[N]$ has an injective cogenerator, say Q (see e.g. [14, 17.12]). Then $K = N \oplus Q$ is the desired Kasch module.

(3) If M is a Kasch module and ${}_RN$ is such that $\sigma[M] = \sigma[N]$, then the module N is not necessarily a Kasch module. To see this, take as N a module which is not Kasch and as M the module K considered in (2).

(4) Clearly if ${}_RM$ is a cogenerator in $\sigma[M]$, then M is a Kasch module. The converse is not true, as the following example shows: let F be a field and denote by R the ring $F[[X]]$ of all formal series in the indeterminate X over F. Then R is a local ring having $P = (X)$ as the only maximal ideal, $R/P \simeq F \leq R$, but R is not a cogenerator of R-Mod since $E(R/P)$ cannot be embedded in R.

2.4 Examples. (1) Any semisimple module M is a Kasch module.

(2) If M is a non-singular module in $\sigma[M]$, i.e. M is *polyform* (see [15]), then M is a Kasch module if and only if M is semisimple. Indeed, one implication is obvious. For the other one, if M is a non-singular module in $\sigma[M]$, then according to [15, 10.2], $N \in \sigma[M]$ is M-singular if and only if $\mathrm{Hom}_R(N, \widehat{M}) = 0$. But, if M is a Kasch module, then such an N must be necessarily zero (see also 2.6). It follows that for any $K \trianglelefteq M$, where "$\trianglelefteq$" means "essential submodule", one obtains an M-singular module M/K, which must be 0. Thus, M has no proper essential submodules, which implies that M is semisimple.

(3) For any nonzero $n \in \mathbb{N}$, the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is a Kasch module, which is polyform if and only if n is square-free, i.e., if and only if it is semisimple.

(4) Any torsion abelian group is a Kasch $\mathbb{Z}$-module. More generally, any usual torsion module over a Dedekind domain D is a Kasch module.

Indeed, it is known (see e.g. [3, Proposition 2.2.3]) that the usual torsion modules over a Dedekind domain D which is not a field are precisely the semi-Artinian D-modules, and moreover, any torsion D-module is the direct sum of its U–primary components (see section 3 for definitions). Apply now 3.2 and 3.6.

2.5 Lemma. *The following assertions are equivalent for $X \in \sigma[M]$:*

(1) $\mathrm{Hom}_R(X, E(M)) = 0$*;*

(2) $\mathrm{Hom}_R(X, \widehat{M})) = 0$*;*

(3) $\mathrm{Hom}_R(C, M) = 0$ *for any (cyclic) submodule C of X.*

Proof: Since for any $f \in \mathrm{Hom}_R(X, E(M))$, one has

$$\mathrm{Im}(f) \in \mathrm{Tr}(\sigma[M], E(M)) = \widehat{M}\,,$$

one deduces that (1) $\Leftrightarrow$ (2).

The equivalence (1) $\Leftrightarrow$ (3) is an immediate consequence of [13, Lemma 3.8, p. 142]. □

2.6 Proposition. *The following properties are equivalent for the module ${}_RM$:*

(1) M is a Kasch module;

(2) Any simple module in $\sigma[M]$ can be embedded in M;

(3) Any simple module in $\sigma[M]$ is cogenerated by M;

(4) $\mathcal{T}_{\widehat{M}} = \{0\}$*;*

(5) $\mathcal{F}_{\widehat{M}} = \sigma[M]$*;*

(6) $\{\, {}_RX \mid X \leq {}_RN$ *and* $\mathrm{Hom}_R(N/X, E(M)) = 0\,\} = \{N\}$ *for any* $N \leq {}_RM$*;*

(7) $F_M \cap D_M = \{R\}$*;*

(8) $\mathrm{Hom}_R(C, M) \neq 0$ *for any nonzero (cyclic) left R-module C from* $\sigma[M]$*;*

(9) $N = Q_{\tau_{\widehat{M}}}(N)$ for any $N \in \sigma[M]$, i.e., any module in $\sigma[M]$ is its own module of quotients with respect to the Lambek torsion theory $\tau_{\widehat{M}}$ in $\sigma[M]$.

