# Module and Comodule Categories - a Survey

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**Abstract**

The theory of modules over associative algebras and the theory of comodules for coassociative coalgebras were developed fairly independently during the last decades. In this survey we display an intimate connection between these areas by the notion of categories subgenerated by an object. After a review of the relevant techniques in categories of left modules, applications to the bimodule structure of algebras and comodule categories are sketched.

**1. Module theory:** Homological classification, the category $\sigma[M]$, Morita equivalence, the functor ring, Morita dualities, decompositions, torsion theories, trace functor.

**2. Bimodule structure of an algebra:** Multiplication algebra, Azumaya rings, biregular algebras, central closure of semiprime algebras.

**3. Coalgebras and comodules:** $C$-comodules and $C^*$-modules, $\sigma$-decomposition, rational functor, right semiperfect coalgebras, duality for comodules.

**4. Bialgebras and bimodules:** The category $\mathcal{M}_B^B$, coinvariants, $B$ as projective generator in $\mathcal{M}_B^B$, fundamental theorem for Hopf algebras, semiperfect Hopf algebras.

**5. Comodule algebras:** ($A$-$H$)-bimodules, smash product $A\#H^*$, coinvariants, $A$ as progenerator in $\mathcal{M}_A^H$.

**6. Group actions and module algebras:** Group actions on algebras, ${}_{A*G}A$ as a progenerator in $\sigma[{}_{A*G}A]$, module algebras, smash product $A\#H$, ${}_{A\#H}A$ as a progenerator in $\sigma[{}_{A\#H}A]$.

# 1 Module theory

In this section we recall mainly those results from module categories which are of interest for the applications to bimodules and comodules given in the subsequent sections. For standard notation we refer to [2], [36] and [37].

Let $R$ be a commutative ring, $A$ an associative $R$-algebra with unit, and $A$-Mod the category of unital left $A$-modules. We usually write morphisms on the opposite side to the scalars.

One of the major tools of ring theory is the characterization of rings by properties of their module categories. The main results in this direction at the end of the sixties were collected by L.A. Skornjakov in an inspiring paper on *Homological classification of rings* [31]. His list of characterizations was continued by various authors and to give a more precise idea of what this means we recall some definitions and results.

A ring $A$ is *left semisimple* if it is a direct sum of simple left ideals. Such rings are also right semisimple in the obvious sense and also called *artinian semisimple*.

$A$ is *von Neumann regular* if for any $a \in A$ there exists $b \in A$ such that $a = aba$. Such rings are characterized by the fact that each finitely generated left (right) ideal is a direct summand.

More generally $A$ is called *left semi-hereditary* if every finitely generated left ideal is projective, and $A$ is *left hereditary* if all its left ideals are projective.

$A$ is *left semiperfect* if ${}_AA$ is a supplemented module (i.e., for every left ideal $I$ there exists a left ideal $K$ which is minimal with respect to the property $I + K = A$). It is known that left semiperfect rings are also right semiperfect. $A$ is *left perfect* if it is left semiperfect and the Jacobson radical of $A$ is right t-nilpotent.

$A$ is *left noetherian* if the ascending chain condition for left ideals is satisfied, and is *left artinian* if the descending chain condition for left ideals holds. $A$ is called *quasi-Frobenius*, or *QF* for short, if it is left noetherian and injective.

All these internal properties of $A$ correspond to properties of $A$-Mod. The following is by no means a complete list of such correspondences. Proofs for these assertions can be found in various books on module theory (e.g., [7], [2], [36]).

**1.1 Homological classification of rings.**

| **The ring $A$ is** | **if and only if** |
|---|---|
| *a skew field* | *every $A$-module is free (i.e., $\simeq A^{(\Lambda)}$, for some set $\Lambda$).* |
| *left semisimple* | *- every simple module in $A$-*Mod *is projective;*<br>*- every module in $A$-*Mod *is injective (projective).* |
| *v. N. regular* | *- every module in $A$-*Mod *is flat;*<br>*- every module in $A$-*Mod *is FP-injective.* |
| *left hereditary* | *- submodules of projective $A$-modules are projective;*<br>*- factor modules of injective $A$-modules are injective.* |
| *left semiperfect* | *- simple $A$-modules have projective covers;*<br>*- finitely generated $A$-modules have projective covers.* |
| *left perfect* | *- every $A$-module has a projective cover;*<br>*- projective $A$-modules are direct sums of local modules.* |
| *left noetherian* | *- direct sums of $A$-injectives are $A$-injective;*<br>*- injective $A$-modules are direct sums of indecomposables.* |
| *left artinian* | *- finitely generated $A$-modules have finite length;*<br>*- injective $A$-modules are direct sums of injective hulls of simple modules.* |
| *quasi-Frobenius* | *- every injective $A$-module is projective;*<br>*- every projective $A$-module is injective.* |

Trying to apply these techniques to non-associative algebras it makes sense to consider such an algebra as a module over the associative enveloping algebra. We will come to this setting later on. The general problem turns out to be the association of a suitable category to any module over an associative algebra.

**1.2 The category $\sigma[M]$.** The aim is to characterize an $A$-module $M$ by a category which should be rich enough to allow important constructions available in $A$-Mod. By the fundamental work of P. Gabriel [10] from 1962 it was known that this is the case for *Grothendieck categories*. It was during a stay at Moscow State University in 1973 that I was led to the notion of the category $\sigma[M]$ as the smallest full subcategory of $A$-Mod which contains $M$ and is Grothendieck.

To describe this category we call an $A$-module $M$*-generated* if it is a homomorphic image of direct sums of copies of $M$ $(=M^{(\Lambda)})$ and denote the class of all $M$-generated modules by $\text{Gen}(M)$. Submodules of $M$-generated modules are said to be $M$*-subgenerated* and these are precisely the objects of $\sigma[M]$ (also denoted by $\overline{\text{Gen}}(M)$).

Soon after I started to investigate such categories I realized that quite a number of results scattered around in the literature found new interpretations in this context. We will come across some examples of this in what follows.

**1.3 Injectivity.** An $A$-module $U$ is said to be *$M$-injective* if any diagram

$$\begin{array}{ccccc} 0 & \to & K & \to & M \\ & & \downarrow & & \\ & & U & & \end{array}$$

can be extended commutatively by some morphism $M \to U$, and $U$ is called *injective for (or in) a class* $\mathcal{C} \subset A$-Mod if it is $M$-injective for any $M \in \mathcal{C}$.

Around 1970 it was shown by de Robert [27] and Azumaya [4] that any $M$-injective module $U$ is also $N$-injective provided $N$ is a submodule, factor module or a direct sum of copies of $M$. This immediately implies that an object is injective in $\sigma[M]$ if and only if it is $M$-injective - a kind of Baer's Lemma in $\sigma[M]$. From the general theory of Grothendieck categories it was known that there are always (enough) injectives in $\sigma[M]$.

**1.4 Projectivity.** The situation for projectives is not quite the same. Of course $M$-projective modules are defined dually to $M$-injectives. Clearly any projective object in $\sigma[M]$ is $M$-projective but the converse implication only holds for finitely generated objects. Moreover, in general there need not be any projectives in $\sigma[M]$. For example, the category $\sigma[\mathbb{Q}/\mathbb{Z}]$ - which is just the category of torsion $\mathbb{Z}$-modules - has no non-zero projectives at all.

An easy argument shows that $\sigma[M] = A/\text{An}(M)$-Mod provided $M$ is a finitely generated module over its endomorphism ring. Faithful modules with this property are called *cofaithful* (in Faith [7]). Of course, in this case $A/\text{An}(M)$ is a projective generator in $\sigma[M]$.

To prepare for the characterization of $M$ by properties of $\sigma[M]$ we have to transfer some definitions from rings to modules.

A module $M$ is *semisimple* if it is a direct sum of simple modules. $M$ is *semiperfect* in $\sigma[M]$ if it is supplemented and is projective in $\sigma[M]$. $M$ is *perfect* in $\sigma[M]$ provided any direct sum $M^{(\Lambda)}$ is semiperfect.

A module $X$ is *hereditary in* $\sigma[M]$ if each submodule of $X$ is projective in $\sigma[M]$.

A module is *noetherian (artinian)* if it has acc (dcc) on submodules and $M$ is *locally noetherian* if every finitely generated submodule of $M$ is noetherian. $M$ has *locally finite length* if every finitely generated submodule has finite length (i.e., is noetherian and artinian).

