Idempotent monads and $\star$-functors

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Dedicated to the memory of Adalberto Orsatti

Abstract

For an associative ring R, let P be an R-module with $S = \mathrm{End}_R(P)$. C. Menini and A. Orsatti posed the question of when the related functor $\mathrm{Hom}_R(P, -)$ (with left adjoint $P \otimes_S -$) induces an equivalence between a subcategory of ${}_R\mathbb{M}$ closed under factor modules and a subcategory of ${}_S\mathbb{M}$ closed under submodules. They observed that this is precisely the case if the unit of the adjunction is an epimorphism and the counit is a monomorphism. A module P inducing these properties is called a $\star$-module.

The purpose of this paper is to consider the corresponding question for a functor $G : \mathbb{B} \to \mathbb{A}$ between arbitrary categories. We call G a *$\star$-functor* if it has a left adjoint $F : \mathbb{A} \to \mathbb{B}$ such that the unit of the adjunction is an *extremal epimorphism* and the counit is an *extremal monomorphism*. In this case (F, G) is an idempotent pair of functors and induces an equivalence between the category $\mathbb{A}_{GF}$ of modules for the monad GF and the category $\mathbb{B}^{FG}$ of comodules for the comonad FG. Moreover, $\mathbb{B}^{FG} = \mathrm{Fix}(FG)$ is closed under factor objects in $\mathbb{B}$, $\mathbb{A}_{GF} = \mathrm{Fix}(GF)$ is closed under subobjects in $\mathbb{A}$.

Key Words: idempotent monads and comonads, $\star$-modules, equivalence of categories, tilting modules, extremal monomorphisms.
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Contents

1 Introduction

Let R and S be associative rings and ${}_RP_S$ an (R, S)-bimodule. In [16], C. Menini and A. Orsatti asked under which conditions on P, the functors $P \otimes_S -$ and $\mathrm{Hom}_R(P, -)$ induce an equivalence between certain subcategories of ${}_R\mathbb{M}$ closed under factor modules (i.e. $\mathrm{Gen}(P)$) and subcategories of ${}_S\mathbb{M}$ closed under submodules (i.e. $\mathrm{Cogen}(\mathrm{Hom}_R(P, Q))$ for some cogenerator Q in ${}_R\mathbb{M}$). Such modules P are called *$\star$-modules* and it is well-known that they are closely related to tilting modules (e.g., [8], [17]).

Because of the effectiveness of these notions in representation theory of finite dimensional algebras (see Assem [2]), various attempts have been made to extend them to more general situations. This was done mostly in categories which do permit some technical tools needed (e.g. additivity, tensor product).

The purpose of this article is to filter out the categorical essence of the theory and to formulate the interesting parts for arbitrary categories. For this we consider a pair (F, G) of adjoint functors between categories $\mathbb{A}$ and $\mathbb{B}$. The crucial step is the observation that these induce functors between the category $\mathbb{B}^{FG}$ of comodules for the comonad FG on $\mathbb{B}$ and the category $\mathbb{A}_{GF}$ of modules for the monad GF on $\mathbb{A}$ (see 3.1). When the comonad FG (equivalently the monad GF) is idempotent, $\mathbb{A}^{FG}$ may be considered as a coreflective subcategory of $\mathbb{A}$ and $\mathbb{B}_{GF}$ becomes a reflective subcategory of $\mathbb{B}$ and these categories are equivalent. To improve the setting one may additionally require $\mathbb{B}^{FG}$ to be closed under factor objects and $\mathbb{A}_{GF}$ to be closed under subobjects. This is achieved by stipulating that the unit of the adjunction is an *extremal epimorphism* in $\mathbb{A}$ and its counit is an *extremal monomorphism* in $\mathbb{B}$. In this case we say that G is a $\star$-functor or that (F, G) is a pair of *$\star$-functors*. Note that no additional structural conditions on the categories are employed.

By definition, an (R, S)-bimodule P is a $\star$-module provided the functor $\mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}$ is a $\star$-functor and our results apply immediately to this situation.

A $\star$-module P is a *tilting module* if (and only if) P is a subgenerator in ${}_R\mathbb{M}$. To transfer this property to a $\star$-functor G, one has to require that every object A in $\mathbb{A}$ permits a monomorphism $A \to G(B)$ for some $B \in \mathbb{B}$. We will not go into this question here.

Central to our investigation are the *idempotent monads* (*comonads*) which have appeared in various places in the literature, e.g., Maranda [15], Applegate and Tierney [1], Isbell [11], Lambek and Rattray [13, 14], and Deleanu, Frei and Hilton [10].

2 Preliminaries

For convenience we recall the basic structures from category theory which will be needed in the sequel.