Proof: (1) $\Rightarrow$ (2) It is known (see [14, 17.12]) that an injective object ${}_RQ$ in $\sigma[M]$ is a cogenerator of $\sigma[M]$ if and only if it contains a copy of each simple module in $\sigma[M]$. So, (1) implies that for any simple object $U \in \sigma[M]$ there exists a monomorphism $\alpha_U : U \longrightarrow \widehat{M}$. It follows that $\mathrm{Im}(\alpha_U) \cap M \neq 0$, and then $\mathrm{Im}(\alpha_U) \cap M = \mathrm{Im}(\alpha_U)$ because $U \simeq \mathrm{Im}(\alpha_U)$

is a simple module. Thus $U \simeq \mathrm{Im}(\alpha_U) \leq M$, which proves the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Let U be an arbitrary simple module in $\sigma[M]$. Since $U \in \mathrm{Cog}(M)$, it follows that there exists a nonzero morphism $f : U \longrightarrow M$, which is necessarily injective because U is a simple module. Thus, any simple module in $\sigma[M]$ can be embedded in M, and so in $\widehat{M}$, showing that $\widehat{M}$ is a cogenerator in $\sigma[M]$.

(1) $\Rightarrow$ (4) If $X \in \mathcal{T}_{\widehat{M}}$ then $\mathrm{Hom}_R(X, \widehat{M}) = 0$. Assume that $X \neq 0$. Then, there exists a nonzero morphism $h : X \longrightarrow \widehat{M}$ because $\widehat{M}$ is a cogenerator in $\sigma[M]$, a contradiction.

(4) $\Leftrightarrow$ (5) and (4) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$ (2) Assume that (2) is not satisfied. Then, there exists a simple module $U \in \sigma[M]$ such that U cannot be embedded in M. Then $\mathrm{Hom}_R(U, M) = 0$, and so $\mathrm{Hom}_R(U, E(M)) = 0$. But every module in $\sigma[M]$ is an epimorphic image of a submodule of M, i.e., it is a subfactor of M. So, there exists $X \leq N \leq M$ such that $U \simeq N/X$. It follows that $\mathrm{Hom}_R(N/X, E(M)) = 0$, and by assumption, we deduce that $X = N$, a contradiction because $U \neq 0$. This proves the desired implication.

(4) $\Rightarrow$ (7), (7) $\Rightarrow$ (8) and (8) $\Rightarrow$ (4) follow from 2.5.

(4) $\Rightarrow$ (9) Assume that $\mathcal{T}_{\widehat{M}} = \{0\}$. Then $\mathcal{F}_{\widehat{M}} = \sigma[M]$. Let $N \in \sigma[M]$. Then, the module of quotients $Q_{\tau_{\widehat{M}}}(N)$ of N with respect to the Lambek torsion theory $\tau_{\widehat{M}}$ in $\sigma[M]$ is

$$Q_{\tau_{\widehat{M}}}(N) = E_{\tau_{\widehat{M}}}(N/\tau_{\widehat{M}}(N))\,.$$

But $\tau_{\widehat{M}}(N) = 0$ and

$$E_{\tau_{\widehat{M}}}(N)/N = \tau_{\widehat{M}}(\widehat{N}/N) = 0\,,$$

by hypothesis. So $E_{\tau_{\widehat{M}}}(N) = N$, and consequently $Q_{\tau_{\widehat{M}}}(N) = N$ for any $N \in \sigma[M]$.

(9) $\Rightarrow$ (5) Suppose that M is such that $Q_{\tau_{\widehat{M}}}(N) = N$ for any $N \in \sigma[M]$. Since $Q_{\tau_{\widehat{M}}}(N) \in \mathcal{F}_{\widehat{M}}$ for all $N \in \sigma[M]$, we deduce that $\mathcal{F}_{\widehat{M}} = \sigma[M]$. □

2.7 Remark. Suppose that ${}_RM$ is such that any simple module from $\sigma[M]$ is M-cyclic, i.e., isomorphic to a factor module of M. This happens e.g. when ${}_RM = {}_RR$ or when M is a self-generator. Then, by the proof of

2.6, one deduces that in this case we can add to the equivalent conditions from 2.6 also the following one:

$$(10)\ \{\,{}_RX \mid X \leq {}_RM \textit{ and } \mathrm{Hom}_R(M/X, E(M)) = 0\,\} = \{M\}\,,$$

in other words, the only rational submodule of M is M itself.

