Lifting theorems for tensor functors on module categories

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Any (co)ring R is an endofunctor with (co)multiplication on the category of abelian groups. These notions were generalised to *monads* and *comonads* on arbitrary categories. Starting around 1970 with papers by Beck, Barr and others a rich theory of the interplay between such endofunctors was elaborated based on *distributive laws* between them and Applegate's lifting theorem of functors between categories to related (co)module categories. Curiously enough some of these results were not noticed by researchers in module theory and thus notions like entwining structures and smash products between algebras and coalgebras were introduced (in the nineties) without being aware that these are special cases of the more general theory.

The purpose of this survey is to explain several of these notions and recent results from general category theory in the language of elementary module theory focussing on functors between module categories given by tensoring with a bimodule. This provides a simple and systematic approach to smash products, wreath products, corings and rings over corings (C-rings). We also highlight the relevance of the Yang-Baxter equation for the structures on the threefold tensor product of algebras or coalgebras (see 3.6).

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Contents.

1. Introduction

The study of the interplay between algebras A and coalgebras C over a commutative ring R led to the definition of *entwining structures* by Brzeziński and Majid in [7] (1998), that is, R-linear maps $\varphi : C \otimes_R A \to A \otimes_R C$ satisfying certain conditions (making $C \otimes_R A$ an A-coring). These can be seen as natural transformation $\varphi \otimes_R - :$ $C \otimes_R A \otimes_R - \to A \otimes_R C \otimes_R -$ between the endofunctors $C \otimes_R -$ and $A \otimes_R -$ of the category $\mathbb{M}_R$ of R-modules. If both A and C are R-algebras (or coalgebras), similar maps (with different conditions) are employed to define a *smash product* or *smash coproduct* on $A \otimes_R C$ (e.g. Caenepeel, Ion, Militaru and Zhu [10,11]).

The corresponding relations between endofunctors (monads and comonads) of different categories were observed in Power and Watanabe [24,25] (1997), Turi and Plotkin [27] (1997), and elsewhere in the context of operational semantics.

These constructions are special cases of *distributive laws* between endofunctors on any categories considered in Beck [2], Barr [1], and elsewhere as early as 1969. They can be interpreted as conditions allowing to lift functors between categories

$\mathbb{A}$ and $\mathbb{B}$ to related module categories. More precisely, for monads (F, μ, η) and (G, μ', η') on the categories $\mathbb{A}$ and $\mathbb{B}$, respectively, and a functor $T : \mathbb{A} \to \mathbb{B}$ we say that $\overline{T} : \mathbb{A}_F \to \mathbb{B}_G$ is a *lifting of* T provided the diagram

$$\begin{tikzcd} \mathbb{A}_F \arrow[r, "\overline{T}"] \arrow[d, "U_F"'] & \mathbb{B}_G \arrow[d, "U_G"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{B} \end{tikzcd}$$

is commutative, where the U_F and U_G denote the forgetful functors.

Now *Applegate's theorem* says that the liftings $\overline{T} : \mathbb{A}_{\mathbb{F}} \to \mathbb{B}_{\mathbb{G}}$ of T are in bijective correspondence with the natural transformations $\lambda : GT \to TF$ inducing commutative diagrams

$$\begin{tikzcd} GGT \arrow[rr, "\mu' T"] \arrow[d, "G\lambda"'] & & GT \arrow[d, "\lambda"] \\ GTF \arrow[r, "\lambda F"] & TFF \arrow[r, "T\mu"] & TF, \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta' T"] \arrow[dr, "T\eta"'] & GT \arrow[d, "\lambda"] \\ & TF. \end{tikzcd}$$

A similar result holds if F and G are comonads and the module categories are replaced by comodule categories. These theorems turned out to be fundamental for the related theory.

In the case $\mathbb{A} = \mathbb{B}$ and $F = G$ we have endofunctors and it is natural to ask if the lifting $\overline{T}$ of a (co)monad T is again a (co)monad or, more generally, which other maps may induce a (co)monad structure on $\overline{T}$.

The purpose of this note is to outline the basic role of this theorem and derived consequences for functors between module categories given by tensoring with a bimodule. The answers to the questions posed above lead to the notions of smash products, wreath products, and corings. We sketch the essentials of the proofs without relying on results from abstract category theory.

The composition of liftable functors is again liftable. However, the question if the lifting of the composition of liftable monads is again a monad employs the Yang-Baxter equations in a general setting.

2. Lifting of functors

Throughout all rings will be associative and with unit unless otherwise stated, and R will be a ring, not necessarily commutative.

In case R is commutative we will tacitly assume that for (R, R)-bimodules M, $rm = mr$ for all $r \in R$, $m \in M$. Moreover, for any R-modules M, N there is a canonical R-isomorphism (*twist map*)

$$\mathsf{tw} : M \otimes_R N \to N \otimes_R M, \quad m \otimes n \mapsto n \otimes m.$$

By I_M, I_T, $I_{\mathbb{M}}$ or just by I we denote the identity morphism of an object M in some category $\mathbb{M}$, a functor T, or the identity functor of $\mathbb{M}$, respectively. Following the usage in category theory we also write T instead of I_T and delete the $\otimes$ between linear maps in case it is convenient (to save space).

Most of the theory to be developed holds for non-commutative rings and functors given by tensoring with a bimodule over them. This reflects the fact that it is largely true more generally for functors on arbitrary categories. However, to find examples satisfying the conditions required, commutativity of R can be of great help.

2.1. ***R-rings.*** A ring A is said to be an *R-ring* provided there is a ring morphism $\iota : R \to A$. Equivalently, this means that A is an (R,R)-bimodule with (R,R)-bilinear multiplication and unit,

$$m_A : A \otimes_R A \to A, \quad \iota_A : R \to A,$$

inducing commutative diagrams for associativity and unitality.

In categorical terminology these conditions describe a *monad* on the category of left (or right) R-modules ${}_R\mathbb{M}$, that is, an (R,R)-bimodule A is an R-ring if and only if $A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a monad (e.g. [3, 3.4]).

If R is commutative then R-rings are just *R-algebras* since - by our convention for bimodules - the image of the morphism $\iota : R \to A$ lies in the centre of A.

A left *A-module* is a left R-module M with an R-linear map

$$\varrho_M : A \otimes_R M \to M, \quad a \otimes m \mapsto am,$$

satisfying the associativity and unitality conditions. A-module morphisms are R-linear maps $f : M \to N$ with $f \circ \varrho_M = \varrho_N \circ (I_A \otimes_R f)$ and the set of those is denoted by $\mathrm{Hom}_A(M,N)$. The category of left A-modules is denoted by ${}_A\mathbb{M}$. It is an abelian category with A as a projective generator.

The (free) functor for the R-ring A,

$$A \otimes_R - : {}_R\mathbb{M} \to {}_A\mathbb{M}, \quad M \mapsto (A \otimes_R M, m_A \otimes I_M),$$

is left adjoint to the forgetful functor $U_A : {}_A\mathbb{M} \to {}_R\mathbb{M}$ by the bijection, for $N \in {}_A\mathbb{M}$,

$$\mathrm{Hom}_A(A \otimes_R M, N) \to \mathrm{Hom}_R(M,N), \quad f \mapsto f \circ (\iota_A \otimes I_M).$$

Right A-modules are defined symmetrically. In this case the R-ring A is considered as functor $- \otimes_R A : \mathbb{M}_R \to \mathbb{M}_R$, that is, the functor symbol is written on the right hand side of the argument.

2.2. ***Lifting functors to module categories.*** Let A be an R-ring, B an S-ring and T an (S,R)-bimodule. Then a functor $\overline{T} : {}_A\mathbb{M} \to {}_B\mathbb{M}$ is called a *lifting* of the functor $T \otimes_R - : {}_R\mathbb{M} \to {}_S\mathbb{M}$ if the diagram

$$\begin{tikzcd} {}_A\mathbb{M} \arrow[r, "\overline{T}"] \arrow[d, "{}_AU"'] & {}_B\mathbb{M} \arrow[d, "{}_BU"] \\ {}_R\mathbb{M} \arrow[r, "T\otimes_R -"] & {}_S\mathbb{M} \end{tikzcd}$$

is commutative, where ${}_AU$ and ${}_BU$ denote the forgetful functors.

It is clear that the functor $\overline{T}$ must act on objects $M \in {}_A\mathbb{M}$ like $T \otimes_R -$, that is, $\overline{T}(M) = T \otimes_R M$, and this should have a left B-module structure.

This can be achieved by an (S, R)-bilinear map $\varphi : B \otimes_S T \to T \otimes_R A$ which assigns to any A-module $\varrho : A \otimes_R M \to M$ a B-action

$$B \otimes_S T \otimes_R M \xrightarrow{\varphi \otimes I} T \otimes_R A \otimes_R M \xrightarrow{I \otimes \varrho} T \otimes_R M. \tag{2.1}$$

To make $T \otimes_R M$ a left B-module some conditions on φ are required.

Associativity of the B-action implies, for $M = A$, commutativity of the inner rectangle in the diagram

$$\begin{tikzcd}
B \otimes_S B \otimes_S T \arrow[rr, "B\otimes\varphi"] \arrow[dr, "B\otimes B\otimes T\otimes\iota_A"] \arrow[ddd, "m_B\otimes T"'] & & B \otimes_S T \otimes_R A \arrow[d, "B\otimes T\otimes A\otimes\iota_A"'] \arrow[dr, "="] & \\
& B \otimes_S B \otimes_S T \otimes_R A \arrow[r, "B\otimes\varphi\otimes A"] \arrow[dd, "m_B\otimes T\otimes A"] & B \otimes_S T \otimes_R A \otimes_R A \arrow[r, "B\otimes T\otimes m_A"'] & B \otimes_S T \otimes_R A \arrow[d, "\varphi\otimes A"] \\
& & & T \otimes_R A \otimes_R A \arrow[d, "T\otimes m_A"] \\
B \otimes_S T \arrow[r, "B\otimes T\otimes\iota_A"] \arrow[drr, "\varphi"'] & B \otimes_S T \otimes_R A \arrow[r, "\varphi\otimes A"] & T \otimes_R A \otimes_R A \arrow[r, "T\otimes m_A"] & T \otimes_R A \\
& & T \otimes_R A \arrow[u, "T\otimes A\otimes\iota_A"] \arrow[ur, "="'] & .
\end{tikzcd}$$

The other inner diagrams are commutative by functoriality of composition and hence the outer diagram yields the commutative diagram

$$\begin{tikzcd}
B \otimes_S B \otimes_S T \arrow[r, "B\otimes\varphi"] \arrow[d, "m_B\otimes T"'] & B \otimes_S T \otimes_R A \arrow[r, "\varphi\otimes A"] & T \otimes_R A \otimes_R A \arrow[d, "T\otimes m_A"] \\
B \otimes_S T \arrow[rr, "\varphi"] & & T \otimes_R A.
\end{tikzcd}$$

