Module categories with linearly ordered closed subcategories

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Abstract

In an earlier paper the first named authors investigated rings whose kernel functors are linearly ordered. The main tool for describing properties of such rings was the filter of ideals associated to a kernel functor. In the present paper more generally closed module categories (i.e. closed under kernels, cokernels and direct sums) with linearly ordered closed subcategories are studied. Properties of these categories are given and they are characterized by conditions on special objects, i.e. cogenerators or generators.

Introduction

Rings whose lattices of kernel functors are linearly ordered were introduced in [3] as a non commutative analogue of valuation rings. The lattice of topologizing filters of left ideals was used there as the main tool for deriving internal properties of those rings, taking advantage of the intertwine between the order and algebraic notions in the lattice.

Here we show how to extend some of those results by working in a more general category, that is, by using as a frame the full subcategory of M-subgenerated modules, for a given module M. Other conclusions obtained here yield new results when applied to the ring itself.

In [3] all filters over chain rings were described. Similarly, for any uniserial module M, Proposition 1.5 provides a characterization of all closed subcategories of $\sigma[M]$. If, in addition, M is a generator in $\sigma[M]$, all closed subcategories are linearly ordered (Proposition 2.1). Also in section 2, when M is a finitely generated self-projective module and closed subcategories of $\sigma[M]$ are linearly ordered, then the ideals of $\mathrm{End}_R(M)$ are linearly ordered by inclusion (Proposition 2.2). Additional necessary conditions follow: if all the closed subcategories of $\sigma[M]$ are linearly ordered, then the fully invariant submodules of every M-injective module are linearly ordered by inclusion (Proposition 2.4) and there is a unique simple module in $\sigma[M]$ (Proposition 2.7).

The modules M such that $\sigma[M]$ has a linearly ordered lattice of closed subcategories are characterized in Theorem 2.5, where this situation is shown to be equivalent to the condition that the fully invariant submodules of any big injective cogenerator in $\sigma[M]$ be linearly ordered by inclusion. In the presence of certain chain conditions (if M is locally artinian), this is the same as to require that arbitrary injective cogenerators in $\sigma[M]$ have linearly ordered lattices of fully invariant submodules, and if, moreover, M is finitely generated and self-projective, the condition on $\sigma[M]$ forces the fully invariant submodules of M to be linearly ordered as well (Corollary 2.6).

Still assuming that $\sigma[M]$ has a linearly ordered lattice of closed subcategories and letting M be finitely generated, we prove the existence of a largest closed subcategory $\mathcal{C}$ in $\sigma[M]$ which, however, need not be closed under extensions in general. For the case where $\mathcal{C}$ is extension-closed and M self-projective, our results improve those of [3]. In fact, Proposition 2.9 and 2.10 show that, in this case, $\mathcal{C}$ coincides with the class of singular modules in $\sigma[M]$, M is strongly prime and the center of $\mathrm{End}_R(M)$ is a valuation domain.

Examples of modules M for which the closed subcategories of $\sigma[M]$ are linearly ordered include all modules over left chain rings, the Prüfer groups $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-modules, homo-uniserial modules of finite length which are self-injective or self-projective, and Azumaya rings with uniserial center (see 2.12).

1 Preliminaries

Throughout the paper R will denote an associative ring with unit, *R-Mod* the category of unital left R-modules, and M a left R-module with $S := \mathrm{End}_R(M)$. For any submodule $K \subset M$, we write $p_K : M \to M/K$ for the canonical projection.

A subcategory of *R-Mod* is called *closed* if it is closed with respect to kernels, cokernels and direct sums.

By $\sigma[M]$ we denote the full subcategory of *R-Mod* whose objects are submodules of M-generated modules. Such modules are said to be *subgenerated by* M, and M is a *subgenerator* of $\sigma[M]$. This is the smallest closed full subcategory of *R-Mod* containing M.

Recall some properties of injectives and projectives (e.g., [9], section 16,17,18).

1.1 Injectives and projectives.

For any R-module N, $E(N)$ will denote the injective hull of N in *R-Mod.* For $N \in \sigma[M]$, $\widehat{N}$ is the injective hull of N in $\sigma[M]$. $\widehat{N}$ is also called the *M-injective hull* of N and is isomorphic to the trace of M in $E(N)$.