As an immediate consequene of 2.6 we obtain the following characterization of left Kasch rings:

2.8 Corollary. *The following are equivalent for the ring R:*
(1) R is a left Kasch ring;
(2) Any left R-module X is its own maximal module of quotients.

2.9 Example. The example from [12] we are going to present now provides a module which is its own module of quotients in the Lambek topology, but which is not Kasch. This shows that in 2.6 (resp. 2.8) we need the condition (9) (resp. (2)) to be fulfilled for *all* $X \in \sigma[M]$, and not only for M (resp. for *all* X in R-Mod, and not only for ${}_RR$).

Let R denote the direct product $\prod_{\lambda\in\Lambda} F_\lambda$ of an infinite family $(F_\lambda)_{\lambda\in\Lambda}$ of fields. Then, according to [11, Proposition 9, p. 100], one has

$$Q_{\max}(R) = Q_{\max}(\prod_{\lambda\in\Lambda} F_\lambda) \simeq \prod_{\lambda\in\Lambda} Q_{\max}(F_\lambda) = \prod_{\lambda\in\Lambda} F_\lambda = R\,,$$

which shows that R is its own maximal ring of quotients. However, R is not a Kasch ring: indeed, if we consider the proper ideal $I = \bigoplus_{\lambda\in\Lambda} F_\lambda$ of R, then clearly $\ell(I) = 0$, and consequently, by 2.1 one deduces that R is not a Kasch ring.

3 Properties of Kasch modules

Denote by $\mathcal{K}$ the class of all Kasch left R-modules. Consider a module ${}_RN$ which is not a Kasch module, let Q be a cogenerator of $\sigma[N]$, and denote $K = N \oplus Q$. Then N is isomorphic to a submodule, as well as to a factor module of the Kasch module K, which shows that the class $\mathcal{K}$ need not to be closed under subobjects nor under factor objects. The above example shows also that a direct summand of a Kasch module is not necessarily a Kasch module.

We are going now to show that the class $\mathcal{K}$ is closed under direct sums. We need first the following:

3.1 Lemma. *Let $(M_\lambda)_{\lambda\in\Lambda}$ be a nonempty family of nonzero left R-modules. Then, for any simple module $U \in \sigma[\bigoplus_{\lambda\in\Lambda} M_\lambda]$ there exists a $\mu \in \Lambda$ such that $U \in \sigma[M_\mu]$.*

Proof: Denote $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ and consider the injective hull $\widehat{U}$ of U in $\sigma[M]$. Then, as known, $\widehat{U}$ is M-generated, so there exists a nonzero morphism $h : M \longrightarrow \widehat{U}$. Denote by $\widetilde{U}$ the image of h. It follows that $U \trianglelefteq \widetilde{U} \trianglelefteq \widehat{U}$, and so, we obtain an epimorphism of R-modules

$$g : \bigoplus_{\lambda\in\Lambda} M_\lambda \longrightarrow \widetilde{U}\,.$$

Denote for each $\lambda \in \Lambda$ by $\varepsilon_\lambda : M_\lambda \longrightarrow M$ the canonical injection. Then, surely there exists a $\mu \in \Lambda$ such that $\varepsilon_\mu g \neq 0$, which produces a nonzero morphism $g_\mu : M_\mu \longrightarrow \widetilde{U}$. Since $U \trianglelefteq \widetilde{U}$ we deduce that U is an epimorphic image of a submodule of M_μ. Thus $U \in \sigma[M_\mu]$. □

3.2 Proposition. *The class $\mathcal{K}$ is closed under arbitrary direct sums and essential submodules.*

Proof: Let $(M_\lambda)_{\lambda\in\Lambda}$ be an arbitrary nonempty family of left R-modules, and $U \in \sigma[\bigoplus_{\lambda\in\Lambda} M_\lambda]$ a simple module. By the previous lemma, there exists a $\mu \in \Lambda$ such that $U \in \sigma[M_\mu]$. Since M_μ is a Kasch module, we deduce that U can be embedded in M_μ, and consequently also in $\bigoplus_{\lambda\in\Lambda} M_\lambda$, proving that $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is a Kasch module.