For the relation between φ and the units of A and B, consider the diagram

$$\begin{tikzcd}
T \arrow[r, "\iota_B\otimes T"] \arrow[d, "T\otimes\iota_A"'] & B \otimes_S T \arrow[r, "\varphi"] \arrow[d, "B\otimes T\otimes\iota_A"'] & T \otimes_R A \arrow[d, "T\otimes A\otimes\iota_A"'] \arrow[dr, "="] & \\
T \otimes_R A \arrow[r, "\iota_B\otimes T\otimes A"'] & B \otimes_S T \otimes_R A \arrow[r, "\varphi\otimes A"'] & T \otimes_R A \otimes_R A \arrow[r, "T\otimes m_A"'] & T \otimes_R A,
\end{tikzcd}$$

in which the inner rectangles are commutative by naturality. The unitality condition for the B-module structure on $T \otimes_R A$ implies that the composition of the maps in the bottom line yields the identity and hence we obtain the commutative diagram

$$\begin{tikzcd}
T \arrow[r, "\iota_B\otimes T"] \arrow[dr, "T\otimes\iota_A"'] & B \otimes_S T \arrow[d, "\varphi"] \\
& T \otimes_R A.
\end{tikzcd}$$

Conversely, given $T \otimes_R A$ as B-module with structure map $\beta : B \otimes_S T \otimes_R A \to T \otimes_R A$, the map

$$\varphi' : \; B \otimes_S T \xrightarrow{B \otimes T \otimes \iota_A} B \otimes_S T \otimes_R A \xrightarrow{\beta} T \otimes_R A$$

has the properties observed under the lifting condition. In the commutative diagram

$$\begin{tikzcd}
T \arrow[r, "\iota_B \otimes T"] \arrow[d, "T\otimes\iota_A"'] & B \otimes_S T \arrow[d, "B\otimes T\otimes\iota_A"'] \arrow[dr, "\varphi'"] & \\
T \otimes_R A \arrow[r, "\iota_B\otimes T\otimes A"'] & B \otimes_S T \otimes_R A \arrow[r, "\beta"'] & T \otimes_R A,
\end{tikzcd}$$

the bottom line yields the identity by unitality and hence $I_T \otimes \iota_A = \varphi' \circ (\iota_B \otimes I_T)$.

By the canonical isomorphism $M \simeq A \otimes_A M$ and associativity of the tensor-product we may assume $\overline{T}(M) = T \otimes_R M = T \otimes_R A \otimes_A M$, that is, $\overline{T}$ is represented by $T \otimes_R A \otimes_A -$ where $T \otimes_R A$ is a (B, A)-bimodule.

Summarising we have proved:

2.3. *Applegate's theorem for rings.* *The liftings $\overline{T} : {}_A\mathbb{M} \to {}_B\mathbb{M}$ of the functor $T \otimes_R - : {}_R\mathbb{M} \to {}_S\mathbb{M}$ are in bijective correspondence with those (S, R)-bilinear maps $\varphi : B \otimes_S T \to T \otimes_R A$ which induce commutativity of the diagrams*

$$\begin{tikzcd}
B \otimes_S B \otimes_S T \arrow[rr, "m_B\otimes T"] \arrow[d, "B\otimes\varphi"'] & & B \otimes_S T \arrow[d, "\varphi"] \\
B \otimes_S T \otimes_R A \arrow[r, "\varphi\otimes A"] & T \otimes_R A \otimes_R A \arrow[r, "T\otimes m_A"] & T \otimes_R A,
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\iota_B\otimes T"] \arrow[dr, "B\otimes\iota_A"'] & B \otimes_S T \arrow[d, "\varphi"] \\
 & T \otimes_R A.
\end{tikzcd} \tag{2.2}$$

Given such a φ, for any $M \in {}_A\mathbb{M}$, the lifting is given by

$$\overline{T}(M) = T \otimes_R M \simeq (T \otimes_R A) \otimes_A M$$

where $T \otimes_R A$ is a (B, A)-bimodule (by 2.1) and φ is a left B-module morphism (by commutativity of the rectangle).

Next we observe that the composition of two liftable functors is again liftable.

2.4. *Composition of liftings to modules.* *Consider an R-ring A, an S-ring B, and a Q-ring H. Let T be an (S, R)-bimodule and U a (Q, S)-bimodule and assume that*

(i) $T \otimes_R - : {}_R\mathbb{M} \to {}_S\mathbb{M}$ lifts to $\overline{T} : {}_A\mathbb{M} \to {}_B\mathbb{M}$ by $\varphi : B \otimes_S T \to T \otimes_R A$,

(ii) $U \otimes_S - : {}_S\mathbb{M} \to {}_Q\mathbb{M}$ lifts to $\overline{U} : {}_B\mathbb{M} \to {}_H\mathbb{M}$ by $\psi : H \otimes_Q U \to U \otimes_S B$.

Then $U \otimes_S T \otimes_R -$ lifts to $\overline{U \otimes_S T} : {}_A\mathbb{M} \to {}_H\mathbb{M}$ by the (Q, R)-bilinear map

$$H \otimes_Q U \otimes_S T \xrightarrow{\psi\otimes T} U \otimes_S B \otimes_S T \xrightarrow{U\otimes\varphi} U \otimes_S T \otimes_R A,$$

and for any $M \in {}_A\mathbb{M}$,

$$\overline{U \otimes_S T}(M) = (U \otimes_S T \otimes_R A) \otimes_A M$$

where $U \otimes_S T \otimes_R A$ is an (H, A)-bimodule.

Proof. The rectangle in (2.2) is now the outer diagram of

$$\begin{tikzcd}
H \otimes_Q H \otimes_Q U \otimes_S T \arrow[r, "m_H \otimes U \otimes T"] \arrow[d, "H \otimes \psi \otimes T"'] & H \otimes_Q U \otimes_S T \arrow[r, "\psi \otimes T"] & U \otimes_S B \otimes_S T \arrow[d, "U \otimes \varphi"] \\
H \otimes_Q U \otimes_S B \otimes_S T \arrow[r, "\psi \otimes B \otimes T"] \arrow[d, "H \otimes U \otimes \varphi"'] & U \otimes_S B \otimes_S B \otimes_S T \arrow[ur, "U \otimes m_B \otimes T"] \arrow[d, "U \otimes B \otimes \varphi"] & U \otimes_S T \otimes_R A \\
H \otimes_Q U \otimes_S T \otimes_R A \arrow[r, "\psi \otimes T \otimes A"] & U \otimes_S B \otimes_S T \otimes_R A \arrow[r, "U \otimes \varphi \otimes A"] & U \otimes_S T \otimes_R A \otimes_R A, \arrow[u, "U \otimes m_A"']
\end{tikzcd}$$

where the pentagons are commutative by the properties of φ and ψ, respectively, and the square is commutative by naturality.

The unitality conditions on ψ and φ yield the commutative triangle in the diagrams

$$\begin{tikzcd}
U \otimes T \arrow[r, "\iota_H \otimes U \otimes T"] \arrow[d, "U \otimes T \otimes \iota_H"'] \arrow[dr, "U \otimes \iota_H \otimes T"] & H \otimes U \otimes T \arrow[d, "\psi \otimes T"] \\
U \otimes T \otimes H & U \otimes H \otimes T, \arrow[l, "U \otimes \varphi"]
\end{tikzcd}$$

showing commutativity of the triangle in (3.1). $\square$

3. Lifting of endofunctors to module categories

We now consider the results of the preceding section for the case $R = S$ and $A = B$.

3.1. ***Lifting endofunctors to modules.*** Let A be an R-ring and T an (R, R)-bimodule. Then a functor $\overline{T} : {}_A\mathbb{M} \to {}_A\mathbb{M}$ is a lifting of $T \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ if the diagram

$$\begin{tikzcd}
{}_A\mathbb{M} \arrow[r, "\overline{T}"] \arrow[d, "{}_AU"'] & {}_A\mathbb{M} \arrow[d, "{}_AU"] \\
{}_R\mathbb{M} \arrow[r, "T \otimes_R -"] & {}_R\mathbb{M}
\end{tikzcd}$$

is commutative where ${}_AU$ denotes the forgetful functor. In this case Applegate's theorem says that the liftings $\overline{T} : {}_A\mathbb{M} \to {}_A\mathbb{M}$ of the functor $T \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ are in bijective correspondence with the (R, R)-bilinear maps $\varphi : A \otimes_R T \to T \otimes_R A$ inducing commutativity of the diagrams

$$\begin{tikzcd}
A \otimes_R A \otimes_R T \arrow[rr, "m_A \otimes T"] \arrow[d, "A \otimes \varphi"'] & & A \otimes_R T \arrow[d, "\varphi"] \\
A \otimes_R T \otimes_R A \arrow[r, "\varphi \otimes A"] & T \otimes_R A \otimes_R A \arrow[r, "T \otimes m_A"] & T \otimes_R A,
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\iota_A \otimes T"] \arrow[dr, "T \otimes \iota_A"'] & A \otimes_R T \arrow[d, "\varphi"] \\
& T \otimes_R A.
\end{tikzcd} \tag{3.1}$$

Such a lifting is given by $T \otimes_R A \otimes_A -$ where $T \otimes_R A$ is an (A, A)-bimodule.

Now both $T \otimes_R -$ and $\overline{T}$ are endofunctors and in this section we will answer the following questions:

(1) If T is an R-ring - when is $\overline{T}$ a monad, that is, $T \otimes_R A$ is an A-ring?

(2) Under which (other) conditions can $\overline{T}$ be made a monad?

(3) If T and U are (R,R)-bimodules such that $T \otimes_R -$ and $U \otimes_R -$ can be lifted to ${}_A\mathbb{M}$, can $U \otimes_R T \otimes_R -$ also be lifted to ${}_A\mathbb{M}$?

(4) If T and U are R-rings such that $T \otimes_R A$ and $U \otimes_R A$ are A-rings, when is $U \otimes_R T$ an A-ring?

We will often write $\otimes$ or $\cdot$ instead of $\otimes_R$ for short (in diagrams).

3.2. ***Tensor product of** R**-rings.*** Given two R-rings (A, m_A, ι_A) and (T, m_T, ι_T), the tensor product $T \otimes_R A$ is again an (R,R)-bimodule. An (R,R)-bilinear map

$$\varphi : A \otimes_R T \to T \otimes_R A$$

allows for the definition of a product m_φ on $T \otimes_R A$,

$$\begin{tikzcd}
 & & T \otimes A \otimes A \arrow[dr, "T\otimes m_A"] & \\
T \otimes A \otimes T \otimes A \arrow[r, "T\otimes\varphi\otimes A"] & T \otimes T \otimes A \otimes A \arrow[ur, "m_T\otimes A\otimes A"] \arrow[dr, "T\otimes T\otimes m_A"'] & & T \otimes A \\
 & & T \otimes T \otimes A \arrow[ur, "m_T\otimes A"'] &
\end{tikzcd},$$

that is, $m_\varphi = (m_T \otimes m_A) \circ (I_T \otimes \varphi \otimes I_A)$. If m_φ is associative and

$$\iota_T \otimes \iota_A = (I_T \otimes \iota_A) \circ \iota_T = (\iota_T \otimes I_A) \circ \iota_A$$

is the unit for this multiplication, then the R-ring $(T \otimes_R A, m_\varphi, \iota_A \otimes \iota_T)$ is called a *smash product* of T and A. For this certain properties of φ are required.