$N \in \sigma[M]$ is injective in $\sigma[M]$ if and only if N is M-injective.

M is called *self-injective* (or *quasi-injective*) if it is M-injective.

If M is self-injective and $U \subset M$ a fully invariant submodule, then the trace of U in M is equal to U, i.e., $Tr(U, M) = U$.

A module $P \in \sigma[M]$ is *projective in* $\sigma[M]$ if it is N-projective with respect to all $N \in \sigma[M]$. In general an M-projective module need not be projective in $\sigma[M]$. However, for finitely generated modules this is true.

By definition the objects of $\sigma[M]$ form a torsion class in *R-Mod.* On the other hand, every torsion class $\mathcal{T}$ in *R-Mod* is of the form $\sigma[M]$ for some module M, e.g., for M equal to the direct sum of all (non-isomorphic) cyclic modules in $\mathcal{T}$.

There is also a one to one correspondence between torsion classes in *R-Mod* and filters of left ideals in R. From this it is clear that the class of all closed subcategories of *R-Mod* is in fact a set. It is partially ordered by inclusion and for $K, L \in \sigma[M]$,

$$\sigma[K] \subset \sigma[L] \text{ if and only if } K \in \sigma[L].$$

1.2 Lattice of closed subcategories

(1) *The set of all closed subcategories of* $\sigma[M]$ *is a complete lattice. For any two* $K, L \in \sigma[M]$,

$$\sigma[K] \vee \sigma[L] = \sigma[K \oplus L],$$

and for any family $\{K_\lambda\}_\Lambda$ *of modules in* $\sigma[M]$,

$$\bigvee_\Lambda \sigma[K_\lambda] = \sigma[\bigoplus_\Lambda K_\lambda].$$

(2) *The minimal closed subcategories of* $\sigma[M]$ *are precisely those of the form* $\sigma[E]$ *with simple modules* $E \in \sigma[M]$.

(3) *If* M *is finitely generated, then there are maximal closed subcategories in* $\sigma[M]$.

Proof: (1) is obvious.

(2) This is clear since every $\sigma[K]$ contains finitely generated modules which have maximal submodules. For any simple module E, $\sigma[E]$ contains no non-trivial closed subcategories.

(3) Consider a chain $\{\sigma[N_\lambda]\}_\Lambda$, Λ an ordered set, of proper closed subcategories of $\sigma[M]$. Assume $\sigma[\bigoplus_\Lambda N_\lambda] = \sigma[M]$, i.e., M is subgenerated by $\bigoplus_\Lambda N_\lambda$. Since M is finitely generated, it is in fact subgenerated by a finite sum $N_{\lambda_1} \oplus \cdots \oplus N_{\lambda_k}$, implying - without restriction - $\sigma[M] = \sigma[N_{\lambda_k}]$, a contradiction.

Hence the set of proper closed subcategories is inductively ordered. By Zorn's Lemma it has maximal elements. □

Trying to derive properties of closed subcategories of $\sigma[M]$ by conditions on the subgenerator M we are led to the following

1.3 Definition. A closed subcategory $\sigma[N]$ of $\sigma[M]$ is called M*-dominated* if it has an M-generated subgenerator, i.e., $\sigma[N] = \sigma[M']$ for some M-generated module M'.

Obviously, if M is a generator in $\sigma[M]$ then every closed subcategory of $\sigma[M]$ is M-dominated.

Recall that for any class of modules $\mathcal{U}$ the *reject of* $\mathcal{U}$ *in* M is defined as (e.g., [9], 14.5)

$$Re(M, \mathcal{U}) = \bigcap \{\text{Ke } f \mid f \in \text{Hom}(M, U), U \in \mathcal{U}\}.$$

1.4 Lemma. *Consider* $N \in \sigma[M]$*.*

(1) $U := Re(M, \sigma[N])$ *is a fully invariant submodule of* M*.*

(2) *If* N *is* M*-generated, then* $\sigma[N] \subset \sigma[M/U]$*.*

(3) *If* M *is artinian and* N *is* M*-generated, then* $\sigma[N] = \sigma[M/U]$*.*

Proof: (1) This is a well-known property of the reject.