Following ideas of Stenström [32], Fieldhouse [8], Mishina-Skornjakov [19] a.o., *pure exact sequences*

$$(*) \qquad 0 \to K \to L \to N \to 0$$

are defined in $\sigma[M]$ by the fact that $\mathrm{Hom}_A(P, -)$ is exact on them for all finitely presented modules $P$ in $\sigma[M]$. Then a module $N$ is called *flat in* $\sigma[M]$ if all sequences $(*)$ (with $N$ fixed) are pure, and $L$ is *regular in* $\sigma[M]$ if all sequences $(*)$ (with $L$ fixed) are pure in $\sigma[M]$.

A noetherian projective module $M$ is called *quasi-Frobenius (QF)* provided it is an injective cogenerator in $\sigma[M]$.

We have the following characterizations of these modules by properties of $\sigma[M]$ (see [36]).

**1.5 Homological classification of modules.**

| **The module $M$ is** | **if and only if** |
|---:|---|
| *simple* | *every module in $\sigma[M]$ is isomorphic to some $M^{(\Lambda)}$.* |
| *semisimple* | *- every simple module (in $\sigma[M]$) is $M$-projective;*<br>*- every module in $\sigma[M]$ is injective (proj.) in $\sigma[M]$.* |
| *regular* | *- every module in $\sigma[M]$ is regular in $\sigma[M]$;*<br>*- every module in $\sigma[M]$ is flat in $\sigma[M]$.* |
| *hereditary* | *$M$ is projective in $\sigma[M]$ and*<br>*- submodules of projectives are projective in $\sigma[M]$, or*<br>*- factor modules of injectives are injective in $\sigma[M]$.* |
| *semiperfect* | *$M$ is projective in $\sigma[M]$ and*<br>*- simple factors of $M$ have proj. covers in $\sigma[M]$, or*<br>*- fin. $M$-generated modules have proj. covers in $\sigma[M]$.* |
| *perfect* | *$M$ is projective in $\sigma[M]$ and*<br>*- $M$-generated modules have proj. covers in $\sigma[M]$, or*<br>*- $M^{(\mathbb{N})}$ is semiperfect in $\sigma[M]$.* |
| *locally noetherian* | *- direct sums of $M$-injectives are $M$-injective;*<br>*- injectives in $\sigma[M]$ are direct sums of indecomposables.* |
| *loc. of finite length* | *- fin. gen. modules in $\sigma[M]$ have finite length;*<br>*- injectives in $\sigma[M]$ are direct sums of injective hulls of simple modules.* |
| *QF* | *$M$ is finitely generated, $M$-projective, and*<br>*- every injective module is projective in $\sigma[M]$, or*<br>*- $M$ is a generator and projectives are inj. in $\sigma[M]$.* |

**1.6 Equivalences.** B. Zimmermann-Huisgen [42] has studied modules $M$ which generate the submodules of $M^{(\mathbb{N})}$ – this characterizes $M$ to be a generator in $\sigma[M]$. K. Fuller's paper [9] shows - in our terminology - that the category $\sigma[M]$ is equivalent to the full module category $\mathrm{End}_A(M)$-Mod, provided $M$ is finitely generated, $M$-projective and a generator in $\sigma[M]$.

**1.7 Morita equivalence for $\sigma[M]$.** *For an $A$-module $M$ with $S = \mathrm{End}({}_AM)$, the following are equivalent:*

*(a)* $\mathrm{Hom}_A(M, -) : \sigma[M] \to S\text{-Mod}$ *is an equivalence;*

*(b)* ${}_AM$ *is a finitely generated projective generator in* $\sigma[M]$*;*

*(c)* ${}_AM$ *is a finitely generated generator in* $\sigma[M]$ *and* $M_S$ *is faithfully flat.*

**1.8 Self-tilting modules.** Generalizing the notion of a projective generator in $\sigma[M]$, a module $M$ is called *self-tilting* if $M$ is projective in the category $\mathrm{Gen}(M)$ and every $N \in \mathrm{Gen}(M)$ is $M$-presented (there is an exact sequence $M^{(\Lambda')} \to M^{(\Lambda)} \to N \to 0$). $M$ is called *tilting in* $A$-Mod provided $M$ is self-tilting and $\sigma[M] = A$-Mod.

Self-tilting modules are precisely the $*$-*modules* considered in Menini-Orsatti [20] and Colpi [6] and hence yield an equivalence between a subcategory of $\sigma[M]$ (which need not be closed under submodules) and a subcategory of $S$-Mod (which need not be closed under factor modules), where $S = \mathrm{End}(M)$. To describe these let $Q$ be any cogenerator in $\sigma[M]$, put $U = \mathrm{Hom}_A(M, Q)$, and denote by $\mathrm{Cog}({}_SU)$ the full subcategory of $S$-Mod determined by all modules which are cogenerated by $U$. Then we have results which generalize the Brenner-Butler theorem from representation theory of algebras (see [3]).

**1.9 Tilting and equivalences.** *For $M$ the following are equivalent:*

*(1)* $M$ *is finitely generated and self-tilting;*

*(2)* $\mathrm{Hom}_A(M, -) : \mathrm{Gen}(M) \to \mathrm{Cog}\,({}_SU)$ *is an equivalence.*

Notice that a self-tilting module $M$ which is a generator in $\sigma[M]$ is projective in $\sigma[M]$ and the above theorem yields the Morita equivalence.

An important property of finitely generated modules $M$ is that the functor $\mathrm{Hom}_A(M, -)$ commutes with direct sums. This is essential in 1.9 and 1.7. For modules which are direct sums of finitely generated modules there is a slight variation of the Hom-functor which still commutes with direct sums.

**1.10 The functor $\widehat{\mathrm{Hom}}_A(M, -)$.** Assume $M = \bigoplus_\Lambda M_\lambda$, where all $M_\lambda$ are finitely generated. For any $A$-module $N$ put

$$\widehat{\mathrm{Hom}}(M, N) = \{f \in \mathrm{Hom}_A(M, N) \mid (M_\lambda)f = 0 \text{ for almost all } \lambda \in \Lambda\}.$$

For $N = M$, we write $T := \widehat{\mathrm{End}}(M) = \widehat{\mathrm{Hom}}(M, M)$. By definition $T$ is a subring of $S := \mathrm{End}_A(M)$ and so $M$ is a right $T$-module. Clearly $T$ has no unit but there are enough idempotents in $T$, i.e., there exists a family $\{e_\lambda\}_\Lambda$ of pairwise orthogonal idempotents $e_\lambda \in T$ such that

$$T = \bigoplus\nolimits_\Lambda e_\lambda T = \bigoplus\nolimits_\Lambda T e_\lambda.$$

We denote by $T$-Mod ($=\sigma[{}_TT]$) the category of all left $T$-modules $N$ with $TN = N$. Then $T$ is a projective generator in $T$-Mod (not finitely generated in general).

For any $A$-module $N$, we have $T\widehat{\text{Hom}}_A(M,N) = \widehat{\text{Hom}}_A(M,N)$, and this yields the adjoint pair of functors,

$$\widehat{\text{Hom}}_A(M,-) : \sigma[M] \to T\text{-Mod}, \quad M \otimes_T - : T\text{-Mod} \to \sigma[M].$$

$\widehat{\text{Hom}}_A(M,-)$ preserves direct sums and direct products, and it preserves direct limits, provided the $M_\lambda$ are finitely presented in $\sigma[M]$. In this setting the Morita equivalence for $\sigma[M]$ from 1.7 has the following form ([36, 51.11]).

**1.11 General Morita equivalence for $\sigma[M]$.** *For $M = \bigoplus_\Lambda M_\lambda$, where all $M_\lambda$ are finitely generated, and $T = \widehat{\text{End}}_A(M)$, the following are equivalent:*

*(a)* $\widehat{\text{Hom}}_A(M,-) : \sigma[M] \to T$-Mod *is an equivalence;*

*(b) $M$ is a projective generator in $\sigma[M]$;*

*(c) $M$ is a generator in $\sigma[M]$ and $M_T$ is faithfully flat.*

The techniques displayed above can be applied to connect $\sigma[M]$ with its functor category in a purely module theoretic way.

**1.12 The functor ring.** Choose a representing set $\{U_\lambda\}_\Lambda$ of the finitely generated modules in $\sigma[M]$, and put $U := \bigoplus_\Lambda U_\lambda$. Then $U$ is a generator in $\sigma[M]$ ($=\sigma[U]$) and $T = \widehat{\text{End}}_A(U)$ is called the *functor ring (of finitely generated modules) of* $\sigma[M]$. By the faithful functor

$$\widehat{\text{Hom}}_A(U,-) : \sigma[M] \to T\text{-Mod},$$

properties of $\sigma[M]$ are closely related to those of $T$-Mod, e.g., $\widehat{\text{Hom}}_A(U,-)$ is exact (an equivalence) if and only if every finitely generated module is projective in $\sigma[M]$, i.e., $M$ is semisimple.