3.3. ***Ring entwinings.*** *For R-rings T and A, and a given (R,R)-bilinear map $\varphi : A \otimes_R T \to T \otimes_R A$, the following are equivalent:*

(a) $T \otimes_\varphi A := (T \otimes_R A, m_\varphi, \iota_T \otimes \iota_A)$ is an A-ring;

(b) φ induces commutativity of the diagrams (3.1) and the diagrams

$$\begin{tikzcd}
A \otimes T \otimes T \arrow[rr, "A\otimes m_T"] \arrow[d, "\varphi\otimes T"'] & & A \otimes T \arrow[d, "\varphi"] \\
T \otimes A \otimes T \arrow[r, "T\otimes\varphi"] & T \otimes T \otimes A \arrow[r, "m_T\otimes A"] & T \otimes A,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "A\otimes\iota_T"] \arrow[dr, "\iota_T\otimes A"'] & A \otimes T \arrow[d, "\varphi"] \\
 & T \otimes A;
\end{tikzcd}
\tag{3.2}$$

(c) φ induces commutativity of the diagrams (3.1) and

$$m_T \otimes I_A : T \otimes T \otimes A \to T \otimes A \text{ and } \iota_T \otimes I_A : A \to T \otimes A$$

are (A,A)-bilinear morphisms.

If these conditions ares satisfied, we call (T, A, φ) a *ring* (or *algebra*) *entwining.*

Proof. Notice that by one of the normality conditions the map $A \to T \otimes_\varphi A$, $a \mapsto 1_T \otimes a$, is a ring morphism. Thus $T \otimes_\varphi A$ is a ring if and only if it is an A-ring (or R-ring).

(a)$\Leftrightarrow$(b) (e.g. [10]) If $(T \otimes_R A, m_T \otimes I_A, \iota_T \otimes \iota_A)$ is a unital associative R-ring, we obtain the *normality conditions*

$$1_T \otimes a = (1_T \otimes a) \cdot_\varphi (1_T \otimes 1_A) = \varphi(a \otimes 1_T), \tag{3.3}$$

$$t \otimes 1_A = (t \otimes 1_A) \cdot_\varphi (1_T \otimes 1_A) = \varphi(1_A \otimes t). \tag{3.4}$$

Applying these, the associativity conditions

$$(1_T \otimes a) \cdot_\varphi ((t \otimes 1_A) \cdot_\varphi (s \otimes 1_A)) = ((1_T \otimes a) \cdot_\varphi (t \otimes 1_A)) \cdot_\varphi (s \otimes 1_A), \tag{3.5}$$

$$((1_T \otimes a) \cdot_\varphi (1_T \otimes b)) \cdot_\varphi (t \otimes 1_A) = (1_T \otimes a) \cdot_\varphi ((1_T \otimes b) \cdot_\varphi (t \otimes 1_A)), \tag{3.6}$$

can be written as

$$\varphi(a \otimes ts) = (m_T \otimes I_A)(I_T \otimes \varphi)(\varphi(a \otimes t) \otimes s), \tag{3.7}$$

$$\varphi(ab \otimes t) = (I_T \otimes m_A)(\varphi \otimes I_A)(a \otimes \varphi(b, t)). \tag{3.8}$$

The conditions (3.3) and (3.5) yield the commutative diagrams in (3.1); similarly, (3.4) and (3.6) lead to the commutative diagrams in (3.2).

On the other hand, multiplying (3.5) by $1_T \otimes d$ from the right shows that $c \otimes d$ associates with the two other elements in (3.5). Continuing with similar arguments one can show that the normality conditions together with (3.5) and (3.6) imply associativity of m_φ. The latter also follows (in view of (d)) from the large diagram in the proof of 3.4.

(b)$\Leftrightarrow$(c) Right A-linearity of $m_T \otimes I_A$ means just commutativity of the quadrangle in 3.2 and right A-linearity of $\iota_T \otimes I_A$ is clear. In the diagram

$$\begin{tikzcd}
A \otimes T \otimes T \otimes A \arrow[rr, "A \otimes m_T \otimes A"] \arrow[d, "\varphi \otimes T \otimes A"'] & & A \otimes T \otimes A \arrow[d, "\varphi \otimes A"] \\
T \otimes A \otimes T \otimes A \arrow[r, "T \otimes \varphi \otimes A"] & T \otimes T \otimes A \otimes A \arrow[r, "m_T \otimes A \otimes A"] \arrow[d, "T \otimes T \otimes m_A"'] & T \otimes A \otimes A \arrow[d, "T \otimes m_A"] \\
& T \otimes T \otimes A \arrow[r, "m_T \otimes A"] & T \otimes A,
\end{tikzcd} \tag{3.9}$$

the bottom rectangle is commutative by functoriality of composition. If (c) holds, then, by (3.2), the upper rectangle is commutative. Commutativity of the outer diagram is just left A-linearity of $m_T \otimes I_A$. Given this, entering the diagram with $I_A \otimes I_T \otimes I_T \otimes \iota_A : A \otimes T \otimes T \longrightarrow A \otimes T \otimes T \otimes A$ leads to commutativity of the rectangle in (3.2).

Now consider the diagram

$$\begin{tikzcd}
 & A\otimes T\otimes A \arrow[d, "\varphi"] \\
A\otimes A \arrow[ur, "A\otimes\iota_T\otimes A"] \arrow[r, "\iota_T\otimes A\otimes A"'] \arrow[d, "m_A"] & T\otimes A\otimes A \arrow[d, "T\otimes m_A"] \\
A \arrow[r, "\iota_T\otimes A"] & T\otimes A.
\end{tikzcd}$$

The outer diagram is commutative if and only if $\iota_T\otimes I_A$ is left A-linear and this is the case if and only if the triangle in (3.2) is commutative. □

In 3.2 we have derived the product and unit on $T\otimes_R A$ from the products and units of A and T. From the diagram there it is obvious that, given m_A and ι_A, product and unit on $T\otimes A$ can also be described by the morphisms $m_T\otimes I_A$ and $\iota_T\otimes I_A$ which have to be (A,A)-bilinear by 3.3. Thus a ring structure on $T\otimes_R A$ can be defined by two (A,A)-bilinear maps $T\otimes T\otimes A\to T\otimes A$ and $A\to T\otimes A$ without requiring a ring structure on T.

3.4. *Liftings as R-rings I.* *Let (A,m_A,ι_A) be an R-ring and T an (R,R)-bimodule such that $T\otimes_R -$ can be lifted to a functor $\overline{T}: {}_A\mathbb{M}\to {}_A\mathbb{M}$ by the entwining $\varphi: A\otimes_R T\to T\otimes_R A$ (see 3.1) and assume there are given (A,A)-bilinear morphisms*

$$\nu: T\otimes T\otimes A\to T\otimes A,\quad \xi: A\to T\otimes A.$$

Then the lifting $\overline{T}$ induces an A-ring structure on $T\otimes_R A$ with multiplication m_ν,

$$T\otimes A\otimes T\otimes A\xrightarrow{T\otimes\varphi\otimes I_A} T\otimes T\otimes A\otimes A\xrightarrow{T\otimes T\otimes m_A} T\otimes T\otimes A\xrightarrow{\nu} T\otimes A,$$

and unit ξ if and only if the data induce commutativity of the diagrams

$$\begin{tikzcd}
T\otimes T\otimes A \arrow[r, "\nu"] & T\otimes A \\
T\otimes A \arrow[u, "T\otimes\xi"] \arrow[ur, "="'] &
\end{tikzcd}\quad,\qquad
\begin{tikzcd}
T\otimes T\otimes A \arrow[r, "\nu"] & T\otimes A \\
T\otimes A\otimes T \arrow[u, "T\otimes\varphi"] & A\otimes T, \arrow[l, "\xi\otimes T"'] \arrow[u, "\varphi"']
\end{tikzcd} \tag{3.10}$$

$$\begin{tikzcd}
T\otimes T\otimes A\otimes T \arrow[r, "\nu\otimes I_T"] \arrow[d, "T\otimes T\otimes\varphi"'] & T\otimes A\otimes T \arrow[r, "T\otimes\varphi"] & T\otimes T\otimes A \arrow[d, "\nu"] \\
T\otimes T\otimes T\otimes A \arrow[r, "T\otimes\nu"] & T\otimes T\otimes A \arrow[r, "\nu"] & T\otimes A.
\end{tikzcd} \tag{3.11}$$

Proof. Left A-linearity of ν is equivalent to commutativity of the diagram (compare

(3.9))

$$
\begin{tikzcd}
A \otimes T \otimes T \otimes A \arrow[r, "\varphi \otimes T \otimes A"] \arrow[d, "A \otimes \nu"'] & T \otimes A \otimes T \otimes A \arrow[r, "T \otimes \varphi \otimes A"] & T \otimes T \otimes A \otimes A \arrow[r, "T \otimes T \otimes m_A"] & T \otimes T \otimes A \arrow[d, "\nu"] \\
A \otimes T \otimes A \arrow[rr, "\varphi \otimes A"] & & T \otimes A \otimes A \arrow[r, "T \otimes m_A"] & T \otimes A,
\end{tikzcd}
\tag{3.12}
$$

whereas right A-linearity of ν corresponds to commutativity of the diagram

$$
\begin{tikzcd}
T \otimes T \otimes A \otimes A \arrow[r, "\nu \otimes A"] \arrow[d, "T \otimes T \otimes m_A"'] & T \otimes A \otimes A \arrow[d, "T \otimes m_A"] \\
T \otimes T \otimes A \arrow[r, "\nu"] & T \otimes A.
\end{tikzcd}
\tag{3.13}
$$

To prove associativity of the product m_ν on $T \otimes A$, consider the diagram (where $\otimes$ is replaced by $\cdot$ between modules and deleted between morphisms)

$$
\begin{tikzcd}
T \cdot A \cdot T \cdot A \cdot T \cdot A \arrow[r, "T\varphi ATA"] \arrow[d, "TAT\varphi A"'] & T \cdot T \cdot A \cdot A \cdot T \cdot A \arrow[rr, "TTm_ATA"] \arrow[d, "TTA\varphi A"'] & \arrow[d, phantom, "(1)"] & T \cdot T \cdot A \cdot T \cdot A \arrow[r, "\nu TA"] \arrow[d, "TT\varphi A"'] \arrow[dr, phantom, "(2)"] & T \cdot A \cdot T \cdot A \arrow[d, "T\varphi A"] \\
T \cdot A \cdot T \cdot T \cdot A \cdot A \arrow[r, "T\varphi TAA"'] \arrow[d, "TA\nu A"'] & T \cdot T \cdot A \cdot T \cdot A \cdot A \arrow[r, "TT\varphi AA"] \arrow[dr, phantom, "(\star\star)"] & T \cdot T \cdot T \cdot A \cdot A \cdot A \arrow[r, "TTTm_AA"] & T \cdot T \cdot T \cdot A \cdot A \arrow[d, "T\nu A"] & T \cdot T \cdot A \cdot A \arrow[d, "\nu A"] \\
T \cdot A \cdot T \cdot A \cdot A \arrow[rr, "T\varphi AA"] \arrow[d, "TATm_A"'] & & T \cdot T \cdot A \cdot A \cdot A \arrow[r, "TTm_AA"] \arrow[d, "TTAm_A"'] & T \cdot T \cdot A \cdot A \arrow[r, "\nu A"] \arrow[d, "TTm_A"'] \arrow[dr, phantom, "(3)"] & T \cdot A \cdot A \arrow[d, "Tm_A"] \\
T \cdot A \cdot T \cdot A \arrow[rr, "T\varphi A"] & & T \cdot T \cdot A \cdot A \arrow[r, "TTm_A"] & T \cdot T \cdot A \arrow[r, "\nu"] & T \cdot A.
\end{tikzcd}
$$

Diagram (1) is commutative by (3.1), diagram $(\star\star)$ is commutative by (3.12) (added T from the left and A from the right), diagram (2) is commutative by assumption (3.11) (applied to A), and commutativity of diagram (3) follows from (3.13). The remaining inner diagrams are commutative by naturality of composition or associativity of multiplication in A. Thus the outer diagram is commutative and this shows associativity of the multiplication m_ν.

