Fuchsian groups

(November 8, 2021 to January 31, 2022)

Oleg Bogopolski
Düsseldorf university, Germany

1 Hyperbolic Geometry

1.1 Hyperbolic Metric (Lecture 1)

The *hyperbolic plane* is the metric space $(\mathbb{H}, \rho)$, where

$$\mathbb{H} := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$$

is the upper open halfplane of complex numbers and $\rho : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is a certain metric, which we define below.

By definition, a C^1-*curve* in $\mathbb{H}$ is any C^1-map $z : [0,1] \to \mathbb{H}$. We write z in the form

$$z(t) = x(t) + iy(t),$$

where x and y are real functions. Recall that the *Euclidean length* of z is the number

$$\ell(z) := \int_0^1 \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\, dt = \int_0^1 \big|z'(t)\big|\, dt.$$

The *hyperbolic length* of z is defined to be the number

$$h(z) := \int_0^1 \frac{\sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}}{y(t)}\, dt = \int_0^1 \frac{\big|z'(t)\big|}{\mathrm{Im}(z(t))}\, dt. \tag{1.1.1}$$

The *hyperbolic distance* from a point $z_1 \in \mathbb{H}$ to a point $z_2 \in \mathbb{H}$ is defined to be the number

$$\rho(z_1, z_2) := \inf h(z), \tag{1.1.2}$$

where the infimum is taken over over all C^1-curves z in $\mathbb{H}$ with $z(0) = z_1$ and $z(1) = z_2$.

Theorem 1.1.1 The function $\rho : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is a metric on $\mathbb{H}$, i.e. for any $z_1, z_2, z_3 \in \mathbb{H}$ we have

(1) $\rho(z_1, z_1) = 0$ and $\rho(z_1, z_2) > 0$ if $z_1 \neq z_2$,

(2) $\rho(z_1, z_2) = \rho(z_2, z_1)$,

(3) $\rho(z_1, z_2) \leqslant \rho(z_1, z_2) + \rho(z_2, z_3)$.

Proof. (1) is clear, (3) will be proved in later (see Corollary 1.2.9). Now we prove (2). Let $z(t) = x(t) + iy(t)$, $t \in [0,1]$ be a C^1-curve from z_1 to z_2. We define $s(t) := 1 - t$. Then

$$w(t) := z(s(t)) = x(s(t)) + y(s(t)),$$

$t \in [0,1]$ is C^1-curve from z_2 to z_1. In the following computation we use the chain rule for the differentiation and the substution rule for the integration.

$$h(w) = \int_0^1 \frac{\left|(w'(t)\right|}{w(t)}\, dt = \int_0^1 \frac{\left|z'(s(t)) \cdot s'(t)\right|}{z(s(t))}\, dt = -\int_0^1 \frac{\left|z'(s(t))\right|}{z(s(t))} \cdot s'(t)\, dt$$
$$= -\int_1^0 \frac{\left|z'(s)\right|}{z(s)}\, ds = \int_0^1 \frac{\left|z'(s)\right|}{z(s)}\, ds = h(z).$$

□

1.2 The group of Möbius transformations

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, i.e. $a, b, c, d \in \mathbb{C},\ ad - bc = 1$. The map

$$\begin{aligned} T_A : \mathbb{C} \cup \{\infty\} &\to \mathbb{C} \cup \{\infty\} \\ z &\mapsto \frac{az+b}{cz+d} \end{aligned}$$

is called the *Möbius transformation* of $\mathbb{C} \cup \{\infty\}$ associated with A. Here we agree that

$$\frac{a\infty + b}{c\infty + d} := \begin{cases} a/c, & \text{of } c \neq 0 \ , \\ \infty, & \text{lf } c = 0 \ . \end{cases}$$

Note that

$$T_A \circ T_B = T_{AB}, \qquad T_E = id, \tag{1.2.1}$$

where E is the identity matrix. The group

$$\text{Möb}_{\mathbb{C}} := \{T_A \mid A \in \mathrm{SL}_2(\mathbb{C})\}$$

with respect to the composition is called the *group of Möbius transformations* of $\mathbb{C} \cup \{\infty\}$. The set

$$\text{Möb}_{\mathbb{R}} := \{T_A \mid A \in \mathrm{SL}_2(\mathbb{R})\}$$

forms a subgroup of $\text{Möb}_{\mathbb{C}}$.

Theorem 1.2.1 Any transformation $T_A \in \text{Möb}_{\mathbb{R}}$ maps the set $\mathbb{H}$ homeomorphically to itself.

Proof. We denote $w = T_A(z)$. Then we have

$$w = \frac{az+b}{cz+d} = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2},$$

$$\mathrm{Im}(w) = \frac{w-\bar{w}}{2i} = \frac{z-\bar{z}}{2i|cz+d|^2} = \frac{\mathrm{Im}(z)}{|cz+d|^2}. \tag{1.2.2}$$

Hence, if $\mathrm{Im}(z) > 0$ then $\mathrm{Im}(w) > 0$. Therefore T_A maps $\mathbb{H}$ to itself. The bijectivity of $(T_A)_{|\mathbb{H}} : \mathbb{H} \to \mathbb{H}$ follows from the formula $(T_A)_{|\mathbb{H}} \circ (T_{A^{-1}})_{|\mathbb{H}} = id_{|\mathbb{H}}$. The continuity of $(T_A)_{|\mathbb{H}}$ and of the inverse map $(T_{A^{-1}})_{|\mathbb{H}}$ is obvious. □

Definition 1.2.2 .