$M$ is called *pure semisimple* if every pure exact sequence in $\sigma[M]$ splits. This is equivalent to every module in $\sigma[M]$ being pure injective (projective) in $\sigma[M]$, and to every module in $\sigma[M]$ being a direct sum of finitely generated modules.

We say $M$ is of *Kulikov type* if $M$ is locally noetherian and submodules of pure projectives are again pure projective in $\sigma[M]$.

$M$ is of *finite (representation) type*, if $M$ is locally of finite length and there are only finitely many non-isomorphic finitely generated indecomposable modules in $\sigma[M]$.

Modules with linearly ordered (by inclusion) submodules are *uniserial* and direct sums of uniserial modules are *serial*. Call $M$ *of serial type* if every module in $\sigma[M]$ is a direct sum of uniserial modules of finite length.

*QF-2 rings* are semiperfect rings whose indecomposable, projective left and right modules have simple essential socles.

**1.13 Classification by the functor ring.**

| A f.g. module $M$ is | if and only if $T$ is |
|---|---|
| *semisimple* | *- (left) semisimple;*<br>*- von Neumann regular;*<br>*- left (semi-) hereditary.* |
| *of Kulikov type* | *left locally noetherian.* |
| *pure semisimple* | *left perfect.* |
| *of finite type* | *- left and right perfect;*<br>*- left locally of finite length.* |
| *of serial type* | *left perfect and QF-2.* |

Similar to the above constructions a *functor ring of finiteley presented modules of* $\sigma[M]$ can be defined and we refer to [36, Chapter 10] for related results. This approach also provides - via *purity* - a connection to model theoretic techniques for algebras (e.g., [26]).

An algebra $A$ is of finite type if and only if $A$ is left *and* right pure semisimple. It is an open question if a left pure semisimple algebra is also right pure semisimple (hence of finite type). By the above characterizations the open problem is if – for the functor ring of $A$-Mod – left perfect implies right perfect.

**1.14 Dualities.** It is well known that a full module category $A$-Mod cannot be dual to a module category. Similarly it is clear that all $\sigma[M]$ is not dual to a module category. Hence for the study of dualities it makes sense to restrict to categories $\sigma_f[M]$ whose objects are submodules of factor modules of *finite* direct sums of copies of $M$.

These type of categories occur in the work of Goursaud [13], and Ohtake [25], for example. They are used to define the spectrum of non-commutative algebras in Rosenberg [28] (see 1.24). The description of Morita dualities for $A$-Mod is extended to the following situation.

**1.15 Morita dualities for $\sigma[M]$.** *For an $A$-module $M$ with $S = \mathrm{End}(_AM)$, the following are equivalent:*

*(a)* $\mathrm{Hom}_A(-, M)$*:* $\sigma_f[M] \to \sigma_f[S_S]$ *is a duality;*

*(b)* $_AM$ *is an injective cogenerator in* $\sigma_f[M]$ *(hence in* $\sigma[M]$*), and* $M_S$ *is an injective cogenerator in* Mod-$S$*;*

*(c)* $_AM$ *is linearly compact, finitely cogenerated and an injective cogenerator in* $\sigma_f[M]$*;*

*(d) all factor modules of* $_AM$ *and* $S_S$ *are M-reflexive.*

Similar to the generalization of progenerators by tilting modules, injective cogenerators can be generalized by *cotilting modules.* We refer to [41] for details.

**1.16 Decomposition of categories.** One of the good properties of any semiperfect algebra $A$ is that it decomposes as a direct sum of ideals $A_1, \ldots, A_n$ yielding a category decomposition

$$A\text{-Mod} = A_1\text{-Mod} \oplus \cdots \oplus A_n\text{-Mod},$$

such that the categories $A_i$-Mod are indecomposable (in this sense).

This is extended to decompositions of categories of type $\sigma[M]$ in the following way (see [40]). Given a family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M]$, we define

$$\sigma[M] = \bigoplus\nolimits_\Lambda \sigma[N_\lambda],$$

provided for every module $L \in \sigma[M]$, $L = \bigoplus_\Lambda \mathrm{Tr}(\sigma[N_\lambda], L)$. We call this a *$\sigma$-decomposition of* $\sigma[M]$, and $\sigma[M]$ is said to be *$\sigma$-indecomposable* if no such (non-trivial) decomposition exists.

By the fact that semiperfect projectives and locally noetherian injectives both have decompositions which *complement direct summands* (see [2, § 12]) we have the following

**1.17 Decomposition of $\sigma[M]$.** *Let $M$ be a locally noetherian $A$-module, or assume $M$ to be a projective generator which is semiperfect in $\sigma[M]$. Then $\sigma[M]$ has a $\sigma$-decomposition with $M_\lambda \in \sigma[M]$,*

$$\sigma[M] = \bigoplus\nolimits_\Lambda \sigma[M_\lambda],$$

*where each $\sigma[M_\lambda]$ is $\sigma$-indecomposable.*

**1.18 Torsion theories in $\sigma[M]$.** Techniques of torsion theory familiar from $A$-Mod also apply to Grothendieck categories. We recall some of these notions for $\sigma[M]$.

Let $M \in A$-Mod. A class $\mathcal{T}$ of modules in $\sigma[M]$ is called a *pretorsion class* if $\mathcal{T}$ is closed under direct sums and factor modules, and a *torsion class* if $\mathcal{T}$ is closed under direct sums, factors and extensions in $\sigma[M]$.

A pretorsion (torsion) class $\mathcal{T}$ is *hereditary* if it is also closed under submodules, and it is *stable* provided it is closed under essential extensions in $\sigma[M]$.

Notice that for any $M \in A$-Mod, $\sigma[M]$ is a hereditary pretorsion class in $A$-Mod, not necessarily closed under extensions. In fact every hereditary pretorsion class in $\sigma[M]$ (or in $A$-Mod) is of type $\sigma[U]$ for some module $U$ (e.g., Viola-Prioli-Wisbauer [34]).

**1.19 Torsion submodules, injectivity.** For any pretorsion class $\mathcal{T}$ and $N \in \sigma[M]$, the submodule

$$\mathcal{T}(N) := \mathrm{Tr}(\mathcal{T}, N) = \sum \{U \subset N \,|\, U \in \mathcal{T}\} \in \mathcal{T},$$

is the *$\mathcal{T}$-torsion submodule* of $N$, and $N$ is said to be *$\mathcal{T}$-torsionfree* if $\mathcal{T}(N) = 0$.

$N$ is called *(M,$\mathcal{T}$)-injective* if $\mathrm{Hom}_A(-, N)$ is exact on exact sequences $0 \to K \to L$ in $\sigma[M]$ with $L/K \in \mathcal{T}$.

Notice that (M,$\mathcal{T}$)-injective is equivalent to $M$-injective provided for every essential submodule $K \subset M$, $M/K \in \mathcal{T}$.

**1.20 Quotient modules.** Let $\mathcal{T}$ be a hereditary torsion class in $\sigma[M]$ and $N \in \sigma[M]$. The $(M, \mathcal{T})$-injective hull of the factor module $N/\mathcal{T}(N)$ is called the *quotient module* of $N$ with respect to $\mathcal{T}$,

$$Q_{\mathcal{T}}(N) := E_{\mathcal{T}}(N/\mathcal{T}(N)).$$

Any $f : N \to L$ in $\sigma[M]$ induces a unique $Q_{\mathcal{T}}(f) : Q_{\mathcal{T}}(N) \to Q_{\mathcal{T}}(L)$ and

$$Q_{\mathcal{T}} : \sigma[M] \to \sigma[M], \quad N \mapsto Q_{\mathcal{T}}(N),$$

is in fact a left exact functor.

**1.21 Singular modules in $\sigma[M]$.** A module $N$ is called *singular in $\sigma[M]$* (or *$M$-singular*) if $N \simeq L/K$ for some $L \in \sigma[M]$ and $K$ essential in $L$. The class $\mathcal{S}_M$ of $M$-singular modules is a hereditary pretorsion class in $\sigma[M]$. If $\mathcal{S}_M(N) = 0$, $N$ is called *non-singular in $\sigma[M]$* or *non-$M$-singular*.