The last statement of the proposition is obvious. □

We are going now to recall some definitions and results from [1], [2], [4] and [9]. For any full subcategory $\mathcal{C}$ of R-Mod we shall denote by $\mathrm{Sim}(\mathcal{C})$ a representative system of all isomorphism classes of simple modules belonging to $\mathcal{C}$. Clearly, $\mathrm{Sim}(\mathcal{C})$ is a set, possibly empty. For any ${}_RX$ we shall denote

$$\mathrm{Sim}(X) := \mathrm{Sim}(\sigma[X])\,.$$

So, $\mathrm{Sim}(R)$ denotes $\mathrm{Sim}(R\text{-Mod})$. We allways shall assume that $\mathrm{Sim}(\mathcal{C}) \subseteq \mathrm{Sim}(R)$ for any full subcategory $\mathcal{C}$ of R-Mod.

Clearly, for any module ${}_RX$ one has:

$$\mathrm{Sim}(X) = \{\, U \in \mathrm{Sim}(R) \mid \exists\, X' \leq X \ \text{ and } \ \exists\, V \leq X/X' \ \text{ with } \ V \simeq U \,\}\,.$$

The next result collects some of the basic properties of "Sim":

3.3 Proposition. *The following assertions hold:*

(1) For an ${}_RX$ *one has* $\mathrm{Sim}(X) = \emptyset \Leftrightarrow X = 0$.

(2) If ${}_RX$ *is a module and* $Y \in \sigma[X]$*, then* $\mathrm{Sim}(Y) \subseteq \mathrm{Sim}(X)$.

(3) For any exact sequence in R-Mod:

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0\,,$$

one has

$$\mathrm{Sim}(X) = \mathrm{Sim}(X') \cup \mathrm{Sim}(X'')\,.$$

(4) For any family of $(M_\lambda)_{\lambda\in\Lambda}$ *of left R-modules one has*

$$\mathrm{Sim}(\bigoplus_{\lambda\in\Lambda} M_\lambda) = \bigcup_{\lambda\in\Lambda} \mathrm{Sim}(M_\lambda)\,.$$

Proof: (1) If $X \neq 0$, then there exists $x \in X$, $x \neq 0$. But, the nonzero cyclic module Rx has a maximal submodule Z, and so, Rx/Z is a simple module in $\sigma[X]$.

(2) is obvious.

(3) Since X' , $X'' \in \sigma[X]$ it is clear that $\mathrm{Sim}(X') \cup \mathrm{Sim}(X'') \subseteq \mathrm{Sim}(X)$. Let now $U \in \mathrm{Sim}(X)$. Without loss of generality, we can suppose that $X' \leq X$ and $X'' = X/X'$. There exists a submodule Y of X and an epimorphism $f : Y \longrightarrow U$.

Two cases arise: $(Y \cap X')f = 0$ and $(Y \cap X')f \neq 0$. In the first case f induces an epimorphism $(Y + X')/X' \simeq Y/(Y \cap X') \longrightarrow U$, and so $U \in \mathrm{Sim}(X'')$.

In the second case, $f|_{Y\cap X'}$ yields an epimorphism $Y \cap X' \longrightarrow U$, and then $U \in \mathrm{Sim}(X')$.

(4) is essentially a reformulation of . □

Recall that a module ${}_RX$ is called a *semi-Artinian* (or *Loewy*) module if any nonzero factor module of X contains a simple submodule.

If $U \in \mathrm{Sim}(R)$, a module ${}_RX$ is said to be U-*primary* whenever X/X' contains a simple module isomorphic to U for any $X' \leq X$, $X' \neq X$.