1) $(\text{Möb}_{\mathbb{R}})_{|\mathbb{H}}$ denotes the set of all maps from $\text{Möb}_{\mathbb{R}}$ restricted to $\mathbb{H}$.

2) By $Z(G)$, we denote the *center* of the group G. We have

$$Z(\mathrm{GL}_2(\mathbb{R})) = \{\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \mid r \in \mathbb{R} \setminus \{0\}\}, \qquad Z(\mathrm{SL}_2(\mathbb{R})) = \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

3) The groups

$$\mathrm{PGL}_2(\mathbb{R}) := \mathrm{GL}_2(\mathbb{R})/Z(\mathrm{GL}_2(\mathbb{R})) \qquad \text{und} \qquad \mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/Z(\mathrm{SL}_2(\mathbb{R})).$$

are called *projective general linear groups* and *projective special linear groups*.

Theorem 1.2.3 $(\text{Möb}_{\mathbb{R}})_{|\mathbb{H}} \cong \mathrm{PSL}_2(\mathbb{R})$.

Beweis. The map

$$\begin{aligned} \phi : \mathrm{SL}_2(\mathbb{R}) &\to (\text{Möb}_{\mathbb{R}})_{|\mathbb{H}} \\ A &\mapsto (T_A)_{|\mathbb{H}} \end{aligned}$$

is an epimorphism with $\mathrm{Ker}\phi = \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$. Therefore we have

$$(\text{Möb}_{\mathbb{R}})_{|\mathbb{H}} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{Ker}\phi \cong \mathrm{PSL}_2(\mathbb{R}).$$

□

Theorem 1.2.4 The transformations from $(\text{Möb}_{\mathbb{R}})_{|\mathbb{H}}$ preserve the hyperbolic lengths of C^1-curves in $\mathbb{H}$.

Proof. Let

$$\begin{aligned} T : \mathbb{H} &\to \mathbb{H}, \\ z &\mapsto \frac{az+b}{cz+d} \end{aligned}$$

be a transformation from $(\text{Möb}_{\mathbb{R}})_{|\mathbb{H}}$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Let $z(t) = x(t) + iy(t)$ be a C^1-curve in $\mathbb{H}$. Denote $w(t) := T(z(t))$. We shall prove that $h(w) = h(z)$:

$$h(w) = \int_0^1 \frac{\left|w'(t)\right|}{\mathrm{Im}(w(t))}\, dt = \int_0^1 \frac{\left|T'(z(t)) \cdot z'(t)\right|}{\mathrm{Im}(T(z(t)))}\, dt. \tag{1.2.3}$$

We have

$$T'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{1}{(cz+d)^2}.$$

Using formula (1.2.2), we deduce

$$\left|T'(z)\right| = \frac{\mathrm{Im}(T(z))}{\mathrm{Im}(z)}. \tag{1.2.4}$$

Substituting into (1.2.3), we obtain

$$h(w) = \int_0^1 \frac{\left|z'(t)\right|}{\mathrm{Im}(z(t))}\, dt = h(z).$$

□

Corollary 1.2.5 The function ρ is $\text{Möb}_{\mathbb{R}}$-invariant, i.e. for any two points $z_1, z_2 \in \mathbb{H}$ and any $T \in \text{Möb}_{\mathbb{R}}$, we have

$$\rho(z_1, z_2) = \rho(T(z_1), T(z_2)).$$

Proof. Using the definition of ρ and Theorem 1.2.4, we have

$$\rho(z_1, z_2) = \inf_{z \in \mathcal{Z}} h(z) = \inf_{z \in \mathcal{Z}} h(T(z)) = \inf_{w \in \mathcal{W}} h(w) = \rho(T(z_1), T(z_2)).$$

Here $\mathcal{Z}$ is the set of all C^1-curves from z_1 to z_2, and $\mathcal{W}$ is the set of all C^1-curves from $T(z_1)$ to $T(z_2)$. □

Definition 1.2.6 (a) For any $r \in \mathbb{R}$, let $\mathbf{A}_r$ be the open Euclidean axis in $\mathbb{H}$, whose closure begins at the point r of the real axis and is perpendicular to this axis.

b) For any two different numbers $r_1, r_2 \in \mathbb{R}$, let $\mathbf{C}_{r_1,r_2}$ be the open Euclidean half-circle in $\mathbb{H}$, whose closure begins at the point r_1 of the real axis and ends at the point r_2.

These open axes and open half-circles are called *geodesic lines* in $\mathbb{H}$.

c) For any two different points z_1, z_2 in $\mathbb{H}$, we define the curve $[z_1, z_2]$ in $\mathbb{H}$ with the beginning z_1 and the end z_2 as follows:

Case 1. $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$.

Then there exists a unique axis $\mathbf{A}_r$ containing z_1, z_2. Let $[z_1, z_2]$ be the curve which goes "inside" of $\mathbf{A}_r$ from z_1 to z_2. More precisely, let

$$[z_1, z_2](t) = (1-t)z + tz.$$

Case 2. $\mathrm{Re}(z_1) \neq \mathrm{Re}(z_2)$.

Then there exists a unique half-circle $\mathbf{C}_{r_1,r_2}$, containing z_1, z_2. Let $[z_1, z_2]$ be the "natural" curve which goes "inside" of $\mathbf{C}_{r_1,r_2}$ from z_1 to z_2.

In both cases we call the curve $[z_1, z_2]$ the *geodesic segment* from z_1 to z_2.

Lemma 1.2.7 .

(a) For any real r, there exists a transformation from $\text{Möb}_{\mathbb{R}}$, which sends $\mathbf{A}_r$ to $\mathbf{A}_0$.