The verification of the unitality conditions is left to the reader. □

Given the morphisms $\nu : T \otimes T \otimes A \to T \otimes A$ and $\xi : A \to T \otimes A$ in 3.4, we may form

$$
\bar{\nu} : \; T \otimes T \xrightarrow{T \otimes T \otimes \iota_A} T \otimes T \otimes A \xrightarrow{\nu} T \otimes A, \quad \sigma : \; R \xrightarrow{\iota_A} A \xrightarrow{\xi} T \otimes A,
$$

from which we can regain the initial maps as

$$
\nu = (I_T \otimes m_A) \circ (\bar{\nu} \otimes I_A), \quad \xi = (I_T \otimes m_A) \circ (\sigma \otimes I_A).
$$

Thus $\bar{\nu}$ and ξ may be used to define a ring structure on $T \otimes_R A$.

3.5. *Liftings as R-rings II.* *Let (A, m_A, ι_A) be an R-ring and T an (R,R)-bimodule. Assume that $T \otimes_R -$ can be lifted to a functor $\overline{T} : {}_A\mathbb{M} \to {}_A\mathbb{M}$ by the entwining $\varphi : A \otimes_R T \to T \otimes_R A$ and that there are given (R,R)-bilinear morphisms*

$$\bar{\nu} : T \otimes T \to T \otimes A, \qquad \sigma : R \to T \otimes A.$$

Then $T \otimes_R A$ has an R-ring structure with multiplication

$$T \cdot A \cdot T \cdot A \xrightarrow{T \otimes \varphi \otimes A} T \cdot T \cdot A \cdot A \xrightarrow{\quad T \otimes T \otimes m_A \quad} T \cdot T \cdot A \xrightarrow{\bar{\nu} \otimes A} T \cdot A \cdot A \xrightarrow{T \otimes m_A} T \cdot A$$

and unit σ provided the data induce commutativity of the diagrams

$$\begin{tikzcd}
T \otimes T \otimes T \arrow[r, "\bar{\nu} \otimes T"] \arrow[d, "T \otimes \bar{\nu}"'] & T \otimes A \otimes T \arrow[r, "T \otimes \varphi"] & T \otimes T \otimes A \arrow[r, "\bar{\nu} \otimes A"] & T \otimes A \otimes A \arrow[d, "T \otimes m_A"] \\
T \otimes T \otimes A \arrow[r, "\bar{\nu} \otimes A"] & T \otimes A \otimes A \arrow[rr, "T \otimes m_A"] & & T \otimes A,
\end{tikzcd} \tag{3.14}$$

$$\begin{tikzcd}
A \otimes T \otimes T \arrow[r, "\varphi \otimes T"] \arrow[d, "A \otimes \bar{\nu}"'] & T \otimes A \otimes T \arrow[r, "T \otimes \varphi"] & T \otimes T \otimes A \arrow[r, "\bar{\nu} \otimes A"] & T \otimes A \otimes A \arrow[d, "T \otimes m_A"] \\
A \otimes T \otimes A \arrow[r, "\varphi \otimes I_A"] & T \otimes A \otimes A \arrow[rr, "T \otimes m_A"] & & T \otimes A,
\end{tikzcd} \tag{3.15}$$

$$\begin{tikzcd}
A \arrow[rr, "\sigma \otimes A"] \arrow[d, "A \otimes \sigma"'] & & T \otimes A \otimes A \arrow[d, "T \otimes m_A"] \\
A \otimes T \otimes A \arrow[r, "\varphi \otimes A"] & T \otimes A \otimes A \arrow[r, "T \otimes m_A"] & T \otimes A,
\end{tikzcd} \tag{3.16}$$

$$\begin{tikzcd}
T \arrow[r, "\sigma \otimes T"] \arrow[d, "T \otimes \iota_A"'] & T \otimes A \otimes T \arrow[r, "T \otimes \varphi"] & T \otimes T \otimes A \arrow[d, "\bar{\nu} \otimes A"] & T \arrow[l, "T \otimes \sigma"'] \arrow[d, "T \otimes \iota_A"] \\
T \otimes A & & T \otimes A \otimes A \arrow[ll, "T \otimes m_A"] \arrow[r, "T \otimes m_A"'] & T \otimes A.
\end{tikzcd} \tag{3.17}$$

(3.14) is known as *cocycle condition*, (3.15) is the so called *twisted condition*, and (3.16), (3.17) are the unitality conditions.

Proof. This is shown with similar methods as 3.3 and 3.4. □

We have seen in 2.4 that the composition of two liftable functors can again be lifted. Another question is if the composition of two ring liftings is again an R-ring. To ensure this we need an extra condition.

3.6. *Yang-Baxter equation.* Let A, T, U be (R,R)-bimodules with linear maps

$$\varphi_{TU} : T \otimes_R U \to U \otimes_R T, \quad \varphi_{AT} : A \otimes_R T \to T \otimes_R A, \quad \varphi_{AU} : A \otimes_R U \to U \otimes_R A.$$

The triple $(\varphi_{TU}, \varphi_{AT}, \varphi_{AU})$ is said to satisfy the *Yang-Baxter equation* if it yields commutativity of the diagram

$$\begin{tikzcd}
A \otimes T \otimes U \arrow[r, "\varphi_{AT}\otimes U"] \arrow[d, "A\otimes\varphi_{TU}"'] & T \otimes A \otimes U \arrow[r, "T\otimes\varphi_{AU}"] & T \otimes U \otimes A \arrow[d, "\varphi_{TU}\otimes A"] \\
A \otimes U \otimes T \arrow[r, "\varphi_{AU}\otimes T"] & U \otimes A \otimes T \arrow[r, "U\otimes\varphi_{AT}"] & U \otimes T \otimes A.
\end{tikzcd}$$

It is well-known that over a commutative ring R the twist map tw satisfies the Yang-Baxter equations for any R-modules A, T, U.

3.7. *Tensor product of three R-rings.* *Let (U, m_U, ι_U), (T, m_T, ι_T) and (A, m_A, ι_A) be R-rings with ring entwinings*

$$\varphi_{TU} : T \otimes_R U \to U \otimes_R T, \quad \varphi_{AT} : A \otimes_R T \to T \otimes_R A, \quad \varphi_{AU} : A \otimes_R U \to U \otimes_R A.$$

The following statements are equivalent:

(a) $(\varphi_{TU}, \varphi_{AT}, \varphi_{AU})$ satisfies the Yang-Baxter equation;

(b) $U \otimes_{\varphi_{TU}} T \otimes -$ lifts to a monad $\overline{U \otimes_{\varphi_{TU}} T} : {}_A\mathbb{M} \to {}_A\mathbb{M}$ by

$$(I_U \otimes \varphi_{AT}) \circ (\varphi_{AU} \otimes I_T) : A \otimes_R U \otimes_{\varphi_{TU}} T \to U \otimes_{\varphi_{TU}} T \otimes_R A;$$

(c) $U \otimes_R T \otimes_R A$ is an R-ring with product

$$(m_U \otimes m_T \otimes m_A) \circ (I_U \otimes \varphi_{TU} \otimes \varphi_{AT} \otimes I_A) \circ (I_U \otimes I_T \otimes \varphi_{AU} \otimes I_T \otimes I_A)$$

and unit $\iota_U \otimes \iota_T \otimes \iota_A$.

Proof. The crucial part is the equivalence between (a) and (b). It was shown in 2.4 that the composition $U \otimes_R T \otimes_R -$ lifts to $\overline{U \otimes_R T} : {}_A\mathbb{M} \to {}_A\mathbb{M}$, that is, (3.1) holds for the R-ring A and the functor $U \otimes_R T \otimes_R -$.

To prove commutativity of the rectangle in (3.2), consider the diagram (where $\otimes$ is replaced by $\cdot$ and obvious identity morphisms are deleted)

$$\begin{tikzcd}
A\cdot U\cdot T\cdot U\cdot T \arrow[r, "\varphi_{TU}"] \arrow[d, "\varphi_{AU}"] & A\cdot U\cdot U\cdot T\cdot T \arrow[r, "m_U"] \arrow[d, "\varphi_{AU}"] \arrow[dr, phantom, "(1)"] & A\cdot U\cdot T\cdot T \arrow[r, "m_T"] \arrow[d, "\varphi_{AU}"] & A\cdot U\cdot T \arrow[d, "\varphi_{AU}"] \\
U\cdot A\cdot T\cdot U\cdot T \arrow[r, "\varphi_{TU}"] \arrow[d, "\varphi_{AT}"] \arrow[dr, phantom, "(2)"] & U\cdot A\cdot U\cdot T\cdot T \arrow[d, "\varphi_{AU}"] & U\cdot A\cdot T\cdot T \arrow[r, "m_T"] \arrow[d, "\varphi_{AT}"] \arrow[dr, phantom, "(3)"] & U\cdot A\cdot T \arrow[d, "\varphi_{AT}"] \\
U\cdot T\cdot A\cdot U\cdot T \arrow[d, "\varphi_{AU}"] & U\cdot U\cdot A\cdot T\cdot T \arrow[d, "\varphi_{AT}"] \arrow[ur, "m_U"] & U\cdot T\cdot A\cdot T \arrow[dr, "\varphi_{AT}"] & U\cdot T\cdot A \\
U\cdot T\cdot U\cdot A\cdot T \arrow[r, "\varphi_{TU}"] \arrow[dr, "\varphi_{AT}"'] & U\cdot U\cdot T\cdot A\cdot T \arrow[ur, "m_U"'] \arrow[drr, "\varphi_{AT}"] & & U\cdot T\cdot T\cdot A \arrow[u, "m_T"'] \\
& U\cdot T\cdot U\cdot T\cdot A \arrow[rr, "\varphi_{TU}"'] & & U\cdot U\cdot T\cdot T\cdot A, \arrow[u, "m_U"']
\end{tikzcd}$$

in which (1) and (3) are commutative since φ_{AU} and φ_{AT} are algebra entwinings. (2) is commutative because of the Yang-Baxter equation, and the other inner diagrams are commutative by naturality of the transformations involved. The outer morphisms yield the rectangle in (3.2) for the entwining between $U \otimes_{\varphi_{TU}} T$ and A.