(2) Every M-generated module in $\sigma[N]$ is generated by M/U.

(3) M/U is cogenerated by elements of $\sigma[N]$. Since M/U is finitely cogenerated by assumption, this implies $M/U \in \sigma[N]$. Now the assertion is clear by (2).

$\square$

A module is called *uniserial* if its submodules are linearly ordered by inclusion. For these modules we obtain:

1.5 Proposition. *Assume* M *is uniserial,* $N \in \sigma[M]$ *and* $U = Re(M, \sigma[N])$*.*

(1) *For every submodule* $U \subset K \subset M$ *with* $U \neq K$*,* $M/K \in \sigma[N]$*.*

(2) *If* N *is* M*-generated, either* $\sigma[N] = \sigma[M/U]$ *or* $\sigma[N] = \sigma[P]$ *with* $P = \bigoplus \{M/V \,|\, U \subset V \subset M,\, U \neq V\}$*.*

Proof: (1) $M/U \in \sigma[N]$ implies $M/K \in \sigma[N]$.

Assume $M/U \notin \sigma[N]$ and $M/K \notin \sigma[N]$. For any $L \subset M$ with $M/L \in \sigma[N]$, $L \not\subset K$ and so $K \subset L$. This implies $K \subset \bigcap\{L \,|\, M/L \in \sigma[N]\} = U$.

(2) If $M/U \in \sigma[N]$ then $\sigma[N] = \sigma[M/U]$ (see 1.4,(2)).

Assume $M/U \notin \sigma[N]$ and consider P as defined in (2). By (1), $P \in \sigma[N]$. On the other hand, $N = \mathrm{Tr}(M, N)$. Since $M/U \notin \sigma[N]$, for every $f \in \mathrm{Hom}_R(M, N)$, U is properly contained in $\mathrm{Ke}\, f$. Hence $M \in \sigma[P]$. $\square$

1.6 Extension of categories

For two R-modules M, N denote by $\mathcal{E}(M, N)$ the full subcategory of *R-Mod* consisting of modules L which have exact sequences

$$0 \to M' \to L \to N' \to 0$$

with $M' \in \sigma[M]$ and $N' \in \sigma[N]$.

$\mathcal{E}(M,N)$ is again a closed subcategory of R-*Mod* and the corresponding filter of left ideals is the product of the filters corresponding to $\sigma[M]$ and $\sigma[N]$ (e.g., [3], [1]).

Obviously $\mathcal{E}(M,M) = \sigma[M]$ if and only if $\sigma[M]$ is closed under extensions in R-*Mod*. In this case the corresponding filter of left ideals in R is idempotent.

1.7 Definition. For any subset $X \subset S = \mathrm{End}_R(M)$ and submodule $K \subset {}_RM$ we put

$$(K:X) := \{m \in M \,|\, ms \in K \text{ for every } s \in X\}.$$

Obviously, for $s \in S$, $(K:s) = \{m \in M \,|\, ms \in K\}$ is the kernel of the morphism $M \xrightarrow{s} M \xrightarrow{p_K} M/K$.

For any $s_1, \ldots, s_k \in S$,

$$(K:\{s_1,\ldots,s_k\}) = \bigcap_{i=1}^{k} (K:s_i).$$

1.8 Lemma. *For submodules $K, L \subset M$ assume there exist $s_1, \ldots, s_k \in S$ satisfying $(K:\{s_1,\ldots,s_k\}) \subset L$.*

Then $M/L \in \sigma[M/K]$.

Proof: By assumption, $p_L : M \to M/L$ yields an epimorphism $M/(K:\{s_1,\ldots,s_k\}) \to M/L$.