(b) For any two different reals r_1, r_2, there exists a transformation from $\text{Möb}_{\mathbb{R}}$, which sends $\mathbf{C}_{r_1,r_2}$ to $\mathbf{A}_0$.

Proposition 1.2.8 Let z be a C^1-curve in $\mathbb{H}$ from z_1 to z_2. Then

$$h(z) \geqslant \left|\ln \frac{\mathrm{Im}(z_2)}{\mathrm{Im}(z_1)}\right|. \tag{1.2.5}$$

In particular,

$$\rho(z_1, z_2) \geqslant \left|\ln \frac{\mathrm{Im}(z_2)}{\mathrm{Im}(z_1)}\right|. \tag{1.2.6}$$

Proof. Denote $z(t) = x(t) + iy(t)$. Then

$$h(z) = \int_0^1 \frac{\sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}}{y(t)}\, dt \geqslant \int_0^1 \frac{\left|y'(t)\right|}{y(t)}\, dt \geqslant \left|\int_0^1 \frac{y'(t)}{y(t)}\, dt\right| = \left|\ln(y(t))\Big|_0^1\right| = \left|\ln \frac{y(1)}{y(0)}\right| = \left|\ln \frac{\mathrm{Im}(z_2)}{\mathrm{Im}(z_1)}\right|.$$

The second formula follows from the first one; use the definition of ρ as infimum of $h(z)$ over all z from z_1 to z_2. $\square$

Theorem 1.2.9 Let z_1, z_2 be two different points in $\mathbb{H}$ and let z be an arbitrary C^1-curve in $\mathbb{H}$ from z_1 to z_2. Then $h(z) \geqslant h([z_1, z_2]) > 0$.

Proof. By Lemma 1.2.7 and Theorem 1.2.4, we can assume that $z_1 = ia$ and $z_2 = ib$ $(b > a > 0)$. Denote $z(t) = x(t) + iy(t)$. Then, using formula (1.2.5), we deduce

$$h(z) \geqslant \left|\ln \frac{\mathrm{Im}(z_2)}{\mathrm{Im}(z_1)}\right| = \ln \frac{b}{a}\ .$$

It suffices to prove that

$$h([z_1, z_2]) = \ln \frac{b}{a}.$$

Denote $w = [z_1, z_2]$ and recall that $w(t) = i(a + t(b - a))$, $t \in [0, 1]$. Then

$$h(w) = \int_0^1 \frac{|w'(t)|}{\mathrm{Im}(w(t))}\, dt = \int_0^1 \frac{b - a}{a + t(b - a)}\, dt = \ln(a + t(b - a))\Big|_0^1 = \ln b - \ln a = \ln \frac{b}{a}\ .$$

$\square$

Corollary 1.2.10 .

(1) For any two points z_1, z_2 in $\mathbb{H}$ holds

$$\rho(z_1, z_2) = h([z_1, z_2]). \tag{1.2.7}$$

(2) For any two real numbers $b > a > 0$, we have

$$\rho(ia, ib) = \ln \frac{b}{a}. \tag{1.2.8}$$

(3) For any three points z_1, z_2, z_3 in $\mathbb{H}$, we have

$$\rho(z_1, z_3) \leqslant \rho(z_1, z_2) + \rho(z_2, z_3).$$

The equality happens exactly in the case where $z_2 \in [z_1, z_3]$.

Proof. Statement (1) follows from Theorem 1.2.9, statement (2) was established in the proof of this theorem. We prove (3). W.l.o.g., $z_1 = ia$, $z_3 = ib$ $(b > a > 0)$. Then

$$\rho(z_1, z_2)+\rho(z_2, z_3) \geqslant \left|\ln \frac{\mathrm{Im}(z_2)}{\mathrm{Im}(z_1)}\right|+\left|\ln \frac{\mathrm{Im}(z_3)}{\mathrm{Im}(z_2)}\right| \geqslant \ln \frac{\mathrm{Im}(z_2)}{\mathrm{Im}(z_1)}+\ln \frac{\mathrm{Im}(z_3)}{\mathrm{Im}(z_2)} = \ln \frac{\mathrm{Im}(z_3)}{\mathrm{Im}(z_1)} = \ln \frac{b}{a} = \rho(z_1, z_3).$$

$\square$

1.3 Some formulas for the hyperbolic metric ρ (Lecture 2)

Lemma 1.3.1 We have $\mathrm{SL}_2(\mathbb{R}) = \langle\{A_r, B_r \,|\, r \in \mathbb{R}\}\rangle = \langle\{A_r \,|\, r \in \mathbb{R}\} \cup \{C\}\rangle$, where

$$A_r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \qquad B_r = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Corollary 1.3.2 We have $\text{Möb}_{\mathbb{R}} = \langle\{\varphi_r \,|\, r \in \mathbb{R}\} \cup \{\psi\}\rangle$, where

$$\begin{aligned} \varphi_r : &\quad z \to z + r, \\ \psi : &\quad z \mapsto -\frac{1}{z}. \end{aligned}$$

Definition 1.3.3 The set $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called *Riemannian sphere*. The *cross-ratio* of four different points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ is defined to be

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

If some z_i equals to ∞, we first cancel the (two) terms containing ∞, using the rule

$$\frac{\infty}{\pm\infty} = \pm 1.$$

We also use some natural rules like

$$\frac{a}{\infty} = 0 \ \ (a \in \mathbb{R}), \qquad \frac{b}{0} = \infty \ \ (b \in \mathbb{R} \setminus \{0\}).$$

Theorem 1.3.4 The cross-ratio is $\text{Möb}_{\mathbb{R}}$-invariant.