The class $\mathcal{L}$ of all semi-Artinian left R-modules is a localizing subcategory of R-Mod, as well as, for each $U \in \mathrm{Sim}(R)$, the class $\mathcal{L}_U$ of all U-primary left R-modules. For any ${}_RX$ and $U \in \mathrm{Sim}(R)$ we shall denote by X_U the greatest U-primary submodule of X, called the U-*primary component* of X.

If X is a left R-module, then the set

$$\mathcal{S}(X) = \{ U \in \mathrm{Sim}(R) \mid X_U \neq 0 \}$$

is called the *support* of X. One says that X is a module with *finite support* in case $\mathcal{S}(X)$ is a finite set.

It is known that if $X \in \mathcal{L}$, then the sum $\sum_{U\in Sim(R)} X_U$ is a direct sum and $\bigoplus_{U\in Sim(R)} X_U \trianglelefteq X$ (cf. [8]), but in general $X \neq \bigoplus_{U\in Sim(R)} X_U$. Following [2], the module X is said to be *Dickson decomposable* if $X = \bigoplus_{U\in Sim(R)} X_U$.

Following [9] (resp. [1]), the ring R is said to be a left T–ring (resp. a left FT–ring) in case any semi-Artinian module (resp. any semi-Artinian module with finite support) in R-Mod is a Dickson decomposable module. By [1, Corollaire 6], any commutative ring is an FT–ring.

We can extend very naturally these definitions as follows:

3.4 Definitions. *The module ${}_RM$ is called a T-module (resp. FT-module) in case any semi-Artinian module (resp. any semi-Artinian module with finite support) in $\sigma[M]$ is Dickson decomposable.*

3.5 Lemma. *Let X be a left R-module and $U \in \mathrm{Sim}(R)$. Then*

$$X \in \mathcal{L}_U \ \Leftrightarrow \ X \in \mathcal{L} \ \text{ and } \ \mathrm{Sim}(X) = \{U\}\,.$$

Proof: If X is U-primary, then obviously X is semi-Artinian. Let $V \in \mathrm{Sim}(X)$. Then some quotient module X/X' of X contains a simple module W isomorphic to V. Then $V \in \mathcal{L}_U$, and consequently $V = U$. The converse implication is clear. □

3.6 Lemma. *For any $U \in \mathrm{Sim}(R)$, $\mathcal{L}_U \subseteq \mathcal{K}$.*

Proof: If X is a nonzero U-primary module, then the socle $\mathrm{Soc}(X)$ of X contains at least a simple submodule of X isomorphic to U, hence any simple module in $\mathrm{Sim}(X) = \{U\}$ can be embedded in X, showing that X is a Kasch module. □

3.7 Proposition. *Let $(U_\lambda)_{\lambda\in\Lambda}$ be a family of simple modules in $\mathrm{Sim}(R)$ and $X_\lambda \in \mathcal{L}_{U_\lambda}$ for each $\lambda \in \Lambda$. Then $\bigoplus_{\lambda\in\Lambda} X_\lambda \in \mathcal{K}$. In particular any Dickson decomposable module is a Kasch module.*

Proof: Apply 3.6 and 3.2. □

3.8 Corollary. *If ${}_RM$ is a T-module (resp. an FT-module), then any semi-Artinian module (resp. semi-Artinian module with finite support) in $\sigma[M]$ is a Kasch module.*

Proof: By definition, any semi-Artinian module (resp. semi-Artinian module with finite support) in $\sigma[M]$ is a Dickson decomposable module. Apply now 3.7. □

3.9 Corollary.

If ${}_RM$ be a semi-Artinian module with finite support, and R is an FT–ring, then any module in $\sigma[M]$ is a Kasch module.

Proof: According to [1, Corollaire 8], for any exact sequence in R-Mod:

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0\,,$$

with X a semi-Artinian module with finite support, one has

$$\mathcal{S}(X) = \mathcal{S}(X') \cup \mathcal{S}(X'')\,.$$

It follows that for each $X \in \sigma[M]$ one has $\mathcal{S}(X) \subseteq \mathcal{S}(M)$, and so X is also with finite support. Note that $\sigma[M] \subseteq \mathcal{L}$ since $M \in \mathcal{L}$. Consequently, any $X \in \sigma[M]$ is Dickson decomposable. Apply now 3.7. □

If R is a commutative ring, then $\mathrm{Ass}(X)$ will denote the "Assasin" of X (see [13]).

