On p-injective rings

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Abstract

A ring R is called *right principally injective (p-injective)* if every R-homomorphism from a principal right ideal to R is left multiplication by an element of R. In a recent paper Nicholson and Yousif showed that a left uniserial ring R is right p-injective if and only if $J(R) = Z(R_R)$. Here we show that the same is true for (two-sided) serial rings by using model theory and the structure of finitely presented modules over serial rings. Another criteria for checking right p-injectivity for serial rings is provided. Moreover we show that a semiperfect right duo right p-injective ring is right continuous.

R is said to be a *completely right p-injective ring* if every factor ring of R is right p-injective. For such rings the lattice of two-sided ideals is distributive. A right duo ring with this property is a direct sum of uniserial rings with nil Jacobson radical provided it has no infinite set of orthogonal idempotents.

AMS Subject classification (1991): Primary 16D50, 16E50, 16L30; Secondary: 16P60

1 Definitions and preliminary results

Throughout this paper R will be an associative ring with unity and all R-modules are unitary. The right (resp. left) annihilator in R of a subset X of a module is denoted by $\mathbf{r}(X)$ (resp. $\mathbf{l}(X)$). The Jacobson radical of R is denoted by $J(R)$, the singular ideals are denoted by $Z(R_R)$ and $Z({}_RR)$ and the socles by $Soc(R_R)$ and $Soc({}_RR)$. For a module M, $E(M)$ and $PE(M)$ denote the injective and pure-injective envelopes of M, respectively. For a submodule $A \subseteq M$, the notation $A \subseteq^{\oplus} M$ will mean that A is a direct summand of M.

A module M_R is called *p-injective* if for every $a \in R$, every R-linear map from aR to M can be extended to an R-linear map from R to M. R is called right *p-injective* if R_R is p-injective. Recall that a module M_R is called *uniserial* if its submodules are linearly ordered by inclusion and *serial* if it is a direct sum of uniserial submodules. A ring R is *right uniserial (serial)* if R_R is uniserial (serial).

We record some well known results on serial and p-injective rings.

1.1 Lemma *[5, 6] Let R be any ring.*

(1) R is right p-injective if and only if $\mathbf{l}(\mathbf{r}(a)) = Ra$ for every $a \in R$.

(2) If R is right p-injective then $J(R) = Z(R_R)$.

(3) If R is left uniserial then R is right p-injective if and only if $J(R) = Z(R_R)$.

(4) If R is right p-injective and $A, B_1, \ldots, B_n$ are two-sided ideals of R then

$$A \cap (B_1 \oplus \ldots \oplus B_n) = (A \cap B_1) \oplus \ldots \oplus (A \cap B_n).$$

1.2 Lemma *[11, p.200, Theorem 3.3] Let R be a serial ring, P a finitely generated projective R-module, and M a finitely generated submodule of P. Then there is a decomposition $P = P_1 \oplus \ldots \oplus P_n$ with indecomposables P_i such that $M = (M \cap P_1) \oplus \ldots \oplus (M \cap P_n)$.*

The next two statements are proved using model theory for modules.

1.3 Lemma *[3] let R be an arbitrary ring and M a finitely presented module over R. Then $PE(M)$ is indecomposable if and only if M has a local endomorphism ring.*

1.4 Lemma *[7] Let R be a serial ring and M a pure-injective indecomposable module over R. Then either M is injective or, for every primitive idempotent $e \in R$ and every nonzero element $m \in Me$, there exists an element $r \in R$ such that $m \in E(M)re$ and $m \notin Mre$.*

1.5 Lemma *[5, Corollary 2.2, Theorem 2.3] Let R be a semiperfect right p-injective ring with $Soc(R_R)$ essential as a right ideal in R. Then $Soc(R_R) = Soc({}_RR)$ is essential as a left ideal and $Z(R_R) = J(R) = Z({}_RR)$.*

Recall that a right R-module M is called *fp-injective* if every R-linear map from a finitely generated submodule of a free R-module F to M can be extended to an R-linear map from F to M. Evidently every fp-injective module is p-injective and the converse is true for some classes of rings including serial rings, see [8]. In the serial ring case we give a short proof to this fact using the above cited Warfield's result.

1.6 Lemma *Every right p-injective module M over a serial ring R is fp-injective.*

Proof. Let N be a finitely generated submodule of a free module P of finite rank and f a homomorphism from N into M. In view of Lemma 1.2 we may assume that N is a finitely generated submodule of an indecomposable projective module eR for some primitive idempotent $e \in R$. Since eR is uniserial, it follows that N is cyclic. Now the existence of the desired extension follows from p-injectivity of M. $\square$

2 Serial p-injective rings

Now we formulate our criteria for serial rings to be right p-injective.