2.1. Monads. A *monad* on a category $\mathbb{A}$ is a triple $\mathbf{T} = (T, \mu, \eta)$ where $T : \mathbb{A} \to \mathbb{A}$ is an endofunctor and $\mu : TT \to T$, $\eta : \mathrm{Id}_{\mathbb{A}} \to T$ are natural transformations inducing commutative diagrams

$$\begin{array}{ccc} TTT & \xrightarrow{T\mu} & TT \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ TT & \xrightarrow{\mu} & T, \end{array} \qquad \begin{array}{ccccc} T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta T} & T \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle =} & \\ & & T & & \end{array} .$$

2.2. Modules for monads. Given a monad $\mathbf{T} = (T, \mu, \eta)$ on the category $\mathbb{A}$, an object $A \in \mathbb{A}$ with a morphism $\rho_A : T(A) \to A$ is called a $\mathbf{T}$-*module* (or $\mathbf{T}$-*algebra*) if $\rho_A \circ \eta_A = \mathrm{Id}_A$ and ρ_A induces commutativity of the diagram

$$\begin{array}{ccc} TT(A) & \xrightarrow{T(\rho_A)} & T(A) \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \rho_A} \\ T(A) & \xrightarrow{\rho_A} & A. \end{array}$$

A *morphism* between $\mathbf{T}$-modules (A, ρ_A) and $(A', \rho_{A'})$ is an $f : A \to A'$ in $\mathbb{A}$ satisfying $f \circ \rho_A = \rho_{A'} \circ T(f)$. We denote the set of these morphisms by $\mathrm{Mor}_{\mathbf{T}}(A, A')$ and the category of $\mathbf{T}$-modules by $\mathbb{A}_{\mathbf{T}}$.

2.3. Comonads. A *comonad* on a category $\mathbb{A}$ is a triple $\mathbf{S} = (S, \delta, \varepsilon)$ where $S : \mathbb{A} \to \mathbb{A}$ is an endofunctor and $\delta : S \to SS$, $\varepsilon : S \to \mathrm{Id}_{\mathbb{A}}$ are natural transformations inducing commutative diagrams

$$\begin{CD} S @>{\delta}>> SS \\ @V{\delta}VV @VV{S\delta}V \\ SS @>{\delta S}>> SSS, \end{CD} \qquad \begin{array}{ccccc} & & S & & \\ & \swarrow^{=} & \downarrow{\scriptstyle \delta} & \searrow^{=} & \\ S & \xleftarrow[S\varepsilon]{} & SS & \xrightarrow[\varepsilon S]{} & S. \end{array}$$

2.4. Comodules for comonads. Given a comonad $\mathbf{S} = (S, \delta, \varepsilon)$ on the category $\mathbb{A}$, an object $A \in \mathbb{A}$ with a morphism $\rho^A : A \to S(A)$ is an $\mathbf{S}$-*comodule* if $\varepsilon_A \circ \rho^A = \mathrm{Id}_A$ and ρ_A induces commutativity of the diagram

$$\begin{CD} A @>{\rho^A}>> S(A) \\ @V{\rho^A}VV @VV{\delta_A}V \\ S(A) @>{S(\rho^A)}>> SS(A). \end{CD}$$

A *morphism* between $\mathbf{S}$-comodules (A, ρ^A) and $(A', \rho^{A'})$ is an $f : A \to A'$ in $\mathbb{A}$ satisfying $\rho^{A'} \circ f = S(f) \circ \rho^A$. We denote the set of these morphisms by $\mathrm{Mor}^{\mathbf{S}}(A, A')$ and the category of $\mathbf{S}$-comodules by $\mathbb{A}^{\mathbf{S}}$.

2.5. Adjoint functors. Let $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ be (covariant) functors between any categories $\mathbb{A}$, $\mathbb{B}$. The pair (F, G) is called *adjoint* (or an *adjunction*) and F (resply. G) is called a *left* (resply. *right*) *adjoint* to G (resply. F) if the two equivalent conditions hold:

(a) there is an isomorphism, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\varphi_{A,B} : \mathrm{Mor}_{\mathbb{B}}(F(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, G(B));$$

(b) there are natural transformations $\eta : \mathrm{Id}_{\mathbb{A}} \to GF$ (called the *unit* of the adjunction) and $\varepsilon : FG \to \mathrm{Id}_{\mathbb{B}}$ (called the *counit* of the adjunction) with commutative diagrams (called the *triangular identities*)

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow_{=} & \downarrow{\scriptstyle \varepsilon F} \\ & & F \end{array}, \qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow_{=} & \downarrow{\scriptstyle G\varepsilon} \\ & & G. \end{array}$$

With unit and counit the mappings are given by

$$\begin{array}{llll} \varphi_{A,B} : & F(A) \xrightarrow{f} B & \longmapsto & A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(f)} G(B), \\ \varphi^{-1}_{A,B} : & A \xrightarrow{g} G(B) & \longmapsto & F(A) \xrightarrow{F(g)} FG(B) \xrightarrow{\varepsilon_B} B. \end{array}$$

2.6. Properties of adjoint functors. *Let* (F, G) *be as in* 2.5. *Then*

(1) (i) *G is faithful if and only if ε_B is an epimorphism for each $B \in \mathbb{B}$.*
(ii) *G is full if and only if ε_B is a coretraction (split monic) for each $B \in \mathbb{B}$.*
(iii) *G is full and faithful if and only if ε is an isomorphism.*

(2) (i) *F is faithful if and only if η_A is a monomorphism for each $A \in \mathbb{A}$.*
(ii) *F is full if and only if η_A is a retraction (split epic) for each $A \in \mathbb{A}$.*
(iii) *F is full and faithful if and only if η is an isomorphism.*

2.7. Adjoint functors and (co)monads. *Let* (F, G) *be as in* 2.5. *Then*

(1) (i) $\mathbf{T} = (GF, G\varepsilon F, \eta)$ *is a monad on* $\mathbb{A}$*;*

(ii) *there is a functor* $\overline{G} : \mathbb{B} \to \mathbb{A}_{GF},\ B \mapsto (G(B), G\varepsilon_B)$.