For $i = 1, \ldots, k$, the kernel of $M \xrightarrow{s_i} M \xrightarrow{p_K} M/K$ is $(K:s_i)$ and hence we have a monomorphism

$$M/(K:\{s_1,\ldots,s_k\}) = M/\textstyle\bigcap_{i=1}^{k}(K:s_i) \longrightarrow (M/K)^k.$$

This implies $M/L \in \sigma[M/K]$. □

2 Closed subcategories linearly ordered

We are going to consider categories $\sigma[M]$ for which all closed subcategories are linearly ordered. It is obvious that this condition is equivalent to

$$(*) \quad \text{for any } K, L \in \sigma[M],\; K \in \sigma[L] \text{ or } L \in \sigma[K].$$

First we observe that the linear order of M-dominated closed subcategories of $\sigma[M]$ can be tested by referring to factor modules of M.

2.1 Proposition. *The following are equivalent for an R-module M:*

(a) *The M-dominated closed subcategories of $\sigma[M]$ are linearly ordered;*

(b) *for every $K, L \subset M$, $M/K \in \sigma[M/L]$ or $M/L \in \sigma[M/K]$.*

In particular, these conditions are satisfied if M is uniserial.

If, in addition, M is a generator in $\sigma[M]$, all closed subcategories of $\sigma[M]$ are linearly ordered.

Proof: $(a) \Rightarrow (b)$ This is obvious since $\sigma[M/K]$ and $\sigma[M/L]$ are M-dominated subcategories.

$(b) \Rightarrow (a)$ Consider $\sigma[N_1]$ and $\sigma[N_2]$ with N_1, N_2 both M-generated and assume $N_1 \notin \sigma[N_2]$. Then there exists $K \subset M$ such that $M/K \in \sigma[N_1]$ but $M/K \notin \sigma[N_2]$. This implies $M/K \notin \sigma[M/L]$ for all $L \subset M$ satisfying $M/L \in \sigma[N_2]$.

By (b), we conclude $M/L \in \sigma[M/K]$ whenever $M/L \in \sigma[N_2]$ and hence $\sigma[N_2] \subset \sigma[M/K] \subset \sigma[N_1]$.

Assume M is uniserial and $K \subset L$. Then $M/L \in \sigma[M/K]$. □

As an application, if R is left uniserial (i.e., R is a left chain ring), the torsion classes in R-*Mod* are linearly ordered (see Corollary 7 in [3]). Also in this case our Proposition 1.5 yields Lemma 6 in [3], as can be seen by analyzing the filters corresponding to $\sigma[R/U]$ and $\sigma[P]$ with the aid of 1.2.

With special conditions on M we obtain more characterizations of the situation described in 2.1:

2.2 Proposition. *Assume M is finitely generated and self-projective and put $S = \mathrm{End}_R(M)$. Then the following are equivalent:*

(a) *The M-dominated closed subcategories of $\sigma[M]$ are linearly ordered;*

(b) *for any submodules $K, L \subset M$ there exists a finite set $X \subset S$ such that $(K : X) \subset L$ or $(L : X) \subset K$.*

In this case the fully invariant submodules of M and the ideals of S are linearly ordered by inclusion.

Proof: $(b) \Rightarrow (a)$ By Lemma 1.8, this holds without restrictions on M.

$(a) \Rightarrow (b)$ Consider $K, L \subset M$ and assume $M/L \in \sigma[M/K]$. Then there exists a submodule $U \xrightarrow{\varepsilon} (M/K)^k$ with an epimorphism $g : U \to M/L$. Denoting by π_i the projections to the components of the product we have the following situation:

$$\begin{array}{ccccccc} M & & & U & \xrightarrow{\varepsilon} & (M/K)^k & \xrightarrow{\pi_i} & M/K \\ & {\scriptstyle p_L}\searrow & & \swarrow{\scriptstyle g} & & & & \\ & & M/L & & & & & \end{array}$$

By self-projectivity of M there exists $h : M \to U$ with $hg = p_L$ and $h\varepsilon\pi_i = s_i p_K$ for some $s_1, \ldots, s_k \in S$. Then

$$\text{Ke } h = \bigcap_{i=1}^{k} \text{Ke } h\varepsilon\pi_i = \bigcap_{i=1}^{k} \text{Ke } s_i p_K = \bigcap_{i=1}^{k} (K : s_i) = (K : \{s_1, \ldots, s_k\}),$$

and $\text{Ke } h \subset \text{Ke } p_L = L$ implies $(K : \{s_1, \ldots, s_k\}) \subset L$.