Let $\mathcal{S}_M^2$ denote the modules $X \in \sigma[M]$ which allow an exact sequence $0 \to K \to X \to L \to 0$, where $K, L \in \mathcal{S}_M$. $\mathcal{S}_M^2$ is a stable hereditary torsion class, called the *Goldie torsion class* in $\sigma[M]$.

It follows from the definition that $(M, \mathcal{S}_M^2)$-injective modules are $M$-injective (since $\mathcal{S}_M \subset \mathcal{S}_M^2$) and $Q_{\mathcal{S}_M^2} : \sigma[M] \to \sigma[M]$ is an exact functor.

**1.22 Lambek torsion theory in $\sigma[M]$.** The hereditary torsion theory $\mathcal{T}_M$ in $\sigma[M]$, whose torsionfree class is cogenerated by the $M$-injective hull $\widehat{M}$ of $M$, is called the *Lambek torsion theory in $\sigma[M]$*,

$$\mathcal{T}_M = \{K \in \sigma[M] \,|\, \mathrm{Hom}_A(K, \widehat{M}) = 0\}.$$

In fact, $\mathcal{T}_M$ is the largest torsion class for which $M$ is torsionfree.

**1.23 Polyform modules.** $M$ is called *polyform* if it is non-$M$-singular. In this case $\mathcal{S}_M$ is closed under extensions and it coincides with $\mathcal{T}_M$. Therefore the quotient module $Q_{\mathcal{S}_M}(M)$ is just the $M$-injective hull $\widehat{M}$ of $M$. It is interesting to observe that $M$ is polyform if and only if $\mathrm{End}_A(\widehat{M})$ is a von Neumann regular ring (and left self-injective). This implies that $\mathrm{End}_A(M)$ is a subring of $\mathrm{End}_A(\widehat{M})$ and that $M$ has finite uniform dimension if and only if $\mathrm{End}_A(\widehat{M})$ is left semisimple. For results about polyform modules see, for example, Clark-Wisbauer [5].

Notice that the above notions applied to $A$ as left $A$-module yield the basic ingredients for Goldie's Theorem (quotient ring of (semiprime) Goldie rings). Applied to $A$ as $(A, A)$-bimodule they allow to construct the (Martindale) central closure of (non-associative ) semiprime rings $A$ (see 2.9).

**1.24 Strongly prime modules.** A module $M$ is called *strongly prime* if each of its non-zero submodules subgenerates $M$, i.e., if for $0 \neq K \subset M$, $M \in \sigma[K]$. This is obviously equivalent to the property that the $M$-injective hull $\widehat{M}$ is generated by each of its non-zero submodules and hence has no non-zero fully invariant submodules.

We mention that a ring $A$ is strongly prime as a left $A$-module if and only if every non-singular $A$-module $N$ is a subgenerator in $A$-Mod (i.e., $\sigma[N] =$ $A$-Mod). Moreover $A$ is strongly prime as an $(A, A)$-bimodule if and only if the central closure of $A$ is a simple ring (see 2.10).

In Rosenberg [28] equivalence classes of strongly prime modules are defined to be the *spectrum* of an abelian category (e.g., $\sigma[M]$ or $A$-Mod), where two $A$-modules $M, N$ are called *equivalent* if $\sigma_f[M] = \sigma_f[N]$.

So far we have been concerned with internal properties of $\sigma[M]$. Discussing the category $\sigma[M]$ with J. Golan in the early eighties in Haifa his first question was if $\sigma[M]$ is a torsion class, i.e., if it is closed under extensions in $A$-Mod. This condition is independent of the internal properties of $\sigma[M]$, and even for a (semi-) simple module $M$, when we have all good properties inside $\sigma[M]$ you may think of, this need not be the case. It was only much later - in particular in connection with the investigation of comodules - when I realized that the *external* behaviour of $\sigma[M]$ is of considerable interest, too.

**1.25 The trace functor.** By definition $\sigma[M]$ is a hereditary pretorsion class in $A$-Mod and we consider the corresponding torsion submodules.

For any $N \in A$-Mod we denote the *trace of* $\sigma[M]$ *in* $N$ by

$$\mathcal{T}^M(N) := \mathrm{Tr}(\sigma[M], N) = \sum \{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_A(K, N),\ K \in \sigma[M]\}.$$

So $\mathcal{T}^M(N)$ is the largest submodule of $N$ which belongs to $\sigma[M]$ and we have a left exact functor

$$\mathcal{T}^M : A\text{-Mod} \to \sigma[M], \quad N \mapsto \mathcal{T}^M(N),$$

which is right adjoint to the inclusion functor $\sigma[M] \to A$-Mod (see [36, 45.11]).

It is natural to investigate properties of this functor. There is a special situation which is of particular interest for comodule categories (see 3.5, 3.8). This is when $\sigma[M]$ is a *cohereditary torsion class*, i.e., $\sigma[M]$ is closed under extensions and the torsionfree class is closed under factor modules.

The trace of $\sigma[M]$ in $A$, $\mathcal{T}^M(A) \subset A$, is an ideal of $A$, called the *trace ideal*. It is useful to describe conditions on the class $\sigma[M]$.

**1.26 $\mathcal{T}^M$ as exact functor.** *For $T := \mathcal{T}^M(A)$ the following are equivalent:*

*(a) The functor $\mathcal{T}^M : A\text{-Mod} \to \sigma[M]$ is exact;*

*(b) $\sigma[M]$ is a cohereditary torsion class;*

*(c) for every $N \in \sigma[M]$, $TN = N$;*

*(d) $T^2 = T$ and ${}_AT$ is a generator in $\sigma[M]$;*

*(e) $TM = M$ and $A/T$ is flat as a right $A$-module.*

# 2 Bimodule structure of an algebra

Extending the module theory for commutative algebras to non-commutative $R$-algebras $A$, the classical approach was to study the category of left (or right) $A$-modules. However by this step some symmetry is lost: for example, the kernels of morphisms in $A$-Mod are no longer the kernels of algebra morphisms. One may ask why the category of $(A, A)$-bimodules - equivalently the category of left $A \otimes_R A^o$-modules - was not considered as an adequate extension of the commutative case. The main reason was probably the fact that in general the algebra $A$ is neither projective nor a generator in $A \otimes_R A^o$-Mod and hence a homological characterization was not possible in this setting.

If $A$ is projective as an $A \otimes_R A^o$-module then $A$ is called a *separable $R$-algebra* and this implies that $A \otimes_R A^o$-Mod is equivalent to $R$-Mod, provided $A$ is a central $R$-algebra. A large part of the literature on bimodules is concentrating on this situation.

Applying ideas and notions from the first section we are able to extend typical results for commutative algebras to the subcategory $\sigma[A]$ of $A \otimes_R A^o$-modules without any a priori conditions on the algebra $A$. Moreover we observe that even associativity of $A$ is not essential and we are going to report about essential parts of the resulting theory.

In this section let $A$ denote an $R$-algebra which is not necessarily associative. For simplicity we will assume that $A$ has a unit $e_A$.

For $a,b,c\in A$ we define *associator* and *commutator* by

$$(a,b,c) := (ab)c - a(bc), \quad [a,b] = ab - ba,$$

and the *centre of* $A$ as the subset

$$Z(A) = \{c \in A \mid (a,b,c) = (a,c,b) = [a,c] = 0 \text{ for all } a,b \in A\}.$$

Clearly $Z(A)$ is a commutative and associative subalgebra containing the unit of $A$. In case $R\,e_A = Z(A)$ we call $A$ a *central $R$-algebra.*

**2.1 Multiplication algebra.** Left and right multiplications by $a \in A$,

$$L_a : A \to A,\ x \mapsto ax, \quad R_a : A \to A,\ x \mapsto xa,$$

define $R$-endomorphisms of $A$, i.e., $L_a, R_a \in End_R(A)$. The $R$-subalgebra of $End_R(A)$ generated by all left and right multiplications in $A$ is called the *multiplication algebra of* $A$, i.e.,

$$M(A) := < \{L_a, R_a \mid a \in A\} > \subset End_R(A).$$

We consider $A$ as a (faithful) left module over $M(A)$. The endomorphism ring of the $M(A)$-module $A$ is called the *centroid* $C(A)$ of $A$. Since we assume $A$ to have a unit, it is easy to see that $C(A)$ is isomorphic to the center $Z(A)$ of $A$, i.e., $Z(A) = \mathrm{End}_{M(A)}(A)$.

By $\sigma[A]$, or $\sigma_{M(A)}[A]$, we denote the full subcategory of $M(A)$-Mod whose objects are submodules of $A$-generated modules. Notice that $\sigma[A] = M(A)$-Mod provided $A$ is finitely generated as $Z(A)$-module.