(2) (i) $\mathbf{S} = (FG, F\eta G, \varepsilon)$ *is a comonad on* $\mathbb{B}$*;*

(ii) *there is a functor* $\overline{F} : \mathbb{A} \to \mathbb{B}^{FG},\ A \mapsto (F(A), F\eta_A)$.

Proof. (1.i), (2.i) are well-known properties of adjoint functors.

(1.ii) describes the *comparison functor*. To show its properties recall that naturality of ε yields the commutative diagram (e.g. [3, Section 3])

$$\begin{array}{ccc} FGFG & \xrightarrow{\varepsilon FG} & FG \\ {\scriptstyle FG\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ FG & \xrightarrow{\varepsilon} & \mathrm{Id}. \end{array}$$

Action of G from the left and application to B yield the commutative diagram

$$\begin{array}{ccc} GFGFG(B) & \xrightarrow{G\varepsilon FG_B} & GFG(B) \\ {\scriptstyle GFG\varepsilon_B}\downarrow & & \downarrow{\scriptstyle G\varepsilon_B} \\ GFG(B) & \xrightarrow{G\varepsilon_B} & G(B). \end{array}$$

This proves the associativity condition for the GF-module $G(B)$. Unitality follows from the triangular identities (2.5). Again by naturality of ε, for any $f \in \mathbb{B}$, $G(f)$ is a GF-module morphism.

The proof of (2.ii) is dual to that of (1.ii). □

2.8. Free functor for a monad. For any monad $\mathbf{T} = (T, \mu, \eta)$ on $\mathbb{A}$ and object $A \in \mathbb{A}$, $(T(A), \mu_A)$ is a $\mathbf{T}$-module, called the *free* $\mathbf{T}$*-module* on A. This yields the *free functor*

$$\phi_{\mathbf{T}} : \mathbb{A} \to \mathbb{A}_{\mathbf{T}}, \quad A \mapsto (T(A), \mu_A),$$

which is left adjoint to the forgetful functor $U_{\mathbf{T}} : \mathbb{A}_{\mathbf{T}} \to \mathbb{A}$ by the isomorphism, for $A \in \mathbb{A}$ and $M \in \mathbb{A}_{\mathbf{T}}$,

$$\mathrm{Mor}_{\mathbf{T}}(T(A), M) \to \mathrm{Mor}_{\mathbb{A}}(A, U_{\mathbf{T}}(M)), \quad f \mapsto f \circ \eta_A.$$

Notice that $U_{\mathbf{T}}\phi_{\mathbf{T}} = T$ and $U_{\mathbf{T}}(M) = M$ on objects $M \in \mathbb{A}_{\mathbf{T}}$. The unit of this adjunction is $\eta : \mathrm{Id}_{\mathbb{A}} \to T = U_{\mathbf{T}}\phi_{\mathbf{T}}$, and for the counit $\tilde{\varepsilon} : \phi_{\mathbf{T}}U_{\mathbf{T}} \to \mathrm{Id}_{\mathbb{A}_{\mathbf{T}}}$ we have $\mu = U_{\mathbf{T}}\tilde{\varepsilon}\phi_{\mathbf{T}}$ (e.g. [3, Theorem 3.2.1], [4, Proposition 4.2.2]).

2.9. Free functor for a comonad. For any comonad $\mathbf{S} = (S, \delta, \varepsilon)$ on $\mathbb{A}$ and object $A \in \mathbb{A}$, $(S(A), \delta_A)$ is an S-comodule, called the *free* $\mathbf{S}$*-comodule* on A. This yields the *free functor*

$$\phi^{\mathbf{S}} : \mathbb{A} \to \mathbb{A}^{\mathbf{S}}, \quad A \mapsto (S(A), \delta_A),$$

which is right adjoint to the forgetful functor $U^{\mathbf{S}} : \mathbb{A}^{\mathbf{S}} \to \mathbb{A}$ by the isomorphism, for $A \in \mathbb{A}$ and $M \in \mathbb{A}^{\mathbf{S}}$,

$$\mathrm{Mor}^{\mathbf{S}}(M, S(A)) \to \mathrm{Mor}_{\mathbb{A}}(U^{\mathbf{S}}(M), A), \quad g \mapsto \varepsilon_A \circ g.$$

Notice that $U^{\mathbf{S}}\phi^{\mathbf{S}} = S$ and $U^{\mathbf{S}}(M) = M$ on objects in $\mathbb{A}^{\mathbf{S}}$. The counit of this adjunction is $\varepsilon : U^{\mathbf{S}}\phi^{\mathbf{S}} = S \to \mathrm{Id}_{\mathbb{A}}$, and for the unit $\tilde{\eta} : \mathrm{Id}_{\mathbb{A}^{\mathbf{S}}} \to \phi^{\mathbf{S}}U^{\mathbf{S}}$ we have $\delta = U^{\mathbf{S}}\tilde{\eta}\phi^{\mathbf{S}}$.

The following observation is the key to our investigation.