Since for any fully invariant submodule $K \subset M$ and $X \subset S$, $K \subset (K : X)$, the first assertion follows from (b).

For every ideal $I \subset S$, $I = \text{Hom}_R(M, MI)$, with MI a fully invariant submodule of M (see [9], 18.4). From this it is clear that the ideals in S are also linearly ordered. $\square$

Since R is a finitely generated projective generator in *R-Mod* our Propositions 2.1 and 2.2 yield Proposition 1 in [3], and tell us that the ideals of R are linearly ordered if this is true for the closed subcategories of *R-Mod* (see Corollary 4 in [3]).

2.3 Proposition. *Assume M is finitely generated, self-projective and*
M is artinian, or
all submodules of M are fully invariant.
Then the following are equivalent:

(a) *The M-dominated closed subcategories of $\sigma[M]$ are linearly ordered;*

(b) *the fully invariant submodules of M are linearly ordered by inclusion.*

Proof: $(a) \Rightarrow (b)$ This is clear by Proposition 2.2.

$(b) \Rightarrow (a)$ First assume M is artinian. For M-generated modules N_1, N_2, consider the fully invariant submodules $U_i := Re(M, \sigma[N_i]) \subset M$, $i = 1, 2$.

By Lemma 1.4, $\sigma[N_i] = \sigma[M/U_i]$. Hence $U_1 \subset U_2$ implies $M/U_2 \in \sigma[M/U_1]$ and $N_2 \in \sigma[N_1]$.

Now assume all submodules are fully invariant. Then (b) implies that M is uniserial and the assertion follows from Proposition 2.1. $\square$

Before characterizing the modules addressed in the title we observe:

2.4 Proposition. *Assume the closed subcategories of $\sigma[M]$ are linearly ordered. Then for every M-injective module $Q \in \sigma[M]$, the fully invariant submodules of Q are linearly ordered by inclusion.*

Proof: Consider fully invariant submodules $U, V \subset Q$. Assume $U \in \sigma[V]$,i.e., U is contained in a V-generated module A. Q injective in $\sigma[M]$ implies

$$U = \mathrm{Tr}(U, Q) \subset \mathrm{Tr}(A, Q) \subset \mathrm{Tr}(V, Q) = V.$$

$\square$

A cogenerator Q in $\sigma[M]$ is called *big* if it contains a copy of every cyclic module in $\sigma[M]$. We are now prepared to show:

2.5 Theorem. *The following are equivalent for an R-module M:*

(a) *The closed subcategories of $\sigma[M]$ are linearly ordered;*

(b) *for any finitely generated (cyclic) $K, L \in \sigma[M]$, $K \in \sigma[L]$ or $L \in \sigma[K]$;*

(c) *for any self-injective $K, L \in \sigma[M]$, $K \in \sigma[L]$ or $L \in \sigma[K]$;*

(d) *for any (some) generator $G \in \sigma[M]$ and $K, L \subset G$, $G/K \in \sigma[G/L]$ or $G/L \in \sigma[G/K]$;*

(e) *for any (some) big injective cogenerator $Q \in \sigma[M]$, the fully invariant submodules of Q are linearly ordered by inclusion.*

Assume M is a progenerator in $\sigma[M]$. Then (a)-(e) are equivalent to

(f) *for any submodules $K, L \subset M$ there exists a finite set $X \subset S$ such that $(K : X) \subset L$ or $(L : X) \subset K$.*

Proof: $(a)\Rightarrow(b)$ and $(a)\Rightarrow(c)$ are trivial.

$(b)\Rightarrow(a)$ For $K, L \in \sigma[M]$ assume $K \notin \sigma[L]$. Then there exists a cyclic submodule $K' \subset K$ with $K' \notin \sigma[L]$ and hence, for every cyclic $L' \subset L$, $K' \notin \sigma[L']$. By (b), this implies $L' \in \sigma[K']$ for all cyclic $L' \subset L$ and so $L \in \sigma[K'] \subset \sigma[K]$.

$(c)\Rightarrow(a)$ For every $K \in \sigma[M]$, $\sigma[K] = \sigma[\bar{K}]$ with $\bar{K}$ self-injective (e.g., $\bar{K}$ the K-injective hull of K).