(c)⇒(a) Assume (d) holds. Then the diagram for (3.2) in the preceding proof has to be commutative. Entering the diagram with the map

$$I_A \otimes \iota_U \otimes I_T \otimes I_U \otimes \iota_T : A \otimes_R T \otimes_R U \to A \otimes_R U \otimes_R T \otimes_R U \otimes_R T,$$

a short argument shows that the Yang-Baxter equation is satisfied.

The remaining assertions are shown by similar arguments. □

3.8. ***Examples.*** For commutative rings R and R-algebras A, T, the twist map $\mathsf{tw} : A \otimes_R T \to T \otimes_R A$ is an algebra entwining. This gives the product usually considered on $T \otimes_R A$. Furthermore, for any R-algebra U, tw satisfies the evolving Yang-Baxter equations thus inducing an algebra structure on $U \otimes_R T \otimes_R A$.

For a non-commutative ring R it is more difficult to find examples. For any R-ring A and $T = A$, the map (e.g. Nuss [22])

$$\varphi : A \otimes_R A \to A \otimes_R A, \quad a \otimes b \mapsto ab \otimes 1_A + 1_A \otimes ab - a \otimes b, \tag{3.18}$$

is a ring entwining satisfying the Yang-Baxter equations thus making $A \otimes_\varphi A$ and multiple tensor products $A \otimes_R A \otimes_R \cdots \otimes_R A$ associative rings.

Other examples evolve in the theory of birings (see 5.4).

3.9. ***Remarks.*** The ring entwining in 3.3 corresponds to the twisted tensor product considered by Čap e.a. [12] and is also known as *smash product* of algebras (e.g. Caenepeel e.a. [10]). It is a special case of the *distributive laws* considered in Beck [2] for functors on arbitrary categories.

The multiplication considered in 3.5 is similar to Brzeziński's construction in [5, Proposition 2.1] which is formulated for functors on categories in [28, 4.8]. These are special cases of the notion of a *wreath* defined as a monad in a particular 2-category by Lack and Street [17]. In the category of sets this may be seen as a generalisation of the wreath product of groups ([17, Example 3.2]). In [17, Example 3.3] it is also outlined how Sweedler's crossed product of Hopf algebras can be described in terms of wreaths.

The product on $T \otimes_R A$ chosen in 3.4 is called *wreath product* in El Kaoutit [15, Proposition 1.11] (for strict monoidal categories) and a universal property of it is formulated in [15, Proposition 1.14].

The general form of the Yang-Baxter equation 3.6 and its application in 3.7 is considered in Bourn [4]. In particular it is used there to insure an algebra structure on multiple products of a monad, generalising the cases considered in Nuss [22] (see 3.8) and Menini and Stefan [19]. Yang-Baxter operators in context with algebras, coalgebras and entwinings are also studied by Brzeziński, Dăscălescu and Nichita in [21], [13], [8].

Given an R-ring (H, m, e) and a ring entwining $\tau : H \otimes_R H \to H \otimes_R H$, it is shown in [20, 6.9] that $(H, m \circ \tau, e)$ is again an R-ring with ring entwining τ, provided τ satisfies the Yang-Baxter equation (holds in general categories).

4. Lifting of endofunctors to comodule categories

First we recall the dual notion of an R-ring.

4.1. ***R-corings.*** Given a ring R, an (R, R)-bimodule C is said to be an R-*coring* provided there are (R, R)-bilinear maps, comultiplication and counit,

$$\Delta_C : C \to C \otimes_R C, \quad \varepsilon_C : C \to R,$$

subject to coassociativity and counitality conditions.

In categorical terminology these conditions describe a *comonad* on the category of left (or right) R-modules ${}_R\mathbb{M}$, that is, an (R, R)-bimodule C is an R-coring if and only if $C \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a comonad (e.g. [3, 4.2]).

4.2. ***C-comodules.*** A left C-*comodule* is a left R-module M with an R-linear map

$$\varrho^M : M \to C \otimes_R M,$$

satisfying the coassociativity and counitality conditions

$$(\Delta \otimes I_M) \circ \varrho^M = (I_C \otimes \varrho^M) \circ \varrho^M, \quad (\varepsilon_C \otimes I_M) \circ \varrho^M = I_M.$$

A C-*comodule morphism* $f : M \to N$ between left C-comodules M and N, is an R-linear map with $(I_C \otimes f) \circ \rho^M = \rho^N \circ f$. The set of all these morphisms is denoted by $\mathrm{Hom}^C(M, N)$; it is an abelian group and hence the category of left C-comodules with these morphisms, denoted by ${}^C\mathbb{M}$, is additive. The C-endomorphisms $\mathrm{End}^C(M) := \mathrm{Hom}^C(M, M)$ form a subring of $\mathrm{End}_R(M)$.

The (free) functor for the R-coring C,

$$C \otimes_R - : {}_R\mathbb{M} \to {}^C\mathbb{M}, \quad M \mapsto (C \otimes_R M, \Delta_C \otimes I_M),$$

is right adjoint to the forgetful functor ${}^CU : {}^C\mathbb{M} \to {}_R\mathbb{M}$ by the bijection, for $N \in {}^C\mathbb{M}$,

$$\mathrm{Hom}^C(N, C \otimes_R M) \to \mathrm{Hom}_R(U^C(N), M), \quad g \mapsto (\varepsilon_C \otimes I_M) \circ g.$$

Right C-comodules are defined symmetrically and given an R-coring C and an S-coring D, an (S, R)-bimodules M is called a (D, C)-*bicomodule* if it has a left D-comodule ${}^M\varrho : M \to D \otimes_S M$ and a right C-comodule structure $\varrho^M : M \to M \otimes_R C$ with a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\varrho^M"] \arrow[d, "{}^M\varrho"'] & M \otimes_R C \arrow[d, "{}^M\varrho \otimes C"] \\
D \otimes_R M \arrow[r, "D \otimes \varrho^M"] & D \otimes_R M \otimes_R C.
\end{tikzcd}$$

4.3. *Cotensor product.* For a right C-comodule $\varrho^N : N \to N \otimes_R C$ and a left C-comodule ${}^M\varrho : M \to C \otimes_R M$, the *cotensor product* $N \otimes^C M$ is defined as the equaliser in $\mathbb{M}_{\mathbb{Z}}$ (e.g. [9, Sections 21]),

$$N \otimes^C M \longrightarrow N \otimes_R M \underset{N \otimes {}^M\varrho}{\overset{\varrho^N \otimes M}{\rightrightarrows}} N \otimes_R C \otimes_R M \; .$$

The cotensor product need not be associative. Nevertheless, one has the comodule isomorphism

$${}^M\varrho : M \to C \otimes^C M, \qquad \varepsilon_C \otimes^C I_M : C \otimes^C M \to M.$$

Moreover, since $M \simeq C \otimes^C M$ is an R-direct summand of $C \otimes_R M$, for any $T \in \mathbb{M}_R$, we get an isomorphism

$$T \otimes_R (C \otimes^C M) \simeq (T \otimes_R C) \otimes^C M.$$

For any $f : N \to N'$ in $\mathbb{M}^C$ and $g : M \to M'$ in ${}^C\mathbb{M}$, the cotensor product of morphisms $f \otimes^C g$ is defined (similar to the module case).

Replacing rings by corings and proceding in a similar way as in 2.3 one obtains

4.4. *Applegate's theorem for corings.* *For rings R, S, let $(C, \Delta_C, \varepsilon_C)$ be an R-coring, $(D, \Delta_D, \varepsilon_D)$ an S-coring, and T an (S,R)-bimodule.*

The liftings $\widehat{T} : {}^C\mathbb{M} \to {}^D\mathbb{M}$ of the functor $T \otimes_R - : {}_R\mathbb{M} \to {}_S\mathbb{M}$ are in bijective correspondence with those (S,R)-bilinear maps $\omega : T \otimes_R C \to D \otimes_S T$ which induce commutativity of the diagrams

$$\begin{tikzcd}
T \otimes_R C \arrow[r, "T\otimes\Delta_C"] \arrow[d, "\omega"'] & T \otimes_R C \otimes_R C \arrow[r, "\omega\otimes C"] & D \otimes_S T \otimes_R C \arrow[d, "D\otimes\omega"] \\
D \otimes_S T \arrow[rr, "\Delta_D\otimes T"] & & D \otimes_S D \otimes_S T,
\end{tikzcd}
\qquad
\begin{tikzcd}
T \otimes_R C \arrow[r, "T\otimes\varepsilon_C"] \arrow[d, "\omega"'] & T \\
D \otimes_S T \arrow[ur, "\varepsilon_D\otimes T"'] &
\end{tikzcd} .$$

For any $M \in {}^C\mathbb{M}$, the lifting is given by

$$\widehat{T}(M) = T \otimes_R M \simeq T \otimes_R (C \otimes^C M) \simeq (T \otimes_R C) \otimes^C M$$

where $T \otimes_R C$ is a (D,C)-bicomodule with left D-comodule structure

$$T \otimes_R C \xrightarrow{T\otimes\Delta} T \otimes_R C \otimes_R C \xrightarrow{\omega\otimes C} D \otimes_S T \otimes_R C.$$

4.5. *Composition of liftings to comodules.* *For rings R, S, Q, consider an R-coring C, an S-coring D, and a Q-coring E. Let T be an (S,R)-bimodule, U a (Q,S)-bimodule, and assume that*

(i) $T \otimes_R - : {}_R\mathbb{M} \to {}_S\mathbb{M}$ lifts to $\widehat{T} : {}^C\mathbb{M} \to {}^D\mathbb{M}$ by $\omega : T \otimes_R C \to D \otimes_S T$,

(ii) $U \otimes_S - : {}_R\mathbb{M} \to {}_Q\mathbb{M}$ lifts to $\widehat{U} : {}^D\mathbb{M} \to {}^E\mathbb{M}$ by $\omega' : U \otimes_S D \to E \otimes_Q U$.

Then $U \otimes_S T \otimes_R -$ lifts to $\widehat{U \otimes_S T} : {}^C\mathbb{M} \to {}^E\mathbb{M}$ by the (Q,R)-bilinear map

$$U \otimes_S T \otimes_R C \xrightarrow{U\otimes\omega} U \otimes_S D \otimes_S T \xrightarrow{\omega'\otimes T} E \otimes_Q U \otimes_S T.$$

For any $M \in {}^C\mathbb{M}$, *the lifting is given by*

$$\widehat{U \otimes_S T}(M) = U \otimes_S T \otimes_R M \simeq (U \otimes_S T \otimes_R C) \otimes^C M$$

where $U \otimes_S T \otimes_R C$ *is an* (E, C)*-bicomodule with left* E*-comodule structure*

$$U \otimes_S T \otimes_R C \xrightarrow{U \otimes T \otimes \Delta} U \otimes_S T \otimes_R C \otimes_R C \xrightarrow{U \otimes \omega \otimes C} U \otimes_S D \otimes_S T \otimes_R C \xrightarrow{\omega' \otimes T \otimes C} E \otimes_Q U \otimes_S T \otimes_R C.$$

We now specialise to the case $R = S$ and $C = D$.