2.10. Idempotent monads. *For a monad* $\mathbf{T} = (T, \mu, \eta)$ *on a category* $\mathbb{A}$*, the following are equivalent:*

(a) *the forgetful functor* $U_{\mathbf{T}} : \mathbb{A}_{\mathbf{T}} \to \mathbb{A}$ *is full (and faithful);*

(b) *the counit* $\tilde{\varepsilon} : \phi_{\mathbf{T}} U_{\mathbf{T}} \to \mathrm{Id}_{\mathbb{A}_{\mathbf{T}}}$ *is an isomorphism;*

(c) *the product* $\mu : TT \to T$ *is an isomorphism;*

(d) *for every* $\mathbf{T}$*-module* (A, ρ_A)*,* $\rho_A : T(A) \to A$ *is an isomorphism in* $\mathbb{A}$*;*

(e) $T\eta$ *(or* ηT*) is an isomorphism;*

(f) $T\eta = \eta T$*;*

(g) $T\mu = \mu T$*.*

Proof. A proof of the equivalences from (a) to (d) can be found in [4, Proposition 4.2.3]. The remaining equivalences are shown in [15, Proposition]. Their proof is based on the diagram

$$\begin{array}{ccc} TT & \xrightarrow{\mu} & T \\ {\scriptstyle TT\eta}\downarrow & & \downarrow{\scriptstyle T\eta} \\ TTT & \xrightarrow{\mu T} & TT \end{array}$$

which is commutative by naturality of μ.

Now, for example, if $T\mu = \mu T$, then $\mu T \circ TT\eta = \mu T \circ T\eta T = TT$ showing that μ (and $T\eta$) is an isomorphism, that is, (g)⇒(c). □

We also need the dual version of this theorem which is shown in Applegate-Tierney [1, Section 6]:

2.11. Idempotent comonads. *For a comonad* $\mathbf{S} = (S, \delta, \varepsilon)$ *on a category* $\mathbb{A}$*, the following are equivalent:*

(a) *the forgetful functor* $U^{\mathbf{S}} : \mathbb{A}^{\mathbf{S}} \to \mathbb{A}$ *is full (and faithful);*

(b) *the unit* $\tilde{\eta} : \mathrm{Id}_{\mathbb{A}^{\mathbf{S}}} \to \phi^{\mathbf{S}} U^{\mathbf{S}}$ *is an isomorphism;*

(c) *the coproduct* $\delta : S \to SS$ *is an isomorphism;*

(d) *for any* $\mathbf{S}$*-comodule* (A, ρ^A)*,* $\rho^A : A \to S(A)$ *is an isomorphism in* $\mathbb{A}$*;*

(e) $S\varepsilon$ *(or* εS*) is an isomorphism;*

(f) $S\varepsilon = \varepsilon S$*;*

(g) $S\delta = \delta S$*.*

3 Idempotent pairs of functors

In this section, we consider an adjoint pair of functors $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ with unit $\eta : \mathrm{Id}_{\mathbb{A}} \to GF$ and counit $\varepsilon : FG \to \mathrm{Id}_{\mathbb{B}}$.

3.1. Related functors. *Let* (F, G) *be as in* 2.5.

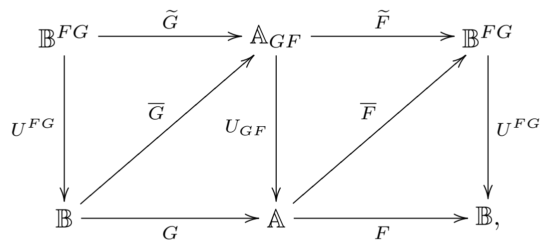
(1) *For the monad* GF *on* $\mathbb{A}$*, composing* U_{GF} *with* $\overline{F}$ *(from 2.7) yields a functor*

$$\widetilde{F} = \overline{F} \circ U_{GF} : \mathbb{A}_{GF} \to \mathbb{B}^{FG}.$$

(2) *For the comonad* FG *on* $\mathbb{B}$*, composing* U^{FG} *with* $\overline{G}$ *(from 2.7) yields a functor*

$$\widetilde{G} = \overline{G} \circ U^{FG} : \mathbb{B}^{FG} \to \mathbb{A}_{GF}.$$

(3) *These functors lead to the commutative diagram*



In general $(\widetilde{F}, \widetilde{G})$ need not be an adjoint pair of functors. As a first observation in this context we state:

3.2. Proposition. *Consider an adjoint pair* (F, G) (*as in 2.5*).

(1) *For* (A, ρ_A) *in* $\mathbb{A}_{GF}$, *the following are equivalent:*
  - (a) $\eta_A : A \to GF(A)$ *is a* GF*-module morphism;*
  - (b) $\eta_A : A \to GF(A)$ *is an epimorphism (isomorphism);*
  - (c) $\rho_A : GF(A) \to A$ *is an isomorphism.*

(2) *For* (B, ρ^B) *in* $\mathbb{B}^{FG}$, *the following are equivalent:*
  - (a) $\varepsilon_B : FG(B) \to B$ *is an* FG*-comodule morphism;*
  - (b) $\varepsilon_B : FG(B) \to B$ *is a monomorphism (isomorphism);*
  - (c) $\rho^B : B \to FG(B)$ *is an isomorphism.*

Proof. (1) (b)⇔(c) for isomorphisms is obvious by unitality of GF-modules.