$(a)\Leftrightarrow(d)$ This follows from Proposition 2.1.

$(a)\Rightarrow(e)$ is shown in Proposition 2.4.

$(e)\Rightarrow(b)$ Consider cyclic modules $K, L \in \sigma[M]$. Without restriction assume $K, L \subset Q$. Since $Tr(K, Q)$ and $Tr(L, Q)$ are fully invariant submodules, we have $K \subset Tr(K, Q) \subset Tr(L, Q) \in \sigma[L]$ or $L \subset Tr(L, Q) \subset Tr(K, Q) \in \sigma[K]$.

$(a)\Leftrightarrow(f)$ This follows from the second part of Proposition 2.2. □

The above characterizations do not involve any finiteness conditions on $\sigma[M]$. For locally artinian modules we can show:

2.6 Corollary. *For a locally artinian R-module M, the following are equivalent:*

(a) *The closed subcategories of $\sigma[M]$ are linearly ordered;*

(b) *for any cyclic $K, L \in \sigma[M]$, $K \in \sigma[L]$ or $L \in \sigma[K]$;*

(c) *for any (some) injective cogenerator $Q \in \sigma[M]$, the fully invariant submodules of Q are linearly ordered by inclusion.*

Proof: $(a)\Leftrightarrow(b)$ and $(b)\Rightarrow(c)$ are clear by 2.5 and 2.4.

$(c)\Rightarrow(b)$ Consider cyclic modules $K, L \in \sigma[M]$ and assume Q is an injective cogenerator in $\sigma[M]$.

M locally artinian and K cyclic implies that K is artinian and therefore finitely cogenerated (see 14.7 in [9]). Since Q is a cogenerator, K embedds in Q^n which gives K finitely cogenerated by $\mathrm{Tr}(K, Q)$.

Likewise, L is finitely cogenerated by $\mathrm{Tr}(L, Q)$.

$Tr(K, Q)$ and $Tr(L, Q)$ being fully invariant submodules, we have $Tr(K, Q) \subset Tr(L, Q)$ or $Tr(L, Q) \subset Tr(K, Q)$.

From this we conclude $K \in \sigma[L]$ or $L \in \sigma[K]$. □

We state some properties for the class of modules under consideration:

2.7 Proposition. *Assume closed subcategories of $\sigma[M]$ are linearly ordered.*

(1) *There is a unique (up to isomorphisms) simple module in $\sigma[M]$.*

(2) *Every projective module in $\sigma[M]$ is a generator in $\sigma[M]$.*

(3) *If M is finitely generated then there exists a largest proper closed subcategory in $\sigma[M]$.*

Proof: (1) This follows from the second statement in 1.2.

(2) Any projective module P in $\sigma[M]$ has a maximal submodule (e.g., [9], 22.3). Hence P generates the unique simple module in $\sigma[M]$ and is a generator in $\sigma[M]$ (by [9], 18.5).

(3) By 1.2, $\sigma[M]$ has a maximal closed subcategory. By assumption, every proper closed subcategory is contained in it. □

For our next proposition we need a lemma which might be of independent interest:

2.8 Lemma. *Assume M is projective in $\sigma[M]$ and $U \subset M$ is a non-zero fully invariant submodule. Then $M \notin \sigma[M/U]$.*

Proof: Assume $M \in \sigma[M/U]$. Then the M-injective hull $\widehat{M}$ of M is generated by M/U. With the canonical embedding $\varepsilon : M \to \widehat{M}$ we have the diagram

$$\begin{array}{ccccc} & & & & M \\ & & & & \downarrow \varepsilon \\ M^{(\Lambda)} & \xrightarrow{p_U^{(\Lambda)}} & (M/U)^{(\Lambda)} & \xrightarrow{f} & \widehat{M}, \end{array}$$

where $p_u^{(\Lambda)}$ and f are epimorphisms.

By projectivity of M, there exists $h : M \to M^{(\Lambda)}$ with $\varepsilon = hp_U^{(\Lambda)} f$ and

$$(U)\varepsilon = (U)hp_U^{(\Lambda)} f = (0)f = 0,$$

contradicting the injectivity of ε. □

The module M is called *strongly prime* if for every non-zero submodule $K \subset M$, $M \in \sigma[K]$ (see [7]).