Proof. The proof follows with the help of Corollary 1.3.2. □

Theorem 1.3.5 Let z, w be two different points in $\mathbb{H}$. Let z^* and w^* be the ends of the geodesic line passing through z and w. We assume that the order of the four points in the completion of this geodesic line is z^*, z, w, w^* (see Fig.1). Then

$$\rho(z, w) = \ln(w, z^*; z, w^*).$$

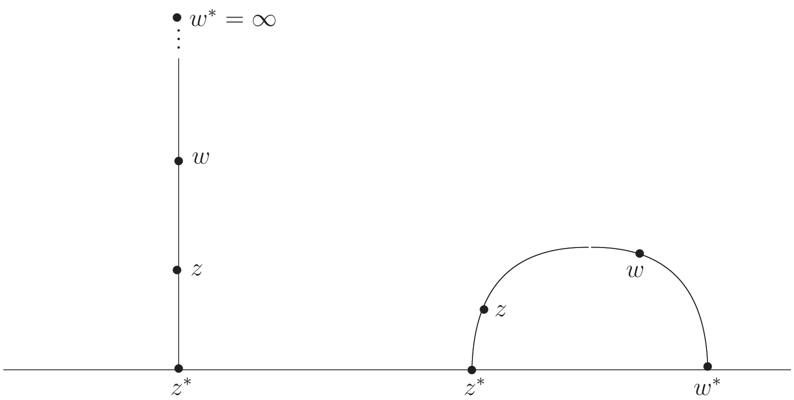

Fig. 1.

Proof. By Lemma 1.2.7, there exist $T \in \text{Möb}_{\mathbb{R}}$ which maps the geodesic line to the imaginary axis $\mathbf{A}_0$. Applying the maps θ_k and ψ, and using Theorem 1.3.4, we may assume that $T(z^*) = 0$, $T(w^*) = \infty$. Then $T(z) = ia$ and $T(w) = ib$ for some $0 < a < b$. By Corollary 1.2.10, we have

$$\rho(z,w) = \rho(T(z),T(w)) = \rho(ia,ib) = \ln\frac{b}{a}.$$

We also have

$$(w,z^*;z,w^*) = (T(w),T(z^*);T(z),T(w^*)) = (ib,0;ia,\infty) = \frac{(ib-0)(ia-\infty)}{(0-ia)(\infty-ib)} = \frac{b}{a}.$$

□

Definition 1.3.6 We define the following three functions from $\mathbb{R}$ to $\mathbb{R}$:

$$\begin{aligned}
\mathrm{ch}(t) :&= \frac{e^t+e^{-t}}{2} && \text{(hyperbolic cosinus)},\\
\mathrm{sh}(t) :&= \frac{e^t-e^{-t}}{2} && \text{(hyperbolic sinus)},\\
\mathrm{th}(t) :&= \frac{\mathrm{sh}(t)}{\mathrm{ch}(t)} = \frac{e^t-e^{-t}}{e^t+e^{-t}} && \text{(hyperbolic tangens)}
\end{aligned}$$

Theorem 1.3.7 For every two points $z, w \in \mathbb{H}$ the following formulas are valid.

$$\begin{aligned}
&(1)\quad \rho(z,w) &&= \ln\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|};\\
&(2)\quad \mathrm{ch}\,\rho(z,w) &&= 1+\frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)};\\
&(3)\quad \mathrm{sh}\left[\frac{1}{2}\,\rho(z,w)\right] &&= \frac{|z-w|}{2\,(\mathrm{Im}(z)\,\mathrm{Im}(w))^{1/2}};\\
&(4)\quad \mathrm{ch}\left[\frac{1}{2}\,\rho(z,w)\right] &&= \frac{|z-\bar{w}|}{2\,(\mathrm{Im}(z)\,\mathrm{Im}(w))^{1/2}};\\
&(5)\quad \mathrm{th}\left[\frac{1}{2}\,\rho(z,w)\right] &&= \left|\frac{z-w}{z-\bar{w}}\right|.
\end{aligned}$$

Proof. One can directly prove that these equations are equivalent. Therefore, we prove only (3).

By Theorem 1.2.4, the left side of (3) is $\text{Möb}_{\mathbb{R}}$-invariant. With the help of Corollary 1.3.2, we first verify that the right side of (3) is also $\text{Möb}_{\mathbb{R}}$-invariant. Since the right side of (3) is evidently φ_r-invariant, it suffices to check that it is ψ-invariant:

$$\frac{|\psi(z)-\psi(w)|}{2\,(\mathrm{Im}(\psi(z))\,\mathrm{Im}(\psi(w)))^{1/2}} = \frac{|\frac{-1}{z}-\frac{-1}{w}|}{2\left(\mathrm{Im}(\frac{-1}{z})\,\mathrm{Im}(\frac{-1}{w})\right)^{1/2}} = \frac{|\frac{-1}{z}-\frac{-1}{w}|}{2\left(\frac{\mathrm{Im}(z)}{|z|^2}\,\frac{\mathrm{Im}(w)}{|w|^2}\right)^{1/2}} = \frac{|z-w|}{2\,(\mathrm{Im}(z)\,\mathrm{Im}(w))^{1/2}}.$$

Therefore, after application of a suitable $T \in \text{Möb}_{\mathbb{R}}$, we may assume that $z = ia, w = ib$ $(a < b)$. Using $\rho(ia,ib) = \ln\frac{b}{a}$, we can easily verify (3). □

1.4 Isometries of $\mathbb{H}$

Definition 1.4.1 A map $f : \mathbb{H} \to \mathbb{H}$ is called an *isometry* if $\rho(f(z_1), f(z_2)) = \rho(z_1, z_2)$ for all $z_1, z_2 \in \mathbb{H}$.