(a)⇒(b) For (A, ρ) in $\mathbb{A}_{GF}$, the condition in (a) requires commutativity of the diagram

$$\begin{array}{ccc} GF(A) & \xrightarrow{GF\eta_A} & GFGF(A) \\ {\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle G\varepsilon F(A)} \\ A & \xrightarrow{\eta_A} & GF(A). \end{array}$$

By the triangular identities (see 2.5), $G\varepsilon F \circ GF\eta \simeq \mathrm{Id}_{GF}$ and hence $\eta_A \circ \rho_A \simeq \mathrm{Id}_{G(A)}$. Since $\rho_A \circ \eta_A \simeq \mathrm{Id}_A$ (by unitality) it follows that η_A (and ρ_A) is an isomorphism.

(b)⇒(a) Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GF(A) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta GF_A} \\ GF(A) & \xrightarrow{GF(\eta_A)} & GFGF(A) \\ {\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle G\varepsilon F(A)} \\ A & \xrightarrow{\eta_A} & GF(A), \end{array}$$

in which the upper square is commutative by naturality of η and the outer rectangle is commutative since the composites of the vertical maps yield the identity. If η_A is an epimorphism, the lower square is also commutative showing that η_A is a GF-module morphism.

(2) These assertions are proved in a similar way. □

3.3. $(\widetilde{F}, \widetilde{G})$ **as an adjoint pair.** *With the notation in 3.1, the following are equivalent:*

(a) *by restriction and corestriction, φ (see 2.5) induces an isomorphism*

$$\tilde{\varphi} : \mathrm{Mor}^{FG}(\widetilde{F}(A), B) \to \mathrm{Mor}_{GF}(A, \widetilde{G}(B)) \ \ \textit{for } A \in \mathbb{A}_{GF},\ B \in \mathbb{B}^{FG},$$

(hence $(\widetilde{F}, \widetilde{G})$ is an adjoint pair of functors);

(b) *$\eta G : G \to GFG$ is an isomorphism;*

(c) *$G\varepsilon F : GFGF \to GF$ is an isomorphism.*

Proof. (a)⇒(b) η_A is the image of $\mathrm{Id} : \widetilde{F}(A) \to \widetilde{F}(A)$ under $\tilde{\varphi}$ and hence a GF-module morphism. By 3.2, this implies that η_A is an isomorphism for all GF-modules A. Since $G(B)$ is a GF-module for any $B \in \mathbb{B}$, we have $\eta_{G(B)} : G(B) \to GFG(B)$ an isomorphism, that is, $\eta G : G \to GFG$ is an isomorphism.

(b)⇒(c) By the triangular identities, (b) implies that $G\varepsilon$ and $G\varepsilon F$ are also isomorphisms.

(c)⇒(a) Unitality and the triangular identities yield the equalities

$$GF(\rho_A) \circ GF\eta_A = G\varepsilon F_A \circ GF\eta_A = G\varepsilon F_A \circ \eta GF_A = \mathrm{Id}_{GF_A}.$$

Given (c), we conclude from these that $GF\eta_A = \eta GF_A$ is an isomorphism and thus $GF(\rho_A) = G\varepsilon F_A$. With this information, the test diagram for η_A being a GF-module morphisms (see proof of 3.2(1)) becomes

$$\begin{array}{ccc} GF(A) & \xrightarrow{\eta GF_A} & GFGF(A) \\ {\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle GF(\rho_A)} \\ A & \xrightarrow{\eta_A} & GF(A), \end{array}$$

and this is commutative by naturality of η. Thus we get an isomorphism

$$\begin{array}{rcl} \tilde{\varphi} : \mathrm{Mor}^{FG}(\widetilde{F}(A), B) & \longrightarrow & \mathrm{Mor}_{GF}(A, \widetilde{G}(B)), \\ \widetilde{F}(A) \xrightarrow{f} B & \longmapsto & A \xrightarrow{\eta_A} \widetilde{G}\widetilde{F}(A) \xrightarrow{\widetilde{G}(f)} \widetilde{G}(B), \end{array}$$

showing that $(\widetilde{F}, \widetilde{G})$ is an adjoint pair of functors. □

Adjoint pairs with the properties addressed in 3.3 are well-known in category theory. Combined with 2.10 and by standard arguments we obtain the following list of characterisations for them.

3.4. Idempotent pair of adjoints. *For the adjoint pair of functors (F, G) (as in 2.5), the following are equivalent.*

(a) *The forgetful functor $U_{GF} : \mathbb{A}_{GF} \to \mathbb{A}$ is full and faithful;*

(b) *the counit $\bar{\varepsilon} : \phi_{GF} U_{GF} \to \mathrm{Id}_{\mathbb{A}_{GF}}$ is an isomorphism;*

(c) *the product $G\varepsilon F : GFGF \to GF$ is an isomorphism;*

(d) *$\varepsilon F : FGF \to F$ is an isomorphism;*

(e) *the forgetful functor $U^{FG} : \mathbb{B}^{FG} \to \mathbb{B}$ is full and faithful;*

(f) *the unit $\bar{\eta} : \mathrm{Id}_{\mathbb{B}^{FG}} \to \phi^{FG} U^{FG}$ is an isomorphism;*

(g) *the coproduct $F\eta G : FG \to FGFG$ is an isomorphism;*

(h) *$\eta G : G \to GFG$ is an isomorphism.*

If these properties hold then (F, G) is called an *idempotent pair* of adjoints.