3.10 Corollary. *Let M be a semi-Artinian module over the commutative ring R. If $\mathrm{Ass}(M)$ is a finite set, then any module in $\sigma[M]$ is a Kasch module.*

Proof: As noted above, any commutative ring is an FT–ring. Since M has finite support if and only if $\mathrm{Ass}(M)$ is a finite set, the result follows now from 3.3. □

3.11 Corollary. *If M be a semi-Artinian module over the commutative semi-local ring R, then any module in $\sigma[M]$ is a Kasch module.*

Proof: By [2, Proposition 1], any $P \in \mathrm{Ass}(M)$ is a maximal ideal of R. Apply now 3.10. □

3.12 Corollary. *If R is a commutative semi-local semi-Artinian ring, then any R-module is a Kasch module.*

3.13 Corollary. *Any module over a commutative Artinian ring is a Kasch module.*

3.14 Remarks. (1) The observation in 3.13 does not hold for noncommutative Artinian rings. For this let R be the ring of upper triangular (2,2)-matrices over a field F. The left R-module

$$M = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix} \text{ has socle } S = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix},$$

and M/S is not isomorphic to S. Hence M is not a Kasch module.

(2) In case any factor module of M is a Kasch module, then M must be necessarily a semi-Artinian module, as this can be shown by considering the ascending Loewy series of M.

(3) The result in 3.10 fails if $\mathrm{Ass}(M)$ is an infinite set. To see this, consider the following example given in [4, 3.34]:

Let F a field and Λ an infinite set. Denote by B the direct product $\prod_\Lambda F_\lambda$, where $F_\lambda = F$ for all $\lambda \in \Lambda$, and by A the subring $\bigoplus_\Lambda F_\lambda + Fe$ of B, where e is the identity element of B. Denote for each $\lambda \in \Lambda$ by $\varepsilon_\lambda : F_\lambda \longrightarrow \bigoplus_{\mu\in\Lambda} F_\mu$ the canonical injection, and $U_\lambda = \varepsilon_\lambda(F_\lambda)$. Then

$$\sum_{\lambda\in\Lambda} U_\lambda = \bigoplus_{\lambda\in\Lambda} U_\lambda = \bigoplus_{\lambda\in\Lambda} F_\lambda\,,$$

is precisely the socle $\mathrm{Soc}(A)$ of A, this is a maximal ideal of A, U_λ's are mutually nonisomorphic simple A-modules, and $U_\lambda \not\simeq U_0$ for all $\lambda \in \Lambda$, where $U_0 = A/\mathrm{Soc}(A)$. The ring A is a semi-Artinian regular ring with the Loewy length 2 which is not semi-simple, the A-module A is not Dickson decomposable, $\mathrm{Ass}(A)$ is an infinite set, and A is not a Kasch ring.

The exact sequence

$$0 \longrightarrow \mathrm{Soc}(A) \longrightarrow A \longrightarrow U_0 \longrightarrow 0$$

of A modules shows also that $\mathcal{K}$ need not to be closed under extensions.

(4) The example considered in 2.9 shows that $\mathcal{K}$ is not closed under direct products. Let R denote the direct product $\prod_{\lambda\in\Lambda} F_\lambda$ of an infinite family $(F_\lambda)_{\lambda\in\Lambda}$ of fields. Each F_λ is a simple R-module in a canonical way, but their product is R itself, which as we have already seen in 2.9, is not a Kasch module.

(5) We are going now to show that a direct sum of two modules which both are not Kasch could be a Kasch module. For this, consider the example due to P.M. Cohn, exhibited in [9]:

Let F be any field possesing an endomorphism $\varphi : F \longrightarrow F$ which is not onto, and denote by A the skew polynomial ring $F[X, \varphi]$ consisting of all polynomials $\sum_{0 \leq i \leq n} X^i a_i$, where $a_i \in F$, with the multiplication $aX = X\varphi(a)$ for any $a \in F$. Then A is a principal right ideal domain.