The set of all Isometries the hyperbolic plane $\mathbb{H}$ is a group.[1] This group is denoted by $\mathrm{Isom}(\mathbb{H})$.

Lemma 1.4.2 The following statements hold.
(1) $(\text{Möb}_{\mathbb{R}})_{|\mathbb{H}} \leqslant \mathrm{Isom}(\mathbb{H})$.
(2) Isometries map geodesic lines to geodesic lines (see Definition 1.2.6).

Proof. (1) follows from Theorem 1.2.4. The proof of (2) can be extracted from the proof of Theorem 1.2.10. $\square$

Notation 1.4.3 We will use the following maps from $\text{Möb}_{\mathbb{R}}$:

$$\theta_k : z \mapsto kz \quad (k \in \mathbb{R}_+).$$

We introduce the map from $\mathbb{H}$ to $\mathbb{H}$ which is not from $\text{Möb}_{\mathbb{R}}$ (show this!).

$$\eta : z \mapsto -\bar{z}.$$

Theorem 1.4.4 We have

$$\begin{aligned} \mathrm{Isom}(\mathbb{H}) &= \langle (\text{Möb}_{\mathbb{R}})_{|\mathbb{H}}, \eta \rangle \\ &\cong \mathrm{PSL}_2(\mathbb{R}) \rtimes \mathbb{Z}_2. \end{aligned}$$

Beweis. We prove $\mathrm{Isom}(\mathbb{H}) \leqslant \langle (\text{Möb}_{\mathbb{R}})_{|\mathbb{H}}, \eta \rangle$. Let $\varphi \in \mathrm{Isom}(\mathbb{H})$. Then φ maps geodesic lines to geodesic lines. Let $I := \{ir \,|\, r > 0\}$. Then $\varphi(I)$ is a geodesic line. By Lemma 1.2.7, there exists $T \in (\text{Möb}_{\mathbb{R}})_{|\mathbb{H}}$ such that $T \circ \varphi(I) = I$. Using the maps $z \mapsto kz$ $(k > 0)$ and $z \mapsto -\frac{1}{z}$ if necessary, we may assume that $T \circ \varphi$ fixes the point i and maps the axes $(0, i]$ and $[i, \infty)$ onto itself. From this, it follows that the isometry $T \circ \varphi$ fixes all points on I. We prove that $T \circ \varphi \in \{id, \eta\}$. Let $z = x + iy \in \mathbb{H}$ and let $T \circ \varphi(z) = u + iv$. Then for all $t > 0$:

$$\rho(x + iy, it) = \rho(T \circ \varphi(z), T \circ \varphi(it)) = \rho(u + iv, it).$$

By Theorem 1.3.7 (3), we have

$$\frac{x^2 + (y-t)^2}{ty} = \frac{u^2 + (v-t)^2}{tv}.$$

It follows that $v = y$ and $x = \pm u$. Thus, $T \circ \varphi(z) \in \{z, -\bar{z}\}$. Since every isometry is continuous, the map $T \circ \varphi$ is either the identity on $\mathbb{H}$, or η. $\square$

[1] This will be clear only after Theorem 1.4.4.

1.5 Hyperbolic Area (Lecture 3)

Definition 1.5.1 Let A be an open set in $\mathbb{H}$. The *hyperbolic area* of A is defined by the formula

$$\mu(A) = \int_A \frac{dx\,dy}{y^2}.$$

In this integral, we consider A as a subset of $\mathbb{R}^2$.

Remark 1.5.2 *(Transformation-formula)*

- Let A be an open subset of $\mathbb{R}^2$.
- Let $\varphi : A \to \mathbb{R}^2$ be an injective differentiable function with continuous partial derivatives. We write $\varphi = (\varphi_1, \varphi_2)$ mit $\varphi_i : A \to \mathbb{R}$ and use the Jacobian of φ:

$$J(\varphi) = \det \begin{pmatrix} \frac{\partial \varphi_1}{\partial x} & \frac{\partial \varphi_1}{\partial y} \\ \frac{\partial \varphi_2}{\partial x} & \frac{\partial \varphi_2}{\partial y} \end{pmatrix}.$$

- Let $f : \varphi(A) \to \mathbb{R}$ be a continuous function. Then the function f has an integral on $\varphi(A)$ if and only if the function $(f \circ \varphi) \cdot \big|J(\varphi)\big|$ has an integral on A. In this case, we have

$$\int_{\varphi(A)} f(x,y)\,dx\,dy = \int_A (f \circ \varphi)(x,y) \cdot \Big|J(\varphi)(x,y)\Big| dx\,dy.$$

Theorem 1.5.3 The function μ is Möb($\mathbb{R}$)-invariant, i.e. for every open subset $A \subseteq \mathbb{H}$ and every $T \in (\text{Möb}_{\mathbb{R}})_{|\mathbb{H}}$ we have

$$\mu(T(A)) = \mu(A).$$

Proof. Let

$$T(z) = \frac{az+b}{cz+d} \qquad (a,b,c,d \in \mathbb{R},\ ad-bc = 1)$$

We write $z = x + iy$. Then there exist functions u, v with $T(z) = u(x,y) + iv(x,y)$. Since T is complex-differentiable (holomorph), u and v satisfy Cauchy-Riemann equations

$$\begin{aligned} \tfrac{\partial u}{\partial x} &= \tfrac{\partial v}{\partial y}, \\ \tfrac{\partial u}{\partial y} &= -\tfrac{\partial v}{\partial x}. \end{aligned}$$