3.5. Remarks. Most of these properties have been considered somewhere in the literature. Perhaps the first hint of idempotent pairs is given in Maranda [15, Proposition] under the name *idempotent constructions* (1966). Isbell discussed their role in [11] calling them *Galois connections* (1971). In Lambek and Rattray [13] they are investigated in the context of localisation and duality (1975). In the same year they were studied in Deleanu, Frei and Hilton [10, Section 2] where it is shown that their Kleisli categories are isomorphic to the category of fractions (of invertible morphisms). Extending these ideas, *idempotent approximations* to any monad are the topic of Casacuberta and Frei [5].

For the adjoint functor pair (F, G) we use the notation (e.g. [13])

$$\begin{aligned} \mathrm{Fix}(GF,\eta) &= \{A \in \mathbb{A} \mid \eta_A : A \to GF(A) \text{ is an isomorphism}\}, \\ \mathrm{Fix}(FG,\varepsilon) &= \{B \in \mathbb{B} \mid \varepsilon_B : FG(B) \to B \text{ is an isomorphism}\}. \end{aligned}$$

We denote the (isomorphic) closure of the image of GF in $\mathbb{A}$ and FG in $\mathbb{B}$ by $GF(\mathbb{A})$ and $FG(\mathbb{B})$, respectively.

3.6. Idempotent pairs and equivalences. *Let (F, G) be an idempotent adjoint pair of functors. Then:*

(i) $\mathbb{A}_{GF} \simeq \mathrm{Fix}(GF,\eta) = GF(\mathbb{A})$ *is a reflective subcategory* $\mathbb{A}$ *with reflector* GF*.*

(ii) $\mathbb{B}^{FG} \simeq \mathrm{Fix}(FG,\varepsilon) = FG(\mathbb{B})$ *is a coreflective subcategory of* $\mathbb{B}$ *with coreflector* FG*.*

(iii) *The (restrictions of the) functors* F*,* G *induce an equivalence*

$$F : GF(\mathbb{A}) \to FG(\mathbb{B}), \qquad G : FG(\mathbb{B}) \to GF(\mathbb{A}).$$

(iv) *The Kleisli category of* GF *is isomorphic to the category of fractions* $\mathbb{A}[S^{-1}]$ *where* S *is the family of morphisms of* $\mathbb{A}$ *rendered invertible by* GF *(or* F*).*

Proof. (i) and (ii) follow from 3.4 (g) and (b), respectively.

(iii) The composition $\widetilde{F}\widetilde{G}$ is isomorphic to the identity on $\mathbb{B}^{FG}$ and $\widetilde{G}\widetilde{F}$ is isomorphic to the identity on $\mathbb{A}_{GF}$.

(iv) This is shown in [10, Theorem 2.6]. □

Of course, if (F, G) induces an equivalence between $\mathbb{A}$ and $\mathbb{B}$, then it is an idempotent pair. More generally, we obtain from 2.6 that (F, G) is idempotent provided the functor F or the functor G is full and faithful.

To consider weaker conditions on the unit and counit, recall that an epimorphism e in any category $\mathbb{A}$ is called *extremal* or a *cover* if whenever $e = m \circ f$ for a monomorphism m then m is an isomorphism. Such epimorphisms are isomorphisms if and only if they are monomorph.

3.7. η_A epimorph. *Let (F, G) be an adjoint pair of functors (as in 2.5).*

(1) *If* $\eta_A : A \to GF(A)$ *is epimorph for any* $A \in \mathbb{A}$*, then*

   (i) (F, G) *is idempotent;*

   (ii) GF *preserves epimorphisms;*

   (iii) *for any coproduct* $\coprod_{i\in I} A_i$ *in* $\mathbb{A}$*, the canonical morphism*

$$\psi : \coprod_I GF(A_i) \to GF(\coprod_I A_i)$$

   is an epimorphism.

(2) *If* $\eta_A : A \to GF(A)$ *is an extremal epimorphism for any* $A \in \mathbb{A}$*, then* $\mathrm{Fix}(GF,\eta)$ *is closed under subobjects in* $\mathbb{A}$*.*

Proof. (1) (i) follows by 3.2.

(ii) For any morphism $f : A \to A'$ in $\mathbb{A}$, we have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & A' \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta'_A} \\ GF(A) & \xrightarrow{GF(f)} & GF(A'). \end{array}$$

If f is epimorph, then so is the composite $\eta'_A \circ f$ and hence $GF(f)$ must also be epimorph.

(iii) We have the commutative diagram

$$\begin{array}{ccc} \coprod_{i\in I} GF(A_i) & \xrightarrow{\quad\psi\quad} & GF(\coprod_{i\in I} A_i) \\ & \nwarrow \quad\quad \nearrow {\scriptstyle \eta_{\coprod_I A_i}} & \\ & \coprod_{i\in I} A_i & \end{array}$$

where $\eta_{\coprod_I A_i}$ is epimorph and hence so is ψ.

(2) In the diagram in the proof of (1)(ii), assume f to be monomorph and $\eta_{A'}$ an isomorphism. Then η_A is monomorph and an extremal epimorphism which implies that it is an isomorphism. □

A monomorphism m in any category $\mathbb{B}$ is called *extremal* if whenever $m = f \circ e$ for an epimorphism e then e is an isomorphism. Such monomorphisms are isomorphisms if and only if they are epimorph.