2.1 Theorem *For a serial ring R with a complete set of primitive orthogonal idempotents $\{e_1. \dots, e_n\}$ the following conditions are equivalent:*

(a) *R is right p-injective.*

(b) *R is right fp-injective.*

(c) $J(R) = Z(R_R)$.

(d) *For any pair of indices $i, j \leq n$ and any $r \in R$ with $0 \neq e_ire_j \in J(Re_j)$ there exists $s \in R$ and $k \leq n$, such that $e_jse_k \neq 0$ and $e_ire_jse_k = 0$.*

Proof. The equivalence between (a) and (b) follows from Lemma 1.6 and the implication (b) $\Rightarrow$ (c) follows from Lemma 1.1.

(c) $\Rightarrow$ (d). If $0 \neq e_i r e_j \in J(Re_j)$ then $e_i r e_j \in J(R) = Z(R_R)$, hence $\mathbf{r}(e_i r e_j)$ is essential in R_R and $\mathbf{r}(e_i r e_j) \cap e_j R \neq 0$. It follows that $e_i r e_j s = 0$ for some $0 \neq e_j s \in e_j R$. Since $e_j R$ is uniserial and $e_j s R = e_j s e_1 R + \ldots + e_j s e_n R$ we obtain $e_j s R = e_j s e_k R$ for some k and $e_j s e_k$ is the desired element.

(d) $\Rightarrow$ (a). Suppose that R_R is not p-injective. Then $e_j R$ is not p-injective as a right R-module for some j. Let M be the pure-injective envelope of $e_j R$. Since $e_j R$ has a local (in fact uniserial) endomorphism ring it follows from Lemma 1.3 that M is an indecomposable pure-injective module. Now if M is injective, it will follow that $e_j R$ is fp-injective since it is a pure submodule of M, a contradiction. By Lemma 1.4, applied to the element $e_j \in M e_j$, we can find an element $r \in R$ such that $e_j \in E(M) r e_j$ and $e_j \notin M r e_j$. If $r e_j \notin J(Re_j)$ then $t r e_j = e_j$ for some $t \in R$. Now, $e_j t \in e_j R \subseteq M$ implies $e_j = e_j t \cdot r e_j \in M r e_j$, a contradiction. Hence we may assume $r e_j \in J(Re_j)$. Since $R e_i r e_j = R r e_j$, for some i, it follows $e_i r e_j \in J(Re_j)$ and hence by assumption $e_i r e_j s e_k = 0$ for some k and some $s \in R$.

Since $e_j \in E(M) r e_j$ we obtain $e_j = m r e_j$ for some $m \in E(M)$. Multiplying this equality by $e_j s e_k$ from the right side we obtain $e_j s e_k = m r e_j \cdot e_j s e_k = 0$, a contradiction. $\square$

2.2 Corollary *Let R be a serial right p-injective ring with essential right socle. Then R is left p-injective with essential left socle.*

Proof. From Lemma 1.5 we obtain $Z(R_R) = J(R) = Z({}_R R)$ and the socle of R is essential in ${}_R R$. From the Theorem 2.1 it follows that R is left p-injective. $\square$

2.3 Example *Let F be an arbitrary field and consider the ring R*

$$\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$

Then R is a (two-sided) serial artinian ring which is neither left nor right p-injective.

Proof. We check this for the right side only. We have $e_{12} \in J(Re_2) \cap e_1 R e_2$ and $e_{12} s \neq 0$ for every nonzero element $s \in e_2 R$ which contradicts (d) of Theorem 2.1. $\square$

Next we provide an example of a ring R which is right uniserial right artinian right duo left p-injective ring which is neither right p-injective nor left uniform. Also every non-invertible element of R has an essential left and right annihilator. Recall that a ring R is *right duo* if every right ideal of R is two-sided.

2.4 Example *Let K be a field and $K(x)$ the field of rational functions over K. Let α be an endomorphism of $K(x)$ which sends x to x^2. Clearly the image of α is $K(x^2)$. Let R be a matrix ring of the form*

$$\left\{ \begin{pmatrix} \alpha(a) & b \\ 0 & a \end{pmatrix} : a, b \in K(x) \right\}.$$

Clearly

$$\begin{pmatrix} 0 & K(x) \\ 0 & 0 \end{pmatrix}$$

is the unique non-trivial right ideal of R. If we view $K(x)$ as a vector space over $K(x^2)$ then every proper left ideal of R has the form

$$\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

where V is a subspace of $K(x)$. It is easy to check that for every $a \in J$, the Jacobson radical of R, $\mathbf{r}(a) = \mathbf{l}(a) = J$. Clearly R is right artinian right uniserial right duo and not left uniserial. It follows from Lemma 1.1 that R is left p-injective and not right p-injective.