**2.2 The centre of an $M(A)$-module.** For any $M(A)$-module $M$, the *centre* $Z_A(M)$ *of* $M$ is defined as

$$\{m \in M \mid L_a m = R_a m,\ L_a L_b m = L_{ab} m,\ R_b R_a m = R_{ab} m, \text{ for all } a,b \in A\}.$$

Obviously, $Z_A(A) = Z(A)$, the centre of the algebra $A$, $Z_A(M)$ is a $Z(A)$-submodule of $M$, and the map

$$\mathrm{Hom}_{M(A)}(A,M) \to Z_A(M), \quad \gamma \mapsto (e_A)\gamma,$$

is a $Z(A)$-module isomorphism.

So $M$ is $A$-generated as an $M(A)$-module if and only if $M = A\,Z_A(M)$. In particular, for any ideal $I \subset A$, $Z_A(I) = I \cap Z(A)$, and $I$ is $A$-generated if and only if $I = A\,(I \cap Z(A))$.

**2.3** $A$ **associative.** *Assume $A$ to be associative. Then the map*

$$A \otimes_R A^o \to M(A), \; a \otimes b \mapsto L_a R_b,$$

*is a surjective $A \otimes_R A^o$ morphism. So $\sigma_{M(A)}[A] = \sigma_{A \otimes_R A^o}[A]$.*

In general $A$ will be neither projective nor a generator in $\sigma[A]$. These properties are related to interesting classes of algebras. A central $R$-algebra $A$ is called an *Azumaya algebra* if $A$ is a (projective) generator in $M(A)$-Mod, and an *Azumaya ring* if it is a projective generator in $\sigma[A]$. An Azumaya ring is an Azumaya algebra if and only if ${}_RA$ is finitely generated, since this implies $\sigma[A] = A$-Mod.

Morita equivalence for the category $\sigma[A]$ has the following form (see 1.7).

**2.4 Azumaya rings.** *For a central $R$-algebra $A$, the following are equivalent:*

*(a) $A$ is an Azumaya ring;*

*(b) $A$ is a generator in $\sigma[A]$ and $A$ is faithfully flat as an $R$-module;*

*(c) for every ideal $I \subset A$,*

$$I = (I \cap R)A \;\; \text{and} \;\; Z(A/I) = R/I \cap R;$$

*(d)* $\text{Hom}_{M(A)}(A, -) : \sigma[A] \to R\text{-Mod}$ *is an equivalence of categories.*

Similar to the commutative case the cogenerator property of $A$ in $\sigma[A]$ has a strong influence on the structure of $A$.

**2.5 Cogenerator algebras.** *For a central $R$-algebra $A$, the following are equivalent:*

*(a) $A$ is a cogenerator in $\sigma[A]$;*

*(b) $A = \widehat{E}_1 \oplus \cdots \oplus \widehat{E}_k$, where the $E_i$ are (up to isomorphism) all the simple modules in $\sigma[A]$, and $\widehat{E}_i$ is the injective hull of $E_i$ in $\sigma[A]$;*

*(c) $A = A_1 \oplus \cdots \oplus A_k$, where the algebras $A_i$ are indecomposable self-injective self-cogenerators as bimodules;*

*Under these conditions, $A_R$ cogenerates all simple $R$-modules and $A_R$ is FP-injective.*

Notice that the decomposition of $A$ in $(c)$ yields a $\sigma$-decomposition of $\sigma[A]$.

To describe dualities recall that $\sigma_f[A]$ denotes the full subcategory of $\sigma[A]$ whose objects are submodules of finitely $A$-generated $M(A)$-modules.

**2.6 Algebras with Morita dualities.** *For a central $R$-algebra $A$, the following are equivalent:*

*(a) $A$ is a cogenerator in $\sigma[A]$ and $A_R$ is injective in $R$-Mod;*

*(b) $\mathrm{Hom}_{M(A)}(-,A) : \sigma_f[A] \to \sigma_f[{}_RR]$ defines a duality;*

*(c) all factor modules of ${}_{M(A)}A$ and $R_R$ are $A$-reflexive.*

Here we have a bimodule version of quasi-Frobenius rings ([37, 27.12]).

**2.7 Quasi-Frobenius algebras**. *For a central $R$-algebra $A$, the following are equivalent:*

*(a) $A$ is a noetherian injective generator in $\sigma[A]$;*

*(b) $A$ is an artinian projective cogenerator in $\sigma[A]$;*

*(c) $A$ is an injective generator in $\sigma[A]$ and $R$ is artinian;*

*(d) $A$ is an Azumaya ring and projectives are injective in $\sigma[A]$.*

The algebra $A$ is called *biregular* if every principal ideal of $A$ is generated by a central idempotent. Clearly any commutative associative ring is biregular if and only if it is von Neumann regular.

Recall that a short exact sequence $(*)\quad 0 \to K \to L \to N \to 0$ is *pure in* $\sigma[A]$, if the functor $\mathrm{Hom}_A(P,-)$ is exact with respect to $(*)$ for every finitely presented module $P$ in $\sigma[A]$, and $L \in \sigma[A]$ is *regular* in $\sigma[A]$ if every exact sequence (with $L$ fixed) of type $(*)$ is pure in $\sigma[A]$.

Notice that a biregular algebra $A$ need not be regular in $\sigma[A]$. However this is true if $A$ is finitely presented in $\sigma[A]$.

**2.8 Biregular Azumaya rings.** *Let $A$ be a central $R$-algebra. Then the following conditions are equivalent:*

*(a) $A$ is a biregular ring and is finitely presented in $\sigma[A]$;*

*(b) $A$ is finitely presented and regular in $\sigma[A]$;*

*(c) $A$ is a biregular ring and is self-projective as $M(A)$-module;*

*(d) $A$ is a biregular ring and is a generator in $\sigma[A]$;*

*(e) $A$ is a generator in $\sigma[A]$ and $R$ is regular.*

Similar to the situation for commutative associative rings we observe that any semiprime algebra $A$ is non-$A$-singular (polyform) as a bimodule. Recall that $\widehat{A}$ denotes the $A$-injective hull of $A$ (in $\sigma[A]$). By results for $\sigma[M]$ we conclude that $\mathrm{End}_{M(A)}(\widehat{A})$ (the *extended centroid*) is a commutative, regular and self-injective ring. The module $\widehat{A}$ can be made an algebra in the following way.

**2.9 Central closure of semiprime algebras.** *Let $A$ be semiprime with $A$-injective hull $\widehat{A}$ and $T := \text{End}_{M(A)}(\widehat{A})$. Then $\widehat{A} = AT$ and a ring structure is defined on $\widehat{A}$ (central closure) by*

$$(as) \cdot (bt) := (ab)st, \quad \text{for } a, b \in A, \; s, t \in T.$$

*(1) $\widehat{A}$ is a semiprime ring with centre $Z(\widehat{A}) = T$.*

*(2) $\widehat{A}$ is self-injective as an $M(\widehat{A})$-module.*

*(3) If $A$ is a prime ring, then $\widehat{A}$ is also a prime ring.*

Calling $A$ *strongly prime* if $A$ is strongly prime as an $M(A)$-module we have from 1.24:

**2.10 Strongly prime algebras.** *For $A$ the following are equivalent:*

*(a) $A$ is a strongly prime algebra;*

*(b) $\widehat{A}$ is generated (as an $M(A)$-module) by any non-zero ideal of $A$;*

*(c) $A$ is a prime ring and the central closure $\widehat{A}$ is a simple algebra.*

# 3 Coalgebras and comodules

The definitions of coalgebras and comodules are formally dual to the definitions of algebras and modules. Whereas the algebra is a projective generator for the left (right) modules, a coalgebra is a *subgenerator* for the right (left) comodules. This indicates that ideas from the categories $\sigma[M]$ might be helpful. Under the condition that the coalgebra is projective over the base ring we can in fact identify the category of comodules with a category of type $\sigma[M]$ (over the dual of the coalgebra).

An $R$-module $C$ is an $R$-*coalgebra* if there is an $R$-linear map

$$\Delta : C \to C \otimes_R C, \text{ with } (id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta.$$

An $R$-linear map $\varepsilon : C \to R$ is a *counit* if $(id \otimes \varepsilon) \circ \Delta$ and $(\varepsilon \otimes id) \circ \Delta$ yield the canonical isomorphism $C \simeq C \otimes_R R$.

We will assume the coalgebra $C$ to be projective as an $R$-module.