4.6. ***Lifting endofunctors to comodules.*** Let C be an R-coring and T an (R, R)-bimodule. Then a functor $\widehat{T} : {}^C\mathbb{M} \to {}^C\mathbb{M}$ is a lifting of $T \otimes_R -$ if and only if the diagram

$$\begin{tikzcd} {}^C\mathbb{M} \arrow[r, "\widehat{T}"] \arrow[d, "{}^CU"'] & {}^C\mathbb{M} \arrow[d, "{}^CU"] \\ {}_R\mathbb{M} \arrow[r, "T \otimes_R -"] & {}_R\mathbb{M} \end{tikzcd}$$

is commutative where ${}^CU$ denotes the forgetful functor. In this case Applegate's theorem says that the liftings $\widehat{T}$ of the functor $T \otimes_R -$ are in bijective correspondence with the (R, R)-bilinear maps $\omega : T \otimes_R C \to C \otimes_R T$ which induce commutativity of the diagrams

$$\begin{tikzcd} T \otimes_R C \arrow[r, "T \otimes \Delta_C"] \arrow[d, "\omega"'] & T \otimes_R C \otimes_R C \arrow[r, "\omega \otimes C"] & C \otimes_R T \otimes_R C \arrow[d, "C \otimes \omega"] \\ C \otimes_R T \arrow[rr, "\Delta_C \otimes T"] & & C \otimes_R C \otimes_R T, \end{tikzcd} \qquad \begin{tikzcd} T \otimes_R C \arrow[r, "T \otimes \varepsilon_C"] \arrow[d, "\omega"'] & T \\ C \otimes_R T \arrow[ur, "\varepsilon_C \otimes T"'] & \end{tikzcd} . \tag{4.1}$$

For any $M \in {}^C\mathbb{M}$ the lifting is given by

$$\widehat{T}(M) = T \otimes_R M \simeq (T \otimes_R C) \otimes^C M,$$

where $T \otimes_R C$ is a (C, C)-bicomodule with left C-comodule structure

$$T \otimes_R C \xrightarrow{T \otimes \Delta} T \otimes_R C \otimes_R C \xrightarrow{\omega \otimes C} C \otimes_S T \otimes_R C.$$

Assume in 4.6, T is also an R-coring. Then the question arises under which conditions the lifting is again an R-coring.

4.7. ***Tensor product of*** R***-corings.*** Given R-corings $(T, \Delta_T, \varepsilon_T)$ and $(C, \Delta_C, \varepsilon_C)$, the tensor product $T \otimes_R C$ is again an (R, R)-module and an (R, R)-bilinear map

$$\omega : T \otimes_R C \to C \otimes_R T,$$

induces a coproduct Δ_ω on $T \otimes_R C$,

$$\begin{tikzcd}
& T\otimes T\otimes C \arrow[dr, "T\otimes T\otimes\Delta_C"] & & \\
T\otimes C \arrow[ur, "\Delta_T\otimes C"] \arrow[dr, "T\otimes\Delta_C"'] & & T\otimes T\otimes C\otimes C \arrow[r, "T\otimes\omega\otimes C"] & T\otimes C\otimes T\otimes C \\
& T\otimes C\otimes C \arrow[ur, "\Delta_T\otimes C\otimes C"'] & & .
\end{tikzcd}$$

If Δ_ω is coassociative and

$$\varepsilon_T \otimes \varepsilon_C = \varepsilon_C \circ (\varepsilon_T \otimes I_C) = \varepsilon_c \circ (I_T \circ \varepsilon_C)$$

is a counit for Δ_ω, the R-coring $(T \otimes_R C, \Delta_\omega, \varepsilon_T \otimes \varepsilon_C)$ is called the *smash coproduct* of T and C. For this ω has to satisfy certain conditions.

4.8. ***Coring entwinings.*** *For R-corings T, C, and an (R,R)-bilinear morphism $\omega : T \otimes_R C \to C \otimes_R T$, the following are equivalent:*

(a) $T \otimes_\omega C := (T \otimes_R C, \Delta_\omega, \varepsilon_T \otimes \varepsilon_C)$ is an R-coring;

(b) ω induces commutativity of (4.1) and the diagrams

$$\begin{tikzcd}
T\otimes C \arrow[r, "\Delta_T\otimes I_C"] \arrow[d, "\omega"'] & T\otimes T\otimes C \arrow[r, "I_T\otimes\omega"] & T\otimes C\otimes T \arrow[d, "\omega\otimes I_T"] \\
C\otimes T \arrow[rr, "I_C\otimes\Delta_T"] & & C\otimes T\otimes T,
\end{tikzcd}
\qquad
\begin{tikzcd}
T\otimes C \arrow[r, "\varepsilon_T\otimes I_C"] \arrow[d, "\omega"'] & C \\
C\otimes T \arrow[ur, "I_C\otimes\varepsilon_T"'] & ;
\end{tikzcd} \tag{4.2}$$

(c) ω induces commutativity of (4.1) and

$$\Delta_T \otimes I_C : T \otimes C \to T \otimes T \otimes C, \quad \varepsilon_T \otimes I_C : T \otimes C \to C,$$

are (C,C)-bicolinear morphisms.

If these conditions hold, the monad $T \otimes_R -$ can be lifted to a comonad $\widehat{T} : {}^C\mathbb{M} \to {}^C\mathbb{M}$, and for $M \in {}^C\mathbb{M}$,

$$\widehat{T}(M) = T \otimes_R M \simeq (T \otimes_R C) \otimes^C M,$$

where $T \otimes_\omega C$ is an R-coring, and (T, C, ω) is called a *coring (*or *coalgebra) entwining.*

Similar to the product on the tensor product for R-rings, we observe that the coproduct defined in 4.7 does not need an explicit coring structure on T but can be expressed by the morphisms $\Delta_T \otimes C$ and $\varepsilon_T \otimes I_C$ which are left and right C-colinear.

4.9. ***Liftings as R-corings I.*** *Let $(C, \Delta_C, \varepsilon_C)$ be an R-coring and T an (R,R)-bimodule such that $T \otimes_R -$ can be lifted to a functor $\widehat{T} : {}^C\mathbb{M} \to {}^C\mathbb{M}$ by the entwining $\omega : T \otimes_R C \to C \otimes_R T$ (see 4.1) and assume there are given (C,C)-bicolinear morphisms*

$$\tau : T \otimes C \to T \otimes T \otimes C, \quad \chi : T \otimes C \to C.$$

Then the lifting $\widehat{T}$ induces an R-coring structure on $T \otimes_R C$ with comultiplication

$$T \otimes C \xrightarrow{\tau} T \otimes T \otimes C \xrightarrow{T\otimes T\otimes \Delta} T \otimes T \otimes C \otimes C \xrightarrow{T\otimes\omega\otimes C} T \otimes C \otimes T \otimes C$$

and counit $\varepsilon_C \circ \chi$ if and only if the data induce commutativity of the diagrams

$$\begin{tikzcd} T \otimes C \arrow[r, "\tau"] \arrow[dr, "="'] & T \otimes T \otimes C \arrow[d, "T\otimes\chi"] \\ & T \otimes C, \end{tikzcd} \qquad \begin{tikzcd} T \otimes C \arrow[r, "\tau"] \arrow[d, "\omega"'] & T \otimes T \otimes C \arrow[d, "T\otimes\omega"] \\ C \otimes T & T \otimes C \otimes T, \arrow[l, "\chi\otimes T"] \end{tikzcd} \tag{4.3}$$

$$\begin{tikzcd} T \otimes C \arrow[r, "\tau"] \arrow[d, "\tau"'] & T \otimes T \otimes C \arrow[r, "T\otimes\tau"] & T \otimes T \otimes T \otimes C \arrow[d, "T\otimes T\otimes\omega"] \\ T \otimes T \otimes C \arrow[r, "T\otimes\omega"] & T \otimes C \otimes T \arrow[r, "\tau\otimes T"] & T \otimes T \otimes C \otimes T. \end{tikzcd} \tag{4.4}$$

Proof. The proof is dual to that of 3.4. □

Given right C-comodule morphisms $\tau : T \otimes C \to T \otimes T \otimes C$ and $\chi : T \otimes C \to C$, one may form

$$\overline{\tau} : T \otimes C \xrightarrow{\tau} T \otimes T \otimes C \xrightarrow{T\otimes T\otimes\varepsilon_C} T \otimes T\ , \qquad \overline{\chi} : T \otimes C \xrightarrow{\chi} C \xrightarrow{\varepsilon_C} R\ ,$$

from which we can regain the initial maps

$$\tau = (\overline{\tau} \otimes I_C) \circ (I_T \otimes \Delta_C), \quad \chi = (\overline{\chi} \otimes I_C) \circ (I_T \otimes \Delta_C).$$

Thus $\overline{\tau}$ and $\overline{\chi}$ may be used to define an R-coring structure on $T \otimes_R C$.

4.10. ***Liftings as R-corings II.*** *Let $(C, \Delta_C, \varepsilon_C)$ be an R-coring and T an (R,R)-bimodule such that $T \otimes_R -$ can be lifted to a functor $\widehat{T} : {}^C\mathbb{M} \to {}^C\mathbb{M}$ by the entwining $\omega : T \otimes_R C \to C \otimes_R T$ (see 4.1) and assume there are given (C,C)-bicolinear morphisms*

$$\overline{\tau} : T \otimes C \to T \otimes T, \qquad \overline{\chi} : T \otimes C \to R.$$

Then the lifting $\widehat{T}$ induces an R-coring structure on $T \otimes_R C$ with comultiplication $\overline{\Delta}$

$$T \cdot C \xrightarrow{T\otimes\Delta_C} T \cdot C \cdot C \xrightarrow{\overline{\tau}\otimes C} T \cdot T \cdot C \xrightarrow{T\otimes T\otimes\Delta_C} T \cdot T \cdot C \cdot C \xrightarrow{T\otimes\omega\otimes C} T \cdot C \cdot T \cdot C$$

and counit $\overline{\chi}$ if and only if the data induce commutativity of the diagrams

$$\begin{tikzcd} T \otimes C \arrow[rr, "T\otimes\Delta_C"] \arrow[d, "T\otimes\Delta_C"'] & & T \otimes C \otimes C \arrow[r, "\overline{\tau}\otimes C"] & T \otimes T \otimes C \arrow[d, "T\otimes\overline{\tau}"] \\ T \otimes C \otimes C \arrow[r, "\overline{\tau}\otimes C"] & T \otimes T \otimes C \arrow[r, "T\otimes\omega"] & T \otimes C \otimes T \arrow[r, "\overline{\tau}\otimes T"] & T \otimes T \otimes T, \end{tikzcd} \tag{4.5}$$

$$\begin{tikzcd} T\otimes C \arrow[rrr, "T\otimes\Delta_C"] \arrow[d, "T\otimes\Delta_C"'] & & & T\otimes C\otimes C \arrow[r, "\omega\otimes C"] & C\otimes T\otimes C \arrow[d, "C\otimes\overline{\tau}"] \\ T\otimes C\otimes C \arrow[r, "\overline{\tau}\otimes C"] & T\otimes T\otimes C \arrow[r, "T\otimes\omega"] & T\otimes C\otimes T \arrow[rr, "\omega\otimes T"] & & C\otimes T\otimes T, \end{tikzcd} \tag{4.6}$$

$$\begin{tikzcd} T\otimes C \arrow[r, "T\otimes\Delta_C"] \arrow[d, "T\otimes\Delta_C"'] & T\otimes C\otimes C \arrow[r, "\omega\otimes C"] & C\otimes T\otimes C \arrow[d, "C\otimes\overline{\chi}"] \\ T\otimes C\otimes C \arrow[rr, "\overline{\chi}\otimes C"] & & C, \end{tikzcd} \tag{4.7}$$

$$\begin{tikzcd} T\otimes C \arrow[r, "T\otimes\Delta_C"] \arrow[d, "T\otimes\varepsilon_C"'] & T\otimes C\otimes C \arrow[d, "\overline{\tau}\otimes C"] & & T\otimes C \arrow[ll, "T\otimes\Delta_C"'] \arrow[d, "T\otimes\varepsilon_C"] \\ T & T\otimes T\otimes C \arrow[l, "T\otimes\overline{\chi}"'] \arrow[r, "T\otimes\omega"] & T\otimes C\otimes T \arrow[r, "\overline{\chi}\otimes T"] & T. \end{tikzcd} \tag{4.8}$$

Proof. The assertions are dual to those in 3.5. □

Recall from 4.5 that the composition of liftable functors is again liftable. Similar to the case of rings (compare 3.7), to lift the composition of two corings, that is, to get a coring structure on the tensor product of three R-corings, we need again the Yang-Baxter equation 3.6.