3 Semiperfect p-injective rings

In this section we show that semiperfect right p-injective right duo rings are right continuous. Recall that a module M_R is called *continuous* if it satisfies the following two conditions: $(C1)$ Every submodule of M is essential in a direct summand, and $(C2)$ If A and B are submodules of M with $A \cong B$ and $B \subseteq^{\oplus} M$ then $A \subseteq^{\oplus} M$.

In [5, Theorem 1.2], it was shown that if R_R is right p-injective then R_R satisfies the $C2$-condition. In particular, if A and B are right ideals of R with $A \subseteq^{\oplus} R_R, B \subseteq^{\oplus} R_R$ and $A \cap B = 0$ then $A \oplus B \subseteq^{\oplus} R_R$. If R is right duo we have the following more general result which is of independent interest.

3.1 Theorem *Let R be right p-injective right duo ring. If A and B are right ideals of R with $A \subseteq^{\oplus} R_R$ and $B \subseteq^{\oplus} R_R$ then $(A \cap B) \subseteq^{\oplus} R_R$ and $(A + B) \subseteq^{\oplus} R_R$.*

Proof. Write $R = A \oplus A_1 = B \oplus B_1$ for some right ideals A_1 and B_1 of R. By Lemma 1.1, $B = B \cap (A \oplus A_1) = (B \cap A) \oplus (B \cap A_1)$. Hence $R = (B \cap A) \oplus (B \cap A_1) \oplus B_1$ and so $(A \cap B) \subseteq^{\oplus} R_R$. Also

$$A + B = A + (B \cap A) \oplus (B \cap A_1) = (A + (B \cap A)) \oplus (B \cap A_1) = A \oplus (B \cap A_1).$$

Since both A and $(B \cap A_1)$ are summands of R_R, it follows from the remark preceding the Theorem that $A \oplus (B \cap A_1)$ is a summand of R_R and so $A + B$ is also a summand of R_R. □

3.2 Lemma *Let R be a local right p-injective ring. Then for any non-zero (two-sided) ideals I and J of R, $I \cap J \neq 0$.*

Proof. Suppose that $I \cap J = 0$ and let $0 \neq u \in I$, $0 \neq v \in J$. Define the map

$$\varphi : (u + v)R \to R, \ (u + v)r \mapsto ur.$$

Clearly φ is a well defined R-homomorphism. By right p-injectivity, φ is given by left multiplication by an element $t \in R$. Hence $t(u + v) = u$, and so $(1 - t)u = tv = 0$. Since R is a local ring it follows that $u = 0$ or $v = 0$, a contradiction. □

3.3 Corollary *Suppose R is a local right p-injective right duo ring. Then R is right uniform.*

3.4 Remark Note that without the condition *right duo* the above result is not true. The ring R given in Example 2.4 is a local left p-injective ring which is not left uniform.

3.5 Theorem *Suppose R is a semiperfect right duo right p-injective ring. Then R is right continuous.*

Proof. By Corollary 3.3, clearly R is a direct sum of local right uniform rings R_i. By [5, Theorem 1.2], any right p-injective ring satisfies the $C2$-condition. We only need to show that R_R satisfies the $C1$-condition. Let A be a non-zero right ideal of R and write $R = R_1 \oplus \ldots \oplus R_n$. By Lemma 1.1, without loss of generality we may write $A = (A \cap R_1) \oplus \ldots \oplus (A \cap R_k)$, for some $k \leq n$ with $A \cap R_i \neq 0, 1 \leq i \leq k$. Since each $A \cap R_i$ is essential as a right ideal in $R_i, 1 \leq i \leq k$, it follows that A_R is essential in $R_1 \oplus \ldots \oplus R_k \subseteq^{\oplus} R_R$. □

3.6 Remark Note that the ring R given in Example 2.4 is a left p-injective right artinian ring which is not left finite dimensional. Hence R can not be left continuous.

4 Completely p-injective rings

A ring R is called *completely right p-injective(right cp-injective)* if every ring homomorphic image of R is right p-injective. R is called *cp-injective* if it is both left and right *cp*-injective. In this section, for right duo rings, we give a characterization for serial rings with nil Jacobson radical in terms of *cp*-injectivity. Recall that a module M is said to be *distributive* if its lattice of submodules is distributive: for all $A, B, C \subset M$, $A \cap (B+C) = A \cap B + A \cap C$.