Notice that here - traditionally - maps are written on the left which may imply anti-isomorphisms of endomorphism rings in comparison with the module theoretic formulation of the previous sections.

The $R$-dual $C^* = \text{Hom}_R(C, R)$ is an associative $R$-algebra with unit $\varepsilon$, where the multiplication of $f, g \in C^*$ is defined by $f * g(c) = f \otimes g(\Delta(c))$, for $c \in C$.

An $R$-module $M$ is a *right $C$-comodule* if there exists an $R$-linear map $\varrho : M \to M \otimes_R C$ such that $(id \otimes \Delta) \circ \varrho = (\varrho \otimes id) \circ \varrho$, and $(id \otimes \varepsilon) \circ \varrho$ yields the canonical isomorphism $M \simeq M \otimes_R R$. *Comodule morphisms* $f : M \to M'$ between right comodules satisfy $\varrho' \circ f = (f \otimes id) \circ \varrho$.

Left $C$-comodules and their morphisms are defined similarly.

$R$-submodules $N \subset M$ are *$C$-sub-comodules* provided $\varrho(N) \subset N \otimes_R C$. Clearly $C$ is a right and left $C$-comodule, and right (left) sub-comodules of $C$ are called right (left) coideals.

An $R$-submodule $D \subset C$ is a *sub-coalgebra* if $\Delta(D) \subset D \otimes_R D$.

Right (left) $C$-comodules together with comodule morphisms form a category which we denote by $\mathcal{M}^C$ (${}^C\mathcal{M}$). These are Grothendieck categories (remember that we assume ${}_RC$ to be projective) which are subgenerated by the right (left) $C$-comodule $C$. They can be identified with categories of type $\sigma[M]$ (see [39, Section 3,4]):

**3.1 $C$-comodules and $C^*$-modules.** *Let $\varrho : M \to M \otimes_R C$ be a right $C$-comodule. Then $M$ is a left $C^*$-module by*

$$\psi : C^* \otimes_R M \to M, \quad f \otimes m \mapsto (id \underline{\otimes} f) \circ \varrho(m).$$

*$C$ is a balanced $(C^*, C^*)$-bimodule, subcoalgebras of $C$ correspond to $(C^*, C^*)$-sub-bimodules, and we identify*

$$\mathcal{M}^C = \sigma[{}_{C^*}C] \subset C^*\text{-Mod}, \quad {}^C\mathcal{M} = \sigma[C_{C^*}] \subset \text{Mod-}C^*.$$

*$C$ is finitely generated as $R$-module if and only if $\sigma[{}_{C^*}C] = C^*$-Mod.*

**3.2 Finiteness conditions.** Of particular importance for the investigation of comodules is the so-called *Finiteness Theorem* which says that every finite subset of a right comodule $M$ is contained in a sub-comodule which is finitely generated as an $R$-module. From this it follows that finiteness properties of the ring $R$ imply finiteness properties of comodules (see [39, 4.9]).

**3.3 Coalgebras over special rings.**

| **If $R$ is** | **then the coalgebra $C$ is/has** |
|---:|---|
| *noetherian* | *locally noetherian.* |
| *perfect* | *dcc on finitely generated coideals.* |
| *artinian* | *locally of finite length.* |
| *injective* | *injective in $\mathcal{M}^C$.* |
| *QF* | *injective and cogenerator in $\mathcal{M}^C$.* |

So if $R$ is noetherian, direct sums of injectives are injective in $\sigma[{}_{C^*}C]$, and applying results on decompositions of closed subcategories we can extend decomposition theorems known for coalgebras over fields to comodules over noetherian (or QF) rings.

**3.4 $\sigma$-decomposition of coalgebras.** *Let $R$ be noetherian. Then there exists a $\sigma$-decomposition $C = \bigoplus_\Lambda C_\lambda$, i.e.,*

$$\mathcal{M}^C = \sigma[{}_{C^*}C] = \bigoplus\nolimits_\Lambda \sigma[{}_{C^*}C_\lambda] = \bigoplus\nolimits_\Lambda \mathcal{M}^{C_\lambda},$$

*with sub-coalgebras $C_\lambda \subset C$, and $\mathcal{M}^{C_\lambda}$ $\sigma$-indecomposable.*

*If $R$ is QF, each fully invariant decomposition of $C$ is a $\sigma$-decomposition.*

**3.5 Rational functor.** Left $C^*$-module which are also right $C$-comodules are traditionally called *rational $C^*$-modules*. The largest submodule of any $C^*$-module $N$ which is a right $C$-comodule is the *rational submodule* of $N$, $\mathcal{T}^C(M) = \mathrm{Tr}_{C^*}(\sigma[{}_{C^*}C], M)$, and this leads to the *rational functor*

$$\mathcal{T}^C : C^*\text{-Mod} \to \mathcal{M}^C,$$

which is right adjoint to the inclusion $\mathcal{M}^C \to C^*$-Mod.

Further properties of this functor depend on the (torsion theoretic) properties of the class $\mathcal{M}^C$ in $C^*$-Mod.

**3.6 The rational functor exact.** *Put $T := \mathcal{T}^C({}_{C^*}C^*)$. The following are equivalent:*

*(a) the functor $\mathcal{T}^C : C^*\text{-Mod} \to \mathcal{M}^C$ is exact;*

*(b) $\mathcal{M}^C$ is a cohereditary torsion class in $C^*$-Mod;*

*(c) for every $N \in \mathcal{M}^C$, $TN = N$;*

*(d) $T^2 = T$ and $T$ is a generator in $\mathcal{M}^C$.*

**3.7 Right semiperfect coalgebras.** We already mentioned that categories of type $\sigma[M]$ need not have projectives. Even for coalgebras $C$ over fields there may be no projectives in $\mathcal{M}^C$. The question arises when there is a projective generator in $\mathcal{M}^C$. By the Finiteness Theorem it is straightforward to see that for coalgebras over fields (or artinian rings) this is equivalent to the fact that every simple module has a projective cover in $\mathcal{M}^C$. If this holds $C$ is called *right semiperfect* (see [18]). Notice that a right semiperfect coalgebra $C$ need not be left semiperfect.

Over a QF ring $R$ there is a certain interplay between left and right properties of $C$. This is based on the observation that, for any right $C$-comodule $N$ which is finitely generated as $R$-module, the dual $N^*$ is a left $C$-comodule. This leads to the following connection between the existence of projectives in $\mathcal{M}^C$ and the exactness of the rational functor.

**3.8 Right semiperfect coalgebras over QF rings.** *Let $R$ be a QF ring and put $T := \mathcal{T}^C({}_{C^*}C^*)$. Then the following are equivalent:*

- *(a) $\mathcal{M}^C$ has a projective generator (projective in $C^*$-Mod);*
- *(b) every simple module has a projective cover in $\mathcal{M}^C$;*
- *(c) injective hulls of simple* left *$C$-comodules are finitely generated $R$-modules;*
- *(d) the functor $\mathcal{T}^C : C^*\text{-Mod} \to \mathcal{M}^C$ is exact.*

In particular one may ask when $C$ itself is projective in $\mathcal{M}^C$. If $R$ is QF then $C$ is an injective cogenerator in $\mathcal{M}^C$ and we have the following characterizations.

**3.9 Projective coalgebras over QF rings.** *Let $R$ be a QF ring. Then the following are equivalent:*

- *(a) $C$ is a submodule of a free left $C^*$-module;*
- *(b) in $\mathcal{M}^C$ every (indecomposable) injective object is projective;*
- *(c) $C$ is projective in $\mathcal{M}^C$ (in $C^*$-Mod).*

*In this case $C$ is a left semiperfect coalgebra and $C$ is a generator in ${}^C\mathcal{M}$.*

By the general Morita equivalence for modules (see 1.11) we have:

**3.10 $C$ as projective generator in $\mathcal{M}^C$.** *Let $R$ be a QF ring and put $T := \mathcal{T}^C({}_{C^*}C^*)$. The following are equivalent:*

- *(a) $C$ is projective in $\mathcal{M}^C$ and ${}^C\mathcal{M}$;*
- *(b) $C$ is a projective generator in $\mathcal{M}^C$ (in ${}^C\mathcal{M}$);*
- *(c) $C$ is a direct sum of finitely generated $C^*$-submodules, $T$ is a ring with enough idempotents, and there is an equivalence*

$$\widehat{\text{Hom}}_{C^*}(C, -) : \mathcal{M}^C \to T\text{-Mod}.$$

For a QF ring $R$ the functor $(-)^* = \text{Hom}_R(-, R)$ defines a duality for the category of finitely generated $R$-modules. Combined with the (covariant) left and right rational functors (both denoted by $\mathcal{T}^C$) we obtain a duality for finitely generated $C$-comodules provided $C$ is left and right semiperfect. Here again categories of type $\sigma_f[M]$ as defined in 1.14 enter the scene. The following extends [14, Theorem 3.5] from base fields to base QF rings.