A module $N \in \sigma[M]$ is called *singular in* $\sigma[M]$ if $N \simeq L/K$, for some $L \in \sigma[M]$ and an essential submodule $K \subset L$. The class of singular modules in $\sigma[M]$ is a torsion class (see [6]).

2.9 Proposition. *Assume M is finitely generated and self-projective and the closed subcategories of $\sigma[M]$ are linearly ordered.*

If the largest closed subcategory $\mathcal{C}$ of $\sigma[M]$ is closed under extensions, then $\mathcal{C}$ coincides with the class of singular modules in $\sigma[M]$, and M is a strongly prime module.

Proof: Denote by $c(M)$ the torsion submodule of M defined by $\mathcal{C}$. Assume $c(M) \neq 0$. This is a fully invariant submodule and hence $M \notin \sigma[M/c(M)]$ by Lemma 2.8, implying $\sigma[M/c(M)] \subset \mathcal{C}$. Since $\mathcal{C}$ is closed under extensions we conclude $M \in \mathcal{C}$, a contradiction.

In general, the Lambek torsion class

$$\mathcal{T}_L = \{K \in \sigma[M] \mid \mathrm{Hom}_R(K, \widehat{M}) = 0\}$$

is the largest torsion class closed under extensions in $\sigma[M]$ for which M is torsion-free (see [6], section 3). Since $\mathcal{C}$ is closed under extensions and M was shown to be torsion-free for $\mathcal{C}$ it follows that $\mathcal{C} = \mathcal{T}_L$.

M being projective in $\sigma[M]$, it is not singular in $\sigma[M]$ and thus $\mathcal{C}$ contains all singular module in $\sigma[M]$.

M is a generator in $\sigma[M]$ (see 2.7) and hence every finitely generated $K \in \mathcal{C}$ is isomorphic to some M^n/L. If $N \subset M^n$ and $N \cap L = 0$, we may assume $N \subset K$ and observe

$$\mathrm{Hom}_R(N, M) \subset \mathrm{Hom}_R(N, \widehat{M}) \subset \mathrm{Hom}_R(K, \widehat{M}) = 0,$$

and hence $N = 0$. Thus K is singular in $\sigma[M]$ and so $\mathcal{C}$ coincides with the class of singular modules in $\sigma[M]$.

Finally, since M is $\mathcal{C}$-torsion free, for every non-zero submodule $H \subset M$, $\sigma[H] \not\subset \mathcal{C}$, which implies that M is strongly prime. □

It should be remarked that there exist finitely generated projective modules M for which $\mathcal{C}$ (as in 2.9) fails to be closed under extensions (See Examples 2, 3 in [3]).

Notice that, applied to $M = R$, the last result gives Proposition 12 in [3].

Our following result not only extends Proposition 19 in [3] but adds information on the center of certain absolutely torsion free rings.

2.10 Proposition.

Assume M is finitely generated and self-projective, and the closed subcategories of $\sigma[M]$ are linearly ordered. Let C be the center of the ring $S = \mathrm{End}_R(M)$. Then

(1) *C is a local ring.*

(2) *If the largest closed subcategory $\mathcal{C}$ of $\sigma[M]$ is closed under extensions, C is a valuation domain.*

Proof: (1) Consider non-invertible $f, g \in C$. Assume $f + g$ is invertible with inverse h. Since $a := fh$ and $b := gh$ are in C, $(M)a$ and $(M)b$ are fully invariant submodules. Therefore, according to Proposition 2.2, we may assume $(M)a \subset (M)b$. By M-projectivity there exists $s \in S$ such that $a = sb$. Applying the argument of Proposition 19 in [3] we arrive at a contradiction. Hence C is local.

(2) Given non-zero elements $f, g \in C$, we may assume $(M)f \subset (M)g$ and so there exists $s \in S$ such that $sg = f$. To see that $s \in C$, notice first that, for every $h \in S$, $gz = zg = 0$ for $z = hs - sh$.