3.8. ε_B monomorph. *Let (F, G) be an adjoint pair of functors (as in 2.5).*

(1) *Assume $\varepsilon_B : FG(B) \to B$ to be monomorph for any $B \in \mathbb{B}$. Then:*
   - (i) *(F, G) is idempotent;*
   - (ii) *FG preserves monomorphisms;*
   - (iii) *for any product $\prod_{i\in I} B_i$ in $\mathbb{B}$, the canonical morphism*

$$\varphi : FG(\prod_I B_i) \to \prod_I FG(B_i)$$

   is a monomorphism.

(2) *If $\varepsilon_B : FG(B) \to B$ is an extremal monomorphism for any $B \in \mathbb{B}$, then* $\mathrm{Fix}(FG, \varepsilon)$ *is closed under factor objects in $\mathbb{B}$.*

Proof. The proof is dual to that of 3.7:

(1) (i) follows by 3.2.

(ii) For any morphism $g : B' \to B$ in $\mathbb{B}$, we have the commutative diagram

$$\begin{array}{ccc} FG(B') & \xrightarrow{FG(g)} & FG(B) \\ {\scriptstyle \varepsilon_{B'}}\downarrow & & \downarrow{\scriptstyle \varepsilon_B} \\ B' & \xrightarrow{\ \ g\ \ } & B. \end{array}$$

If g is monomorph, then $g \circ \varepsilon_{B'}$ is monomorph and so is $FG(g)$.

(iii) We have the commutative diagram in $\mathbb{B}$,

$$\begin{array}{ccc} FG(\prod_{i\in I} B_i) & \xrightarrow{\varphi} & \prod_I FG(A_i) \\ & \searrow^{\varepsilon_{\prod_I B_i}} \quad \swarrow & \\ & \prod_{i\in I} B_i, & \end{array}$$

where $\varepsilon_{\prod_I B_i}$ is monomorph and hence so is φ.

(2) In the diagram in (ii), we now have g an epimorphism and $\varepsilon_{B'}$ an isomorphism. Thus ε_B is epimorph and an extremal monomorphism, hence an isomorphism. □

3.9. Definition. An adjoint pair (F, G) of functors with unit η and counit ε is said to be a *pair of $\star$-functors* provided

$\eta_A : A \to GF(A)$ is an extremal epimorphism for all $A \in \mathbb{A}$ and
$\varepsilon_B : FG(B) \to B$ is an extremal monomorphism for all $B \in \mathbb{B}$.

Combining the information from 3.6, 3.7 and 3.8, we obtain the following.

3.10. Theorem. *For a pair of $\star$-functors (F, G), the functors (see 3.1)*

$$\widetilde{F} : \mathbb{A}_{GF} \to \mathbb{B}^{FG}, \quad \widetilde{G} : \mathbb{B}^{FG} \to \mathbb{A}_{GF}$$

induce an equivalence where $\mathbb{A}_{GF} = \mathrm{Fix}(GF, \eta)$ is a reflective subcategory of $\mathbb{A}$ closed under subobjects in $\mathbb{A}$ and $\mathbb{B}^{FG} = \mathrm{Fix}(FG, \varepsilon)$ is a coreflective subcategory of $\mathbb{B}$ closed under factor objects in $\mathbb{B}$.

4 $\star$-modules

In this section let R, S be rings and P be an (R, S)-bimodule. The latter provides the adjoint pair of functors

$$T_P := P \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad H_P := \mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M},$$

with unit and counit

$$\eta_X : X \to H_P T_P(X),\ x \mapsto [p \mapsto p \otimes x], \quad \varepsilon_N : T_P H_P(N) \to N,\ p \otimes f \mapsto (p)f,$$

where $N \in {}_R\mathbb{M}$ and $X \in {}_S\mathbb{M}$. Associated to this pair of functors we have the monad and comonad

$$H_P T_P : {}_S\mathbb{M} \to {}_S\mathbb{M}, \quad T_P H_P : {}_R\mathbb{M} \to {}_R\mathbb{M}.$$

It is well-known that in module categories all monomorphism and all epimorphisms are extremal.

Recall that $N \in {}_R\mathbb{M}$ is said to be *P-static* if ε_N is an isomorphism, and $X \in {}_S\mathbb{M}$ is *P-adstatic* if η_X is an isomorphism (e.g. [18]).

An R-module N is called *P-presented* if there exists an exact sequence of R-modules

$$P^{(\Lambda')} \to P^{(\Lambda)} \to N \to 0, \quad \Lambda, \Lambda' \text{ some sets.}$$

Let Q be any injective cogenerator in ${}_R\mathbb{M}$ and $P^* := \mathrm{Hom}_R(P, Q)$. An S-module X is said to be *P^*-copresented* if there exists an exact sequence of S-modules

$$0 \to X \to P^{*\Lambda'} \to P^{*\Lambda}, \quad \Lambda, \Lambda' \text{ some sets.}$$

When $S = \mathrm{End}_R(P)$, there are canonical candidates for fixed modules for $T_P H_P$ and for $H_P T_P$, namely

$$P \in \mathrm{Fix}(T_P H_P, \varepsilon) \text{ and } S,\ P^* \in \mathrm{Fix}(H_P T_P, \eta),$$

and hence the description of the fixed classes can be related to these objects.