**3.11 Duality for comodules.** *If $R$ is QF the following are equivalent:*

*(a) $C$ is left and right semiperfect;*

*(b) the following functors are exact*

$$\mathcal{T}^C \circ (-)^* : \mathcal{M}^C \to {}^C\mathcal{M}, \quad \mathcal{T}^C \circ (-)^* : {}^C\mathcal{M} \to \mathcal{M}^C;$$

*(c) the left and right trace ideals coincide and form a ring $T$ with enough idempotents and there is a duality*

$$\mathcal{T}^C \circ (-)^* : \sigma_f[{}_{C^*}C] \to \sigma_f[T_T].$$

# 4 Bialgebras and bimodules

**4.1 Bialgebras.** Let $B$ be an $R$-coalgebra and algebra with counit and unit,

$$\begin{array}{ll} \Delta : \ B \to B \otimes_R B, & \varepsilon : B \to R, \\ \mu : B \otimes_R B \to B & \iota : R \to B \end{array}$$

$B$ is called a *bialgebra* if $\Delta$ and $\varepsilon$ are algebra morphisms (equivalently - $\mu$, $\iota$ are coalgebra morphisms).

An immediate implication of this definition is the fact that $R$ itself has a right $B$-comodule structure $R \to R \otimes_R B$, $r \mapsto r \otimes 1_B$. As a consequence $R$ is a direct summand of $B$ and hence $B$ is a generator for $R$-Mod.

An $R$-module $M$ is called a *right $B$-bimodule* if $M$ is a right $B$-module and a right $B$-comodule by $\varrho : \ M \to M \otimes_R B$, such that $\varrho$ is $B$-linear in the sense $\varrho(mb) = \varrho(m)\Delta b$, for all $m \in M, b \in B$.

The reader should be aware of the fact that the notion *bimodule* applies in different situations. We have already talked about $(A, A)$-bimodules over associative algebras $A$. Moreover, for any bialgebra $B$ we may consider left $B$-comodules which are right $B$-modules, right $B$-comodules which are left $B$-modules, etc. Of course $B$ itself belongs to any of these categories but it may have different properties in the distinct settings.

**4.2 The category $\mathcal{M}^B_B$.** Let $B$ be an $R$-bialgebra. By $\mathcal{M}^B_B$ we denote the category whose objects are the right $B$-bimodules and the morphisms are maps which are both $B$-comodule and $B$-module morphisms ($= \mathrm{Bim}_B(-,-)$).

One link to our previous considerations is given by the observation that the right $B$-module $B \otimes_R B$ is a subgenerator in $\mathcal{M}^B_B$.

We will assume that $B_R$ is projective which implies that $\mathcal{M}^B_B$ is a Grothendieck category and the objects in $\mathcal{M}^B_B$ may be considered as left modules over the *smash product* $B\#B^*$, which is defined as the $R$-module $B \otimes_R B^*$ with multiplication

$$(a \otimes f)(b \otimes g) := ((\Delta b)(a \otimes f))\,(1_B \otimes g).$$

Similar to the case of comodules we may identify

$$\mathcal{M}_B^B = \sigma_{B\#B^*}\,[B \otimes_R B] \subset B\#B^*\text{-Mod},$$

where $\mathcal{M}_B^B = B\#B^*$-Mod if and only if $B_R$ is finitely generated.

**4.3 Coinvariants.** The *coinvariants* of a right $B$-bimodule $M$ are

$$M^{coB} := \{m \in M \mid \; \varrho(m) = m \otimes 1_B\}\ ,$$

and there is an $R$-module isomorphism

$$\nu_M:\ \mathrm{Bim}_B(B, M) \to M^{coB},\ f \mapsto f(1_B).$$

In particular, $\mathrm{Bim}_B(B, B) \to B^{coB} = R\,1_B$ is a ring isomorphism.

It is interesting to compare the next theorem with results from Section 2. Recall that a finitely generated generator in $\sigma[M]$ is projective in $\sigma[M]$ if and only if it is faithfully flat over its endomorphism ring. As noticed in 4.1, $B$ is a generator for modules over $\mathrm{Bim}_B(B, B) = R$ and hence we have:

**4.4 B as generator in $\mathcal{M}_B^B$.** *The following are equivalent:*

*(a) $B$ is a (projective) generator in $\mathcal{M}_B^B$;*

*(b) $\mathrm{Bim}_B(B, -) : \mathcal{M}_B^B \to R$-Mod is an equivalence (with inverse $- \otimes_R B$);*

*(c) for every $M \in \mathcal{M}_B^B$, we have an isomorphism*

$$M^{coB} \otimes_R B \to M,\ m \otimes b \mapsto mb;$$

*(d) $B \otimes_R B \to B \otimes_R B,\ a \otimes b \mapsto (a \otimes 1)\Delta b$, is an isomorphism.*

Similar to the multiplication defined on the dual of any coalgebra, for a bialgebra $B$ a *convolution product* for $f, g \in \mathrm{End}_R(B)$ is defined by

$$f * g(b) = \mu(f \otimes g(\Delta(b))),\ \text{for } b \in B.$$

**4.5 Antipodes and Hopf algebras.** Let $(B, \Delta, \varepsilon, \mu, \iota)$ be a bialgebra. An element $S \in \mathrm{End}_R(B)$ is called an *antipode* of $B$ if it is inverse to $id_B$ with respect to the convolution product $*$. So by definition,

$$\mu \circ (S \otimes id_B) \circ \Delta = \mu \circ (id_B \otimes S) \circ \Delta = \iota \circ \varepsilon\,.$$

Notice that as an endomorphism of $B$, $S$ need neither be injective nor surjective.

A bialgebra with an antipode is called a *Hopf algebra*. Again we will assume that ${}_RH$ is projective. One of the intrinsic properties of such an algebra is that $H$ is a generator for the $H$-bimodules.

**4.6 Fundamental Theorem.** *Let $H$ be a Hopf $R$-algebra. Then $H$ is a projective generator in $\mathcal{M}_H^H$, so for any right $H$-Hopf module $M$,*

$$M^{coH} \otimes_R H \to M, \ m \otimes h \mapsto mh \,,$$

*is a bimodule isomorphism, and*

$$\mathrm{Bim}_H(H, -) : \mathcal{M}_H^H \to R\text{-Mod}$$

*is an equivalence of categories.*

It follows from 4.6 that Hopf algebras behave similarly to algebras $A$ which are (projective) generators in $\sigma[A]$ (Azumaya rings, 2.4).

The fundamental theorem describes properties of $H$ as a bimodule. It does not give information about the right (or left) comodule structure of $H$. However it brings a certain symmetry to $H$ and we observe, for example, that over a QF ring right semiperfect Hopf algebras are also left semiperfect.

For these characterizations the coinvariants of the (left) trace ideal $\mathcal{T}^H(H^*)$ are of importance, which are called *left integrals* for $H$ (see [21]).

**4.7 Semiperfect Hopf algebras.** *Let $R$ be a QF ring and $T := \mathcal{T}^H(H^*)$. Then the following are equivalent:*

*(a) $H$ is right semiperfect;*

*(b) $T$ (or $T^{coH}$) is a faithful and flat $R$-module;*

*(c) $T$ is a projective generator in $\mathcal{M}^H$;*

*(d) $H$ is a projective (generator) in $\mathcal{M}^H$;*

*(e) $H$ is left semiperfect.*

Algebras of this type are called *left co-Frobenius Hopf algebras* in [14].

# 5 Comodule algebras

The results on Hopf algebras and their bimodules can be transferred to a more general setting.

Let $H$ be an $R$-Hopf algebra. An $R$-algebra $A$ with a right $H$-comodule structure $\varrho : A \to A \otimes_R H$ is a *right $H$-comodule algebra* provided $\varrho$ is an algebra morphism.

**5.1** $(A\text{-}H)$**-bimodules.** An $R$-module $M$ is called a *right* $(A\text{-}H)$*-bimodule* if $M$ is a right $A$-module and a right $H$-comodule $\varrho_M : M \to M \otimes_R H$, such that $\varrho_M$ is $A$-linear, i.e., for $a \in A$, $m \in M$,

$$\varrho_M(ma) = \varrho_M(m) \cdot a \ (= \varrho_M(m)\varrho_A(a))\,.$$

We denote by $\mathcal{M}_A^H$ the category of $(A\text{-}H)$-bimodules with morphisms those maps which are both $A$-module and $H$-comodule morphisms ($= \mathrm{Bim}_A^H(-,-)$).