Let $\beta \in F \setminus \varphi(F)$ and consider the elements $a = X$, $b = X + \beta$. If we denote $U = A/aA$ and $V = A/bA$ then $V \simeq aA/abA$ and $U \simeq bA/baA$, U and V are simple right A-modules which are not isomorphic, and the canonical exact sequences

$$0 \longrightarrow aA/abA \longrightarrow A/abA \longrightarrow A/aA \longrightarrow 0$$

$$0 \longrightarrow bA/baA \longrightarrow A/baA \longrightarrow A/aA \longrightarrow 0$$

are not splitting. This shows that both the right A-modules A/abA and A/baA are not Kasch modules, but their direct sum is a Kasch module.

3.15 Proposition. *Any faithful left R-module over a left Kasch ring R is a Kasch module.*

Proof: If N is a faithful module over the Kach ring R, then the module ${}_RR$ can be embedded in N^N, hence any simple left R-module is cogenerated by N, proving that N is a Kasch module. □

We have proved so far that $\mathcal{K}$ is closed under direct sums and under essential subobjects, but need not to be closed under subobjects, nor factor objects, nor extensions and nor direct products.

Some natural questions arise:

Question 1. *For which rings R is the class $\mathcal{K}$ of all left Kasch R-modules $\mathcal{K}$ closed under extensions resp. direct products?*

Question 2. *Let M be a Kasch module. When is any submodule (resp. factor module) of M again a Kasch module?*

Question 3. *For which modules M is any module in $\sigma[M]$ a Kasch module? In particular, for which rings R are all left R-modules Kasch modules?*

If all modules in $\sigma[M]$ are homo-serial then they are all Kasch modules (see [14, 56.7, 56.8]). As a special case all left (and right) R-modules are Kasch provided R is left and right an artinian and principal ideal ring (see

[14, 56.9]). Moreover all modules over commutative (semi-) local (semi-) Artinian rings are Kasch (by 3.12).

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References

[1] T. Albu, *Modules de torsion à support fini* , C. R. Acad. Sci. Paris **273**, Série A (1971), 335-338.

[2] T. Albu, *Modules décomposables de Dickson* , C. R. Acad. Sci. Paris **273**, Série A (1971), 369-372.

[3] T. Albu, *On some classes of modules I* (in Romanian), Stud. Cerc. Mat. **24** (1972), 1329-1392.

[4] T. Albu and C. Năstăsescu, *Décompositions primaires dans les catégories de Grothendieck commutatives I* , J. Reine Angew. Math. **280** (1976), 172-194.

[5] T. Albu and C. Năstăsescu, "*Relative Finiteness in Module Theory*", Marcel Dekker, Inc., New York and Basel, 1984.

[6] T. Albu and R. Wisbauer, *M-density, M-adic completion and M-subgeneration*, Rend. Sem. Mat. Univ. Padova **98** (1997), to appear.

[7] F.W. Anderson and K.R. Fuller, "*Rings and Categories of Modules*", Springer-Verlag, New York Heidelberg Berlin (1992).

[8] S.E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc. **121** (1966), 223-235.

[9] S.E. Dickson, *Decomposition of modules II. Rings without chain conditions*, Math. Z. **104** (1969), 349-357.

[10] D.N. Dikranjan, E. Gregorio and A. Orsatti, *Kasch bimodules*, Rend. Sem. Mat. Univ. Padova **85** (1991), 147-160.

[11] J. Lambek, "*Lectures on Rings and Modules*", Blaisdell Publishing Company (1966).

[12] J. Lambek, "*Torsion Theories, Additive Semantics, and Rings of Quotients*", Springer-Verlag, New York Heidelberg Berlin (1971).

[13] B. Stenström, "*Rings of Quotients*", Springer-Verlag, Berlin Heidelberg New York (1975).

[14] R. Wisbauer, "*Foundations of Module and Ring Theory*", Gordon and Breach, Reading (1991).

[15] R. Wisbauer, "*Modules and Algebras: Bimodule Structure and Group Actions on Algebras*", Pitman Monographs 81, Longman (1996).

[16] W. Xue, *On Kasch duality*, Algebra Colloq. **1** (1994), 257-266.

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