4.1. (T_P, H_P) **idempotent.** *The following are equivalent:*

(a) $H_P \varepsilon T_P : H_P T_P H_P T_P \to H_P T_P$ *is an isomorphism;*

(b) *for any* $X \in {}_S\mathbb{M}$, $\varepsilon T_P(X) : P \otimes_S \mathrm{Hom}_R(P, P \otimes_S X) \to P \otimes_S X$ *is an isomorphism* (*that is,* $P \otimes_S X$ *is* P*-static*)*;*

(c) $T_P \eta H_P : T_P H_P \to T_P H_P T_P H_P$ *is an isomorphism;*

(d) *for any* $N \in {}_R\mathbb{M}$, $\eta H_P(N) : \mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, P \otimes_S \mathrm{Hom}_R(P, N))$ *is an isomorphism* (*that is,* $\mathrm{Hom}_R(P, N)$ *is* P*-adstatic*)*.*

If we assume $S = \mathrm{End}_R(P)$*, then* (a)-(d) *are also equivalent to:*

(e) *every* P*-presented* R*-module is* P*-static;*

(f) *every* P^**-copresented module is* P*-adstatic.*

Proof. The equivalences (a)-(d) follow from 3.4. For the remaining equivalences see, for example, [18, 4.3]. □

4.2. Idempotence and equivalence. *With the notation above, let* (T_P, H_P) *be an idempotent pair. Then these functors induce an equivalence*

$$\widetilde{T_P} : {}_S\mathbb{M}_{H_P T_P} \to {}_R\mathbb{M}^{T_P H_P}, \quad \widetilde{H_P} : {}_R\mathbb{M}^{T_P H_P} \to {}_S\mathbb{M}_{H_P T_P},$$

where ${}_R\mathbb{M}^{T_P H_P} = \mathrm{Fix}(T_P H_P, \varepsilon)$ *is a coreflective subcategory of* ${}_R\mathbb{M}$ *and* ${}_S\mathbb{M}_{H_P T_P} = \mathrm{Fix}(H_P T_P, \eta)$ *is a reflective subcategory of* ${}_S\mathbb{M}$*:*

If $S = \mathrm{End}_R(P)$*, then* ${}_R\mathbb{M}_{T_P H_P}$ *is precisely the subcategory of* P*-presented* R*-modules and* ${}_S\mathbb{M}_{H_P T_P}$ *the subcategory of* P^**-copresented* S*-modules.*

Proof. The first part is a special case of 3.6. For the final remark we again refer to [18, 4.3]. □

Note that the corresponding situation in complete and cocomplete abelian categories is described in [6, Theorem 1.6].

Recall that the module P is *self-small* if, for any set Λ, the canonical map

$$\mathrm{Hom}_R(P, P)^{(\Lambda)} \to \mathrm{Hom}_R(P, P^{(\Lambda)})$$

is an isomorphism, and P is called *w-Σ-quasiprojective* if $\mathrm{Hom}_R(P, -)$ respects exactness of sequences

$$0 \to K \to P^{(\Lambda)} \to N \to 0,$$

where $K \in \mathrm{Gen}(P)$, Λ any set.

The following observations are known from module theory.

4.3. Proposition. *For an* R*-module* P *with* $S = \mathrm{End}_R(P)$*, the following are equivalent:*

(a) $\eta_X : X \to H_P T_P(X)$ *is surjective, for all* $X \in {}_S\mathbb{M}$*;*

(b) P *is self-small and w-Σ-quasiprojective;*

(c) (T_P, H_P) *is an idempotent functor pair and* ${}_S\mathbb{M}_{H_P T_P}$ *is closed under submodules in* ${}_S\mathbb{M}$*..*

For the proof we refer to [17], [7]. The assertions where shown by Lambek and Rattray for a self-small object in a cocomplete additive category (see [14, Theorem 4], [12, Proposition 1]).

The following corresponds to [18, 4.4].

4.4. Proposition. *For an R-module P with $S = \mathrm{End}_R(P)$, the following are equivalent:*

(a) $\varepsilon_N : T_P H_P(N) \to N$ *is monomorph (injective), for all* $N \in {}_R\mathbb{M}$*;*

(b) (T_P, H_P) *is idempotent and* ${}_R\mathbb{M}^{T_P H_P}$ *is closed under factor modules in* ${}_R\mathbb{M}$*.*

As suggested in 3.9, we call H_P a $\star$-functor provided the unit $\eta_{S\mathbb{M}} : \mathrm{Id} \to H_P T_P$ is an epimorphism and the counit $\varepsilon : T_P H_P \to \mathrm{Id}_{R\mathbb{M}}$ is a monomorphism. In this case, the module P is called a *$\star$-module* ([16], [8]) and we obtain:

4.5. Theorem. *For an R-module P with $S = \mathrm{End}_R(P)$, the following are equivalent:*

(a) P *is a $\star$-module;*

(b) H_P *is a $\star$-functor;*

(c) (T_P, H_P) *induces an equivalence*

$$T_P : {}_S\mathbb{M}_{H_P T_P} \to {}_R\mathbb{M}^{T_P H_P}, \quad H_P : {}_R\mathbb{M}^{T_P H_P} \to {}_S\mathbb{M}_{H_P T_P},$$

where ${}_R\mathbb{M}^{T_P H_P}$ *is closed under factor modules in* ${}_R\mathbb{M}$ *and* ${}_S\mathbb{M}_{H_P T_P}$ *is closed under submodules in* ${}_S\mathbb{M}$*.*

The equivalence of (a) and (b) is shown in [7, Theorem 4.1] (see also [16], [8], [2], [17]). For objects in any Grothendieck category they are shown in Colpi [9, Theorem 3.2].

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