This is obviously an additive category which is closed under infinite direct sums and homomorphic images and it is easy to show that $A \otimes_R H$ is a subgenerator in $\mathcal{M}_A^H$. Assuming $H_R$ to be projective, $\mathcal{M}_A^H$ is a Grothendieck category and the $(A\text{-}H)$-bimodules can be considered as left modules over the *smash product* $A\#H^*$, which is defined as the $R$-module $A \otimes_R H^*$ with multiplication

$$(a \otimes k)(b \otimes h) = \sum\nolimits_j b_j a \otimes (\tilde{b}_j \cdot k) * h,$$

where $a, b \in A$, $k, h \in H^*$ and $\varrho_A(b) = \sum_j b_j \otimes \tilde{b}_j$.

Then $(A\text{-}H)$-bimodule morphisms are precisely the $A\#H^*$-module morphisms and we can identify

$$\mathcal{M}_A^H = \sigma_{A\#H^*}\,[A \otimes_R H] \subset A\#H^*\text{-Mod}.$$

If ${}_RH$ is finitely generated, then $\mathcal{M}_A^H = A\#H^*$-Mod.

**5.2 Coinvariants.** Similar to the case of Hopf algebras, *coinvariants* for $M \in \mathcal{M}_A^H$ are defined by

$$M^{coH} = \{m \in M \mid \ \varrho_M(m) = m \otimes 1_H\}\,,$$

and there is an $R$-module isomorphism

$$\mathrm{Bim}_A^H(A, M) \to M^{coH}, \ f \mapsto f(1_A).$$

In particular we have a ring (anti-) isomorphism

$$\mathrm{End}_A^H(A) := \mathrm{Bim}_A^H(A, A) \to A^{coH} = \{a \in A \mid \ \Delta(a) = a \otimes 1_H\}\,.$$

Applying our results about progenerators in module categories we obtain an extension of Menini-Zuccoli [22, Theorem 3.29] from base fields to rings, which in turn was an extension of Schneider's result [29, Theorem 1].

**5.3** $A$ **as progenerator in** $\mathcal{M}_A^H$**.** *The following are equivalent:*

(a) *$A$ is a projective generator in $\mathcal{M}_A^H$;*

(b) $\mathrm{Bim}_A^H(A, -) : \mathcal{M}_A^H \to \mathrm{Mod}\text{-}A^{coH}$ *is a category equivalence;*

*(c)* ${}_{A^{coH}}A$ *is faithfully flat and we have an isomorphism*

$$A \otimes_{A^{coH}} A \longrightarrow A \otimes_R H, \;\; a \otimes b \mapsto (a \otimes 1_H)\varrho_A(b)\,;$$

*(d) for any* $M \in \mathcal{M}_A^H$ *and* $N \in \text{Mod-}A^{coH}$, *we have isomorphisms*

$$\begin{array}{ll} \mathrm{Bim}_A^H(A, M) \otimes_{A^{coH}} A \to M, & f \otimes a \mapsto f(a), \\ N \to \mathrm{Bim}_A^H(A, N \otimes_{A^{coH}} A), & n \mapsto [a \mapsto n \otimes a]. \end{array}$$

# 6 Group actions and module algebras

**6.1 Group actions on algebras.** A group $G$ *acts on* an associative $R$-algebra $A$ if there is a group homomorphism from $G$ to the group of all $R$-algebra automorphisms of $A$.

The *skew group algebra* $A * G$ is defined as the direct sum $A^{(G)}$ with multiplication given by

$$(ag) \cdot (bh) = a({}^g b)gh, \;\text{ for } a, b \in A \text{ and } g, h \in G.$$

$A * G$ is an associative algebra and we consider $A$ as left $A * G$-module. The $A * G$-submodules of $A$ are the $G$-invariant left ideals of $A$. $A$ is a cyclic $A * G$-module and it is a finitely presented $A * G$-module if and only if the group $G$ is finitely generated.

For any $A * G$-module $M$ the set of *$G$-invariant elements of $M$* is denoted by

$$M^G := \{m \in M \mid gm = m, \text{ for every } g \in G\},$$

and the (evaluation) map

$$\mathrm{Hom}_{A*G}(A, M) \to M^G, \;\; f \mapsto f(1),$$

is an isomorphism of left $A^G$-modules. Hence $M$ is $A$-generated as an $A * G$-module if and only if $M = AM^G$.

$A^G$ is called the *fixed ring of* $A$ and we have an algebra isomorphism

$$\mathrm{End}_{A*G}(A) \to A^G, \;\; f \mapsto f(1).$$

Properties of the $A * G$-module $A$, in particular the connection with the subcategory $\sigma[{}_{A*G}A] \subset A * G\text{-Mod}$ are investigated in [35] and [37, Chap. 10]. As an example we recall the characterization of $A$ being a progenerator which parallels properties of Azumaya rings.

**6.2 $_{A*G}A$ as a progenerator in $\sigma[_{A*G}A]$.** *The following are equivalent:*

(a) $_{A*G}A$ *is a progenerator in* $\sigma[_{A*G}A]$;

(b) $\mathrm{Hom}_{A*G}(A,-) : \sigma[_{A*G}A] \to A^G$*-Mod is an equivalence of categories;*

(c) *for every $G$-invariant left ideal $I \subset A$,*

$$I = A \cdot I^G \quad \textit{and} \quad (A/I)^G \simeq A^G/I^G.$$

**6.3 Module algebras.** Let $H$ be a Hopf algebra. An (associative) $R$-algebra $A$ with unit is said to be a *left $H$-module algebra* if it has a $H$-module structure such that the map

$$A \to \mathrm{Hom}_R(H, A), \quad a \mapsto [x \mapsto x \cdot a],$$

is an algebra morphism with respect to the convolution product.

For a left H-module algebra $A$, the *smash product* $A\#H$ is defined as the $R$-module $A \otimes_R H$ with multiplication

$$(a\#h)(b \otimes g) := \sum\nolimits_i [a(h_i \cdot b)] \otimes \tilde{h}_i g, \text{ for } h, g \in H,\ a, b \in A,$$

where $\Delta(h) = \sum_i h_i \otimes \tilde{h}_i$.

If $G$ is a group acting on an algebra $A$, then $A * G \simeq A\#R[G]$, where the group ring $R[G]$ is a Hopf algebra. So the theory of module algebras generalizes the action of groups on algebras.

We consider $A$ as a left $A\#H$-module, and in particular the category $\sigma[_{A\#H}A]$. The *invariants* of any $M \in A\#H$-Mod are defined by

$$M^H := \{m \in M \mid h \cdot m = \varepsilon(h)m \text{ for all } h \in H\},$$

and the canonical map

$$\mathrm{Hom}_{A\#H}(A, M) \to M^H, \quad f \mapsto f(1_A),$$

is an $A^H$-isomorphism, where $\mathrm{End}_{A\#H}A \cong A^H \subset A$ (as algebras).

Here the Morita equivalence for module categories implies:

**6.4 $_{A\#H}A$ as a progenerator in $\sigma[_{A\#H}A]$.** *For a left $H$-module algebra $A$ the following are equivalent:*

(a) $_{A\#H}A$ *is a progenerator in* $\sigma[_{A\#H}A]$;

(b) $_{A\#H}A$ *is self-projective and for simple modules* $E \in \sigma[_{A\#H}A]$, $E^H \neq 0$;

(c) *$A$ is a generator in $\sigma[_{A\#H}A]$ and*

  (i) $A \cdot I \neq A$ *for each (maximal) left ideal* $I \subset A^H$, *or* *(ii)* $_{A^H}A$ *is faithfully flat;*

*(d)* $Hom_{A\#H}(A,-) : \sigma[_{A\#H}A] \to A^H$-Mod *is an equivalence;*

*(e) for any H-stable left ideal* $I \subset A$,

$$I = A \cdot (I \cap A^H) \text{ and } (A/I)^H \cong A^H/I^H.$$

For more detailed results concerning the $A\#H$-module structure of $A$ we refer to Gruschka [16].

Notice that in the last sections we have mainly reported about (pro-) generator properties in various comodule situations. The application of other techniques from $\sigma[M]$ (as presented in the first section) to comodule and module algebras is still under investigation.

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