Using these equations, we compute the Jacobian of the map

$$\varphi : (x,y) \mapsto (u(x,y), v(x,y)).$$

$$J(\varphi) := \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\right|^2 = \left|\frac{dT}{dz}\right|^2 = \frac{1}{|cz+d|^4}.$$

Then[2]

$$\mu(T(A)) = \int_{T(A)} \frac{dx\,dy}{(\mathrm{Im}(z))^2} = \int_A \frac{1}{(\mathrm{Im}(T(z)))^2} \cdot |J(T)|\,dx\,dy$$

$$\overset{(1.2.2)}{=} \int_A \frac{|cz+d|^4}{(\mathrm{Im}(z))^2} \cdot \frac{1}{|cz+d|^4}\,dx\,dy = \mu(A).$$

□

1.6 Angles in $\mathbb{H}$

Definition 1.6.1 Let $\gamma_1 : [c_1, d_1] \to \mathbb{H}$ and $\gamma_2 : [c_2, d_2] \to \mathbb{H}$ be two injective diffrentiable curves in $\mathbb{H}$, which pass through some common point z. The *hyperbolic angle* between γ_1 and γ_2 at the point z is the Euclidian angle between the tangent lines ζ_1 and ζ_2 to these curves at the point z:

$$\angle(\gamma_1, \gamma_2; z) := \angle_{\mathrm{e}}(\zeta_1, \zeta_2; z).$$

Theorem 1.6.2 The transformations $T \in (\mathrm{M\ddot{o}b}_{\mathbb{R}})_{|\mathbb{H}}$ are conform, i.e. they preserve the orientation and angles between C^1-curves:

$$\angle(T(\gamma_1), T(\gamma_2); T(z)) = \angle(\gamma_1, \gamma_2; z).$$

1.7 Gauß-Bonnet formula

A *hyperbolic n-gon* in $\mathbb{H}$ is a closed subset of $\mathbb{H}$, which is bounded by n hyperbolic segments of the form $[z, w]$. We also consider *hyperbolic n-gons* in the extension $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$.

Theorem 1.7.1 (Gauß-Bonnet) Let Δ be a hyperbolic triangle in $\overline{\mathbb{H}}$ with the angles α, β, γ. Then

$$\mu(\Delta) = \pi - \alpha - \beta - \gamma.$$

Proof. Let $\Delta = ABC$.

Case 1. Let $A, B \in \mathbb{H}$ and $C \in \mathbb{R} \cup \{\infty\}$.

Since Möbius transformations preserve areas and angles, we may assume that $C = \infty$.

Then the side AB lies on the halfcircle $\mathbf{C}_{r_1,r_2}$. We may assume that 0 is the center of $\mathbf{C}_{r_1,r_2}$. Let R be the radius of this halfcircle. Sine the sides AC and BC are geodesic lines and $C = \infty$, they are vertical axes. Let a and b be x-coordinates of these axes. Then

$$\mu(\Delta) = \int_\Delta \frac{dxdy}{y^2} = \int_a^b dx \int_{\sqrt{R^2-x^2}}^\infty \frac{dy}{y^2} = \int_a^b \frac{dx}{\sqrt{R^2-x^2}}.$$

After substitution $x = R\cos\theta$, we obtain

$$\mu(\Delta) = \int_{\pi-\alpha}^{\beta} \frac{-\sin\theta\,d\theta}{\sin\theta} = \pi - \alpha - \beta.$$

[2]In this case, we have $f : (x, y) \mapsto \frac{1}{y^2}$ and $\varphi : (x, y) \mapsto (u, v)$.

Case 2. Let $A, B, C \in \mathbb{H}$.
Case 3. Let $A \in \mathbb{H}$ und $B, C \in \mathbb{R} \cup \{\infty\}$.
Case 4. Let $A, B, C \in \mathbb{R} \cup \{\infty\}$.

These cases can be reduced to Case 1. □

1.8 Hyperbolic trigonometry

Theorem 1.8.1 Let Δ be a geodesic triangle in $\mathbb{H}$ with finite hyperbolic lengths a, b, c of its sides and with the non-zero angles α, β, γ (opposite to the corresponding sides). Then

(1) $\dfrac{\operatorname{sh} a}{\sin \alpha} = \dfrac{\operatorname{sh} b}{\sin \beta} = \dfrac{\operatorname{sh} c}{\sin \gamma}$. (sinus theorem)

(2) $\operatorname{ch} c = \operatorname{ch} a \operatorname{ch} b - \operatorname{sh} a \operatorname{sh} b \cos \gamma$. (first cosinus theorem)

(3) $\operatorname{ch} c = \dfrac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}$. (second cosinus theorem)

Theorem 1.8.2 If two geodesic triangles in $\mathbb{H}$ have the same angles, then there exists an isometry which maps one triangle to the other.

Theorem 1.8.3 (Pythagoras Theorem for $\mathbb{H}$) Let Δ be a geodesic triangle in $\mathbb{H}$ with finite hyperbolic lengths a, b, c of its sides and with the non-zero angles α, β, γ (opposite to the corresponding sides). If $\gamma = \frac{\pi}{2}$, then

$$\operatorname{ch} c = \operatorname{ch} a \operatorname{ch} b.$$

2 Fuchsian groups

2.1 Classification of elements of $\mathrm{PSL}_2(\mathbb{R})$

Definition 2.1.1 Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of the group $\mathrm{SL}_2(\mathbb{R})$ different from $\pm E$. The number

$$\mathrm{Tr}(A) := a + d$$

is called the *trace* of A.