According to Lemma 2.8, $M \notin \sigma[M/(M)g]$ and so $\sigma[M/(M)g]$ is contained in $\mathcal{C}$. However, $(M)z$ is an onto image of $M/(M)g$ and thus $(M)z \in \mathcal{C}$, which forces $(M)z = 0$ and consequently $s \in C$.

Finally, for any $f \in C$, $f^2 = 0$ would imply $(M)f \in \sigma[M/(M)f] \subset \mathcal{C}$ and so $(M)f = 0$. This means that the valuation ring C is a domain. □

A generalization of Proposition 2 in [3] will be obtained next.

2.11 Proposition. *Let M be an R-module and $f : R \to T$ a surjective ring homomorphism. Assume the closed subcategories of $\sigma[M]$ are linearly ordered. Then the closed subcategories of T-modules in $\sigma[T \otimes_R M]$ are linearly ordered.*

Proof: Put $I := \mathrm{Ke}\, f$. Then $T \otimes_R M \simeq M/IM$, both as R- and T-modules, and so $T \otimes_R M \in \sigma[M]$. Since every R-module annihilated by I is a T-module, it can be seen that every closed subcategory of $\sigma[M/IM]$ in T-*Mod* is a torsion class in R-*Mod*. From this our assertion follows. □

We finally give some examples for the theory considered.

2.12 Examples.

Modules over left uniserial rings.

For any left uniserial (left chain) ring R, the closed subcategories of R-*Mod* are linearly ordered. Hence for any R-module M the closed subcategories of $\sigma[M]$ are linearly ordered.

Prüfer groups.

For any prime $p \in \mathbb{N}$, consider the Prüfer group $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module. Then $\sigma[\mathbb{Z}_{p^\infty}]$ consists of all abelian p-groups.

The closed subcategories of $\sigma[\mathbb{Z}_{p^\infty}]$ either contain an indecomposable module with maximal finite length or are equal to $\sigma[\mathbb{Z}_{p^\infty}]$. From this we see that they are linearly ordered.

$\mathbb{Z}_{p^\infty}$ is an injective cogenerator in $\sigma[\mathbb{Z}_{p^\infty}]$ and its (fully invariant) submodules are linearly ordered.

There are no non-trivial $\mathbb{Z}_{p^\infty}$-dominated closed subcategories in $\sigma[\mathbb{Z}_{p^\infty}]$.

$\sigma[\mathbb{Z}_{p^\infty}]$ has no finitely generated subgenerator and has no maximal closed subcategories.

Uniserial modules.

Let M be any homo-uniserial R-module of finite length, i.e., M is uniserial and all composition factors are isomorphic (see [9], 56.1). If M is self-projective or self-injective, then the closed subcategories of $\sigma[M]$ are linearly ordered.

In fact, under the given conditions M is a projective generator or an injective cogenerator and the assertion follows from 2.1 and 2.6.

Azumaya rings with uniserial center.

Let A be any (non-associative) ring with unit and center C. Denote by $M(A)$ the multiplication ring of A, i.e., the subring of $\mathrm{End}_{\mathbb{Z}}(A)$ generated by left and right multiplications with elements of A. Then A is a left $M(A)$-module and ideals in A are (fully invariant) $M(A)$-submodules. We denote by $\sigma[A]$ the smallest closed subcategory of $M(A)$-*Mod* containing A.

Assume the (two-sided) ideals of A are linearly ordered by inclusion. Then, by 2.1, the A-dominated closed subcategories of $\sigma[A]$ are linearly ordered. Such rings are also investigated in Osofsky-Resco [2].

A is called an *Azumaya ring* if A is self-projective and self-generator as an $M(A)$-module. For such rings $\mathrm{Hom}_{M(A)}(A, -) : \sigma[A] \to M(A)$-*Mod* is an equivalence of categories (e.g. [5]).

Combining the Propositions 2.1 and 2.3 we can extend the characterization of uniserial commutative rings to Azumaya rings:

For an Azumaya ring A, the following are equivalent:

(a) *Closed subcategories of $\sigma[A]$ are linearly ordered;*

(b) *the ideals of A are linearly ordered by inclusion;*

(c) *C is a uniserial ring.*

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