4.11. *Tensor product of three corings.* *Let* $(C,\Delta_C,\varepsilon_C)$, $(T,\Delta_T,\varepsilon_T)$, *and* $(U,\Delta_U,\varepsilon_U)$ *be three R-corings with coring entwinings*

$$\omega_{TC}:T\otimes_R C\to C\otimes_R T,\quad \omega_{UC}:U\otimes_R C\to C\otimes_R U,\quad \omega_{UT}:U\otimes_R T\to T\otimes_R U.$$

The following statements are equivalent:

(a) $(\omega_{TC},\omega_{UC},\omega_{UT})$ *satisfies the Yang-Baxter equation;*

(b) $U\otimes_R T\otimes_R C$ *is an R-coring with coproduct*

$$(I_U\otimes I_T\otimes\omega_{UC}\otimes I_T\otimes I_C)\circ(I_U\otimes\omega_{UT}\otimes\omega_{TC}\otimes I_C)\circ(\Delta_U\otimes\Delta_T\otimes\Delta_C)$$

and counit $\varepsilon_U\otimes\varepsilon_T\otimes\varepsilon_C$;

(c) $(\omega_{UC}\otimes I_C)\circ(I_U\otimes\omega_{TC}):U\otimes_R T\otimes_R C\to C\otimes_R U\otimes_R T$ *is a coring entwining.*

4.12. *Remarks.* For coalgebras C,T, the coring entwining in 4.8 is known as *smash coproduct* (e.g. Caenepeel e.a. [10]) and is just the dual of Beck's *distributive laws*. It can also be found in [9, 2.14]. The constructions listed in 4.9 and 4.10 dualise the wreath product defined in Lack and Street [17]. This is called *cowreath* in El Kaoutit [14,15] and, for example, the situation considered in 4.9 corresponds to [14, Proposition 2.2] (for strict monoidal categories). A universal property of the cowreath product is given in [15, Proposition 1.7]. The use of the Yang-Baxter equation 3.6 for the tensor product of three R-corings is an obvious dualisation of the ring case. Given an R-coring (H,δ,ε) and a coring entwining $\tau:H\otimes_R H\to H\otimes_R H$, it follows from [20, 6.9] that $(H,\tau\circ\delta,\varepsilon)$ is also a coring with coring entwining τ, provided τ satisfies the Yang-Baxter equation (holds in general categories).

5. Mixed liftings

Given an R-ring (A, m_A, ι_A) and an R-coring $(C, \Delta_C, \varepsilon_C)$, we may consider $T = C$ in the diagram in 3.1, and $T = A$ in the diagram 4.6. This yields the diagrams

$$\begin{tikzcd} {}_A\mathbb{M} \arrow[r, "\overline{C}"] \arrow[d, "{}_AU"'] & {}_A\mathbb{M} \arrow[d, "{}_AU"] \\ {}_R\mathbb{M} \arrow[r, "C\otimes_R -"] & {}_R\mathbb{M}, \end{tikzcd} \qquad \begin{tikzcd} {}^C\mathbb{M} \arrow[r, "\widehat{A}"] \arrow[d, "{}^CU"'] & {}^C\mathbb{M} \arrow[d, "{}^CU"] \\ {}_R\mathbb{M} \arrow[r, "A\otimes_R -"] & {}_R\mathbb{M}. \end{tikzcd}$$

In both cases the lifting properties are related to an (R, R)-bilinear map

$$\psi : A \otimes_R C \to C \otimes_R A.$$

The lifting in the left hand case requires commutativity of the diagrams (see 3.1)

$$\begin{tikzcd} A\otimes_R A\otimes_R C \arrow[rr, "m_A\otimes C"] \arrow[d, "A\otimes\psi"] & & A\otimes_R C \arrow[d, "\psi"] \\ A\otimes_R C\otimes_R A \arrow[r, "\psi\otimes A"] & C\otimes_R A\otimes_R A \arrow[r, "C\otimes m_A"] & C\otimes_R A, \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\iota_A\otimes C"] \arrow[dr, "C\otimes\iota_A"'] & A\otimes_R C \arrow[d, "\psi"] \\ & C\otimes_R A, \end{tikzcd} \tag{5.1}$$

whereas the lifting to ${}^C\mathbb{M}$ needs commutativity of (see 4.1)

$$\begin{tikzcd} A\otimes_R C \arrow[r, "A\otimes\Delta_C"] \arrow[d, "\psi"] & A\otimes_R C\otimes_R C \arrow[r, "\psi\otimes C"] & C\otimes_R A\otimes_R C \arrow[d, "C\otimes\psi"] \\ C\otimes_R A \arrow[rr, "\Delta_C\otimes A"] & & C\otimes_R C\otimes_R A, \end{tikzcd} \qquad \begin{tikzcd} A\otimes_R C \arrow[r, "A\otimes\varepsilon_C"] \arrow[d, "\psi"] & A \\ C\otimes_R A \arrow[ur, "\varepsilon_C\otimes A"'] & \end{tikzcd} . \tag{5.2}$$

The functor $\overline{C}$ is just $C\otimes_R A\otimes_A -$ where $C\otimes_R A$ is considered as (A, A)-bimodule (see 3.1). The conditions to make it an A-coring by the coproduct $\underline{\Delta}$

$$C\otimes_R A \xrightarrow{\Delta_C\otimes A} C\otimes_R C\otimes_R A \xrightarrow{\simeq} (C\otimes_R A)\otimes_A (C\otimes_R A)$$

turns out to be commutativity of the diagrams in (5.2). For example, left A-linearity of $\underline{\Delta}$ means commutativity of the diagram

$$\begin{tikzcd} A\otimes_R C\otimes_R A \arrow[r, "A\otimes\Delta_C\otimes A"] \arrow[d, "\psi\otimes A"] & A\otimes_R C\otimes_R C\otimes A \arrow[r, "\psi\otimes C\otimes A"] & C\otimes_R A\otimes_R C \arrow[d, "C\otimes\psi"] \\ C\otimes_R A\otimes_R A \arrow[rr, "\Delta_C\otimes A\otimes A"] \arrow[d, "C\otimes m_A"] & & C\otimes_R C\otimes_R A\otimes_R A \arrow[d, "C\otimes C\otimes m_A"] \\ C\otimes_R A \arrow[rr, "\Delta_C\otimes A"] & & C\otimes_R C\otimes_R A, \end{tikzcd}$$

in which the upper part is commutative if the rectangle in 5.2 is so, and the lower part is commutative by naturality.

On the other hand, to define a product and a unit for the lifting $\widehat{A}$, That is, to make it a monad, the diagrams in (5.1) are to be commutative: for the product we need that

$$m_A\otimes I_C : A\otimes_R A\otimes_R C \to A\otimes C$$

is left C-colinear and (dual to the argument above) this is equivalent to commutativity of the rectangle in (5.1).

By 4.6, the functor $\widehat{A}$ is equal to $A \otimes_R C \otimes^C -$ with $A \otimes_R C$ a (C, C)-bicomodule. Then $A \otimes_R C$ has a structure of the following type introduced in Brzeziński [6, Section 6] and also addressed as *semialgebras* in Positselski [23]:

5.1. *C-rings.* Let C be an R-coring. A (C, C)-bicomodule $(H, {}^H\varrho, \varrho^H)$ is called a *C-ring* if $(H \otimes^C H) \otimes^C H = H \otimes^C (H \otimes^C H)$, and there are (C, C)-bicomodule morphisms

$$\mu_H : H \otimes^C H \to H, \quad \eta_H : C \to H,$$

inducing commutativity of the diagrams

$$\begin{tikzcd} H \otimes^C H \otimes^C H \arrow[r, "\mu_H \otimes^C H"] \arrow[d, "H \otimes^C \mu_H"'] & H \otimes^C H \arrow[d, "\mu_H"] \\ H \otimes^C H \arrow[r, "\mu_H"] & H, \end{tikzcd} \qquad \begin{tikzcd} C \otimes^C H \arrow[r, "\eta_H \otimes^C H"] & H \otimes^C H \arrow[d, "\mu_H"] & H \otimes^C C \arrow[l, "H \otimes^C \eta_H"'] \\ H \arrow[u, "{}^H\varrho"] \arrow[r, "="] & H \arrow[r, "="] & H. \arrow[u, "\varrho^H"'] \end{tikzcd} \tag{5.3}$$

Summarising these observations we obtain:

5.2. *Mixed entwinings.* *Let (A, m_A, ι_A) an R-ring, $(C, \Delta_C, \varepsilon_C)$ an R-coring, and $\psi : A \otimes_R C \to C \otimes_R A$ an (R, R)-bilinear map. Then the following are equivalent:*

(a) *With the structures induced by ψ, $C \otimes_R A$ is an A-coring;*

(b) *the diagrams in (5.1) and (5.2) are commutative;*

(c) *with the structures induced by ψ, $A \otimes_R C$ is a C-ring;*

(d) *ψ induces a lift of $C \otimes_R -$ to a comonad $\overline{C} : {}_A\mathbb{M} \to {}_A\mathbb{M}$;*

(e) *ψ induces a lift of $A \otimes_R -$ to a monad $\widehat{A} : {}^C\mathbb{M} \to {}^C\mathbb{M}$.*

If these conditions hold we call (C, A, ψ) a *mixed entwining.*

Proof. We only prove (b)⇒(c), that is, we show that $A \otimes_R C$ has the structure of a C-ring. Its right comodule structure is the obvious one and its left C-comodule structure is given by

$$A \otimes_R C \xrightarrow{A \otimes \Delta_C} A \otimes_R C \otimes_R C \xrightarrow{\psi \otimes C} C \otimes_R A \otimes_R C$$

From the isomorphisms for the cotensor product (see 4.3) we obtain for any $M \in {}^C\mathbb{M}$ the associativity property

$$((A \otimes_R C) \otimes^C (A \otimes_R C)) \otimes^C M \simeq (A \otimes_R C) \otimes^C ((A \otimes_R C) \otimes^C M).$$

Multiplication on $A \otimes_R C$ is given by

$$(A \otimes_R C) \otimes^C (A \otimes_R C) \xrightarrow{A \otimes \varepsilon_C \otimes^C A \otimes C} A \otimes_R A \otimes_R C \xrightarrow{m_A \otimes C} A \otimes_R C$$

and we have a morphism $\eta_{A\otimes C} = \iota_A \otimes I_C : C \to A \otimes_R C$ which obviously is right C-colinear and is left C-colinear by commutativity of the diagram (apply (5.1))

$$\begin{tikzcd}
C \arrow[r, "\iota_A\otimes C"] \arrow[d, "\Delta"'] & A\otimes C \arrow[r, "A\otimes\Delta"] & A\otimes C\otimes C \arrow[d, "\psi\otimes C"] \\
C\otimes C \arrow[urr, "\iota_A\otimes C\otimes C"'] \arrow[rr, "C\otimes\iota_A\otimes C"'] & & C\otimes A\otimes C.
\end{tikzcd}$$

As outlined above, left C-colinearity of multiplication follows by the commutative rectangle in (5.2).