4.1 Theorem *Let R be a right cp-injective ring. Then the lattice of two-sided ideals of R is distributive.*

Proof. Suppose the lattice of two-sided ideals of R is a non-distributive (modular) lattice. It follows from [2, Theorem 2] that it contains a minimal non-distributive modular sublattice consisting of five elements. Hence we can find three noncomparable two-sided ideals I, J and K in R such that $I \cap J = I \cap K = J \cap K$ and $I + J = I + K = J + K$. Then factorizing by the common intersection we may suppose that all these sums are direct and all these intersections are zero. Now by Lemma 1.1 it follows that $0 \neq I = I \cap (J \oplus K) = (I \cap J) \oplus (I \cap K) = 0$, a contradiction. □

4.2 Corollary *Every right duo right cp-injective ring is right and left distributive.*

Proof. The right distributivity follows from the above Theorem and we can apply the following result from [9, Corollary 2.10]: every right distributive right p-injective ring is left distributive. □

Recall that a ring R is *strongly regular* if for every $a \in R$ there exists $b \in R$ such that $a = ba^2$.

4.3 Lemma *For a ring R the following are equivalent:*

(a) R is strongly regular;

(b) R is right p-injective with no non-zero nilpotent elements;

(c) R is a semiprime right p-injective right duo ring.

Proof. $(a) \Rightarrow (b), (c)$ is standard.

$(c) \Rightarrow (a)$ We adopt the argument given in Example 6 of [5]. Let $a \in R$ and set $T = aR \cap \mathbf{r}(a)$. Then clearly T is a two-sided ideal of R with $T^2 = 0$. Since R is semiprime, $T = 0$ and hence $\mathbf{r}(a^2) = \mathbf{r}(a)$. By Lemma 1.1 we get $Ra = Ra^2$ and hence R is (strongly) regular.

(b) $\Rightarrow$ *(a)* Note that in rings without non-zero nilpotent elements for every $a \in R$, $\mathbf{r}(a) = \mathbf{l}(a)$. Now the same argument as before applies. $\Box$

4.4 Remark More results of the type given in Lemma 4.3 may be found in some of Yue Chi Ming's work on p-injectivity (e.g., [14]).

A ring R is *π-regular* if every descending chain of the form $aR \supseteq a^2R \supseteq \ldots$ becomes stationary.

4.5 Lemma *Let R be right duo and right cp-injective. Then R is π-regular.*

Proof. Let $a \in R$ and consider the following ascending chain of right annihilators $\mathbf{r}(a) \subseteq \mathbf{r}(a^2) \subseteq \ldots$. Let $I = \bigcup_{i=1}^{\infty} \mathbf{r}(a^i)$ and consider the ring $\bar{R} = R/I$. Clearly $\mathbf{r}_{\bar{R}}(\bar{a}) = \bar{0}$ and hence it follows from Lemma 1.1 that $\bar{R}\bar{a} = \bar{R}$. So $1 - sa \in \mathbf{r}(a^m)$ for some $s \in R$ and $m > 0$. Since R is right duo there exists $t \in R$ such that $sa = at$ and hence $a^m = a^{m+1}t$ from which we infer that R is π-regular. $\Box$

4.6 Theorem *For a right duo ring R the following conditions are equivalent:*

(a) R is right cp-injective with no infinite set of orthogonal idempotents.

(b) R is cp-injective with no infinite set of orthogonal idempotents.

(c) R is a finite direct sum of (two-sided) uniserial rings with nil Jacobson radical.

Proof. (a) $\Rightarrow$ (c) By Lemma 4.5, R is π-regular and hence $J(R)$ is a nil ideal and so idempotents can be lifted modulo $J(R)$. By assumption and Lemma 4.3, it follows that $R/J(R)$ is semisimple artinian and hence R is semiperfect. Hence $R = R_1 \oplus \ldots \oplus R_n$ where each R_i is a local ring which is left and right distributive by Corollary 4.2. Since local right distributive rings are right uniserial we are done.

(c) $\Rightarrow$ (b) we may assume that R is uniserial with nil radical J. Let I be any (two-sided) ideal of R and consider the ring $\bar{R} = R/I$. Clearly, every element of $J(\bar{R})$ has a nonzero left and right annihilator. Hence by [6, Lemma 1], $\bar{R}$ is right and left p-injective.

(b) $\Rightarrow$ (a) is trivial. $\Box$

Notice that any von Neumann regular ring which is not right noetherian is *cp*-injective with an infinite set of orthogonal idempotents.

Acknowledgement: This paper was written while the first and third authors were visiting the Mathematical Institute of the University of Düsseldorf. Both authors would like to thank the Institute for the warm hospitality. The first author was supported by Heinrich-Hertz-Stiftung des Ministerium für Wissenschaft und Forschung des Landes Nordrhein-Westfalen. The third author was supported by a research grant from the Ohio State University.

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