- A is called *elliptic* if $|\mathrm{Tr}(A)| < 2$.
- A is called *parabolic* if $|\mathrm{Tr}(A)| = 2$.
- A is called *hyperbolic* if $|\mathrm{Tr}(A)| > 2$.

A nontrivial element from $\mathrm{PSL}_2(\mathbb{R})$ is called elliptic, parabolic, or hyperbolic if some (equivalently any) its preimage in $\mathrm{SL}_2(\mathbb{R})$ is elliptic, parabolic, or hyperbolic, respectively.

We write $A \underset{\mathrm{SL}_2(\mathbb{R})}{\sim} B$ if A and B are conjugate by a matrix from $\mathrm{SL}_2(\mathbb{R})$.

Lemma 2.1.2 Suppose that $\pm E \neq A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, and let λ_1, λ_2 be eigenvectors of A. Then the following holds.

1) A is hyperbolic if and only if $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$. In this case

$$A \underset{\mathrm{SL}_2(\mathbb{R})}{\sim} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$, $\lambda \neq \pm 1$.

2) A is parabolic if and only if $\lambda_1 = \lambda_2 \in \{-1, 1\}$. In this case

$$A \underset{\mathrm{SL}_2(\mathbb{R})}{\sim} \pm \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

for some $\alpha \in \{-1, 1\}$.

3) A is elliptic if and only if $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. In this case $\lambda_2 = \overline{\lambda}_1$, $|\lambda_1| = |\lambda_2| = 1$ and

$$A \underset{\mathrm{SL}_2(\mathbb{R})}{\sim} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in (0, \pi) \cup (\pi, 2\pi)$.

Definition 2.1.3 The element $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}$ as the map $z \mapsto \frac{az+b}{cz+d}$. The set of *fixed points* of A in $\mathbb{H}$ is

$$\mathrm{Fix}(A) := \{ z \in \mathbb{H} \,|\, z = \frac{az+b}{cz+d} \}.$$

The action of A on $\mathbb{H}$ can be naturally extended to an action of A on the compactified hyperbolic plane $\widehat{\mathbb{H}} := \mathbb{H} \cup \partial\mathbb{H}$, where $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. The subset $\partial\mathbb{H}$ is called the *boundary* of $\mathbb{H}$. The set of fixed points of A in $\widehat{\mathbb{H}}$ is denoted by $\widehat{\mathrm{Fix}}(A)$.

Theorem 2.1.4 For any $A \in \mathrm{PSL}_2(\mathbb{R})$ the following holds.

1) If A is hyperbolic, then $\widehat{\mathrm{Fix}}(A)$ consists of two points in $\partial\mathbb{H}$. One of them is *attracting* and the other one is *repelling*.

2) If A is parabolic, then $\widehat{\mathrm{Fix}}(A)$ consists of a single point in $\partial\mathbb{H}$.

3) If A elliptic, then $\widehat{\mathrm{Fix}}(A)$ consists of a single point in $\mathbb{H}$.

Definition 2.1.5 Suppose that $A \in \mathrm{PSL}_2(\mathbb{R})$ is hyperbolic. The geodesic line in $\mathbb{H}$ connecting two fixed points of A is called the *axis* of A and is denoted by $\mathrm{Axis}(A)$.

Bemerkung 2.1.6 The axis of a hyperbolic element $A \in \mathrm{PSL}_2(\mathbb{R})$ is A-invariant.

Bemerkung 2.1.7 If we know the type of A (hyperbolic, parabolic, or elliptic) and the fixed points of A, we can describe the action of A on $\widehat{\mathbb{H}}$ on a qualitative level, i.e. with the help of pictures.

Bemerkung 2.1.8 The element $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ fixes ∞ if and only if $c = 0$. In this case the corresponding transformation $\varphi_A : \mathbb{C} \to \mathbb{C}$ has the form $z \mapsto a^2 z + ba$ and we have the following.

- If $a = \pm 1$, then A is parabolic.
- If $a \neq \pm 1$, then A is hyperbolic with $\mathrm{Fix}(A) = \{\infty, \dfrac{ba}{1-a^2}\}$.

Subsections 2.2 and 2.3 contain necessary information about topological spaces and topological groups.

2.2 Topological spaces

Definition 2.2.1 (Topology, topological space, open and closed sets)

Let X be a set. A *topology* $\mathfrak{T}$ on X is a set of some subsets of X (each such subset is called an *open set* in X), which satisfies the following axioms:

(1) The empty set $\varnothing$ and the set X are open.

(2) The intersection of finitely many open sets is open.

(3) the union of arbitrary set of open sets is open

The pair $(X, \mathfrak{T})$ is called a *topological space*. Sometimes we simply write X for the topological space if the topology $\mathfrak{T}$ is defined. A subset U of X is called *closed* if $X \setminus U$ is open.

Definition 2.2.2 (Basis of a topology)

Let $(X, \mathfrak{T})$ be a topological space. A subset $B \subseteq \mathfrak{T}$ is called a *basis* of the topology $\mathfrak{T}$ if each open set of X is a union of some open sets belonging to B.

Definition 2.2.3 (Neighborhood) Let $(X, \mathfrak{T})$ be a topological space and let x be a point of X. A subset U of X is called a *neighborhood of x* if there exists an open subset $\mathcal{O}$ such that $x \in \mathcal{O} \subseteq U$.

Definition 2.2.4 (Continuous maps)

Let $(X_1, \mathfrak{T}_1)$ and $(X_2, \mathfrak{T}_2)$ be two topological spaces. A map $f : X_1 \to X_2$ is called *continuous* if for every open set U in X_2 the full preimage $f^{-1}(U)$ is open in X_1.