For the left hand diagram for $\eta_{A\otimes C}$ in (5.3) consider the diagram

$$\begin{tikzcd}
C\otimes A\otimes C \arrow[r, "\iota_A\otimes C\otimes A\otimes C"] & A\otimes C\otimes A\otimes C \arrow[dr, "A\otimes\varepsilon_C\otimes A\otimes C"] & \\
A\otimes C\otimes C \arrow[u, "\psi\otimes C"] \arrow[r, "\iota_A\otimes C\otimes C"] & A\otimes A\otimes C\otimes C \arrow[u, "A\otimes\psi\otimes C"] \arrow[r, "A\otimes A\otimes\varepsilon_C\otimes C"'] \arrow[d, "m_A\otimes C"'] & A\otimes A\otimes C \arrow[d, "m_A\otimes C"] \\
A\otimes C \arrow[u, "A\otimes\Delta_C"] \arrow[r, "A\otimes\Delta_C"] & A\otimes C\otimes C \arrow[r, "A\otimes\varepsilon_C\otimes C"] & A\otimes C,
\end{tikzcd}$$

in which commutativity of the triangle follows from (5.2) and the other diagrams are commutative by naturality. Noticing that the image of $(\psi \otimes I_C) \circ (I_A \otimes \Delta_C)$ is equal to $C \otimes^C (A \otimes_R C)$ we obtain the conditions required.

Commutativity of the right hand rectangle for $\eta_{A\otimes C}$ in (5.3) is seen by commutativity of the diagram

$$\begin{tikzcd}
A\otimes C\otimes C \arrow[rr, "A\otimes C\otimes\iota_A\otimes C"] \arrow[dr, "A\otimes\varepsilon_C\otimes C"] & & A\otimes C\otimes A\otimes C \arrow[d, "A\otimes\varepsilon_C\otimes A\otimes C"] & \\
A\otimes C \arrow[u, "A\otimes\Delta_C"] \arrow[r, "="] & A\otimes C \arrow[r, "A\otimes\iota_A\otimes C"] & A\otimes A\otimes C \arrow[r, "m_A\otimes C"] & A\otimes C,
\end{tikzcd}$$

taking into account that the image of $I_A \otimes \Delta$ is $(A \otimes C) \otimes^C C$.

For the remaining assertions and more details about mixed entwinings the reader is referred to [9]. □

5.3. *Entwining two R-rings with an R-coring.* *Let (A, m_A, ι_A), (B, m_B, ι_B) be R-rings with a ring entwining $\varphi_{BA} : B \otimes_R A \to A \otimes_R B$, and $(C, \Delta_C, \varepsilon_C)$ an R-coring with mixed entwinings*

$$\psi_{AC} : A \otimes_R C \to C \otimes_R A, \quad \psi_{BC} : B \otimes C \to C \otimes_R B.$$

Then the following are equivalent:

(a) $(\varphi_{BA}, \psi_{BC}, \psi_{AC})$ satisfies the Yang-Baxter equation;

(b) $(\psi_{AC} \otimes I_B) \circ (I_A \otimes \psi_{BC}) : A \otimes_{\varphi_{BA}} B \otimes_R C \to C \otimes_R A \otimes_{\varphi_{BA}} B$ is a mixed entwining;

(c) $C \otimes_R A \otimes_R B$ has an $A \otimes_{\varphi_{BA}} B$-coring structure (induced by the ψ's);

(d) $A \otimes_R B \otimes_R C$ *has a* C*-ring structure (induced by the* ψ*'s).*

Proof. Recall from 3.3 that a ring entwining φ_{BA} yields an R-ring $A \otimes_{\varphi_{BA}} B$.

The crucial step in (a)⇒(b) is to prove commutativity of the rectangle in (5.1). For this consider the diagram (where $\otimes_R$ is replaced by $\cdot$ and obvious identity maps are deleted)

$$\begin{tikzcd}
A{\cdot}B{\cdot}A{\cdot}B{\cdot}C \arrow[r, "\varphi_{BA}"] \arrow[d, "\psi_{BC}"'] & A{\cdot}A{\cdot}B{\cdot}B{\cdot}C \arrow[r, "m_B"] \arrow[d, "\psi_{BC}"'] \arrow[dr, phantom, "(1)"] & A{\cdot}A{\cdot}B{\cdot}C \arrow[r, "m_A"] \arrow[d, "\psi_{BC}"'] & A{\cdot}B{\cdot}C \arrow[d, "\psi_{BC}"'] \\
A{\cdot}B{\cdot}A{\cdot}C{\cdot}B \arrow[r, "\varphi_{BA}"] \arrow[d, "\psi_{AC}"'] \arrow[dr, phantom, "(2)"] & A{\cdot}A{\cdot}B{\cdot}C{\cdot}B \arrow[d, "\psi_{BC}"'] & A{\cdot}A{\cdot}C{\cdot}B \arrow[r, "m_A"] \arrow[d, "\psi_{AC}"] \arrow[dr, phantom, "(3)"] & A{\cdot}C{\cdot}B \arrow[d, "\psi_{AC}"] \\
A{\cdot}B{\cdot}C{\cdot}A{\cdot}B \arrow[d, "\psi_{BC}"'] & A{\cdot}A{\cdot}C{\cdot}B{\cdot}B \arrow[d, "\psi_{AC}"'] \arrow[ur, "m_B"] & A{\cdot}C{\cdot}A{\cdot}B \arrow[dr, "\psi_{AC}"] & C{\cdot}A{\cdot}B \\
A{\cdot}C{\cdot}B{\cdot}A{\cdot}B \arrow[r, "\varphi_{BA}"] \arrow[dr, "\psi_{AC}"'] & A{\cdot}C{\cdot}A{\cdot}B{\cdot}B \arrow[ur, "m_B"'] \arrow[drr, "\psi_{AC}"] & & C{\cdot}A{\cdot}A{\cdot}B \arrow[u, "m_A"'] \\
& C{\cdot}A{\cdot}B{\cdot}A{\cdot}B \arrow[rr, "\varphi_{BA}"'] & & C{\cdot}A{\cdot}A{\cdot}B{\cdot}B, \arrow[u, "m_B"']
\end{tikzcd}$$

in which (1) and (3) are commutative because ψ_{AC} and ψ_{BC} are mixed entwinings; (2) is commutative because of the Yang-Baxter equation, and the other inner diagrams are commutative because of naturality of the transformations involved. The outer morphisms yield the rectangle in (5.1) for the mixed entwining between $A \otimes_{\varphi_{BA}} B$ and C.

The remaining assertions follow by arguments similar to those used for rings and corings. $\square$

5.4. ***R-birings.*** An R*-biring* $\mathbf{B} = (\underline{B}, \overline{B}, \lambda)$ is an (R,R)-bimodule B which is an R-ring $\underline{B} = (B, m, e)$ and an R-coring $\overline{B} = (B, \Delta, \varepsilon)$ with commutative diagrams

$$\begin{tikzcd}
B \otimes_R B \arrow[r, "B\otimes\varepsilon"] \arrow[d, "m"'] & B \arrow[d, "\varepsilon"] \\
B \arrow[r, "\varepsilon"] & R,
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "e"] \arrow[d, "e"'] & B \arrow[d, "\Delta"] \\
B \arrow[r, "e\otimes B"'] & B \otimes_R B,
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "e"] \arrow[dr, "="'] & B \arrow[d, "\varepsilon"] \\
& R,
\end{tikzcd}
\tag{5.4}$$

and a mixed entwining $\lambda : \overline{B} \otimes_R \underline{B} \to \underline{B} \otimes_R \overline{B}$ inducing commutativity of the diagram

$$\begin{tikzcd}
B \otimes_R B \arrow[r, "m"] \arrow[d, "B\otimes\Delta"'] & B \arrow[r, "\Delta"] & B \otimes_R B \\
B \otimes_R B \otimes_R B \arrow[rr, "\lambda\otimes B"'] & & B \otimes_R B \otimes_R B. \arrow[u, "B\otimes m"']
\end{tikzcd}
\tag{5.5}$$

The first diagrams in (5.4) just mean that $\varepsilon : B \to R$ is a ring morphism or a B-module morphism and $e : R \to B$ is a coring morphism or a B-comodule morphism.

The commutativity of (5.5) can be read as m being left B-colinear with respect to the left B-comodule structure on $B \otimes_R B$ induced by λ or else as left B-linearity of Δ with respect to the left B-module structure on $B \otimes_R B$ induced by λ.

It follows from 5.2 that these conditions imply that $\overline{B} \otimes_R \underline{B}$ is a $\underline{B}$-coring and $\underline{B} \otimes_R \overline{B}$ is a $\overline{B}$-ring.

The conditions required in 5.4 make the functor $B \otimes_R -$ a bimonad on ${}_R\mathbb{M}$ in the sense of [20, Definition 4.1]. In general they do not imply the same property for the functor $- \otimes_R B$ on $\mathbb{M}_R$. The definition of a *bimonoid* in monoidal categories given in [15, Definition 2.3] corresponds to the τ*-bimonad* for a double entwining $\tau : B \otimes_R B \to B \otimes_R B$ defined in [20, 6.2]. To ensure that the tensor product $B \otimes_R B$ is again of this type, τ has to satisfy the Yang-Baxter equation (see [20, 6.8]).

Over commutative rings R, the definition 5.4 is close to the notion of *bialgebras*. Classically their definition is based on the twist map $\mathsf{tw} : B \otimes_R B \to B \otimes_R B$ and the mixed entwining is given by (e.g. [3, 7.1])

$$B \otimes B \xrightarrow{\Delta \otimes B} B \otimes B \otimes B \xrightarrow{B \otimes \mathsf{tw}} B \otimes B \otimes B \xrightarrow{m \otimes B} B \otimes B.$$

In this case the compatibility conditions can be expressed by requiring that m and e are coalgebra morphisms, equivalently, Δ and ε are algebra morphisms (with product and coproduct on $B \otimes_R B$ induced by tw).

Among bialgebras, Hopf algebras are characterised by the fact that the functor $B \otimes_R - : {}_R\mathbb{M} \to {}^B_B\mathbb{M}$ is an equivalence (e.g. [20, Theorem 6.12]).

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