Definition 2.2.5 (Induced topologie und quotient topology)

(1) Let $(X, \mathfrak{T})$ be a topological space and let Y be a subset of X. We define on Y the *induced topology* $\mathfrak{T}_Y$ as follows: A subset $S \subseteq Y$ is defined to be open in Y if there exists an open set $\mathcal{O}$ in X such that $S = \mathcal{O} \cap Y$.

(2) Let $(X, \mathfrak{T})$ be a topological space and let Y be a set. Let $f : X \to Y$ be a map. We define on Y the *quotient topology* as follows: A subset $U \subseteq Y$ is defined to be open in Y if the full preimage $f^{-1}(U)$ is open in X.

Remark 2.2.6

(1) Let $(X, \mathfrak{T})$ be a topological space and let Y be a subset of X. The inclusion map $i : Y \to X$, $y \mapsto y$, becomes continuous if we endow Y with the induced topology. Moreover, the induced topology on Y is the weakest topology on Y for which the map i is continuous.

(2) Let $(X, \mathfrak{T})$ be a topological space and let Y be a set. Let $f : X \to Y$ be a map. The map $f : X \to Y$ becomes continuous if we endow Y with the quotient topology. Moreover, the quotient topology on Y is the weakest topology on Y for which the map f is continuous.

Definition 2.2.7 (Discrete topological space and a discrete subset of a topological space)

(1) A topological space $(X, \mathfrak{T})$ is called *discrete* if one from two equivalent statements is valid:

   (a) For every point $x \in X$ the set $\{x\}$ is open.

   (b) Every subset of X is open.

(2) A subset X of a topological space Y is called *discrete* if X with the induced topology is discrete. Equivalently, if for every point $x \in X$ there exists an open set $\mathcal{O}_x$ in Y such that $\mathcal{O}_x \cap X = \{x\}$.

Definition 2.2.8 (Hausdorff space) A topological space $(X, \mathfrak{T})$ is called Hausdorff space if for every two different points x_1 and x_2 in X there exist open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that

$$x_1 \in \mathcal{O}_1, \;\; x_2 \in \mathcal{O}_2 \;\text{ and }\; \mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing.$$

Definition 2.2.9 (First countable topological spaces) A topological space $(X, \mathfrak{T})$ is called *first countable* if the following holds:

For every point $x \in X$, there exists a countable collection of neihborhoods $U_1, U_2, \ldots$ of x such that, for every neighborhood U of x, there exists $i \in \mathbb{N}$ such that $U_i \subseteq U$.

Remark 2.2.10 (Continuity and countable sequences)

(1) Let $(X, \mathfrak{T})$ be a topological space and let $(Y, \mathfrak{T}_Y)$ be a subspace with the induced topology. Then the following holds:

(i) If $(X, \mathfrak{T})$ is first countable, then $(Y, \mathfrak{T}_Y)$ is first countable.

(ii) Let $(y_n)_{n\in\mathbb{N}} \subseteq Y$ be a sequence and $y \in Y$ a point. Then $y_n \to y$ in $(Y, \mathfrak{T}_Y)$ if and only if $y_n \to y$ in $(X, \mathfrak{T})$.

(2) Let $(X_1, \mathfrak{T}_1)$ and $(X_2, \mathfrak{T}_2)$ be two topological spaces. Let $f : X_1 \to X_2$ be a map. We will compare the following two conditions.

(a) The map f is continuous.

(b) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X_1 and every $x \in X_1$ the following holds: If $x_n \to x$ in $(X_1, \mathfrak{T}_1)$, then $f(x_n) \to f(x)$ in $(X_2, \mathfrak{T}_2)$.

Condition (b) follows from condition (a), but not conversely, in general.
If $(X_i, \mathfrak{T}_i)$, $i = 1, 2$, are first countable then (a) and (b) are equivalent.

Remark 2.2.11 (Criterium of discreteness) Let $(X, \mathfrak{T})$ be a first countable topological space. Then this space is discrete if for every point $x \in X$ and every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \to x$ we have $x_n = x$ for all sufficiently large n.

Definition 2.2.12 (Compact spaces and compact subsets)

(1) A topological space $(X, \mathfrak{T})$ is called *compact* if for each covering $X = \bigcup\limits_{i\in I} U_i$ with $U_i \in \mathfrak{T}$, there exists a finite subset $I_0 \subseteq I$ such that $X = \bigcup\limits_{i\in I_0} U_i$.

(2) A subset Y of a topological space $(X, \mathfrak{T})$ is called *compact* if one of the following equivalent conditions is satisfied:

(a) $(Y, \mathfrak{T}_Y)$ is a compact space.

(b) For each covering $Y \subseteq \bigcup\limits_{i\in I} U_i$ with $U_i \in \mathfrak{T}$, there exists a finite subset $I_0 \subseteq I$ such that $Y \subseteq \bigcup\limits_{i\in I_0} U_i$.

Remark 2.2.13 (Compactness for metric spaces)
Let (X, d) be a metric space. For $r > 0$ and $x \in X$ the set

$$B_r(x) := \{x' \in X \,|\, d(x, x') < r\}$$

is called an *open ball* with the center x and radius r. A subset Y of X is called *bounded* if Y lies in some open ball.

Let $\mathfrak{T}$ be the topology on X for which the set $\{B_r(x) \,|\, x \in X, r > 0\}$ is a basis. Then

(1) The topological space $(X, \mathfrak{T})$ is Hausdorff and first countable.
(2) A subset Y in $\mathbb{R}^n$ is compact if and only if Y is closed and bounded.

Lemma 2.2.14 Every discrete and closed subset M of a compact topological space X is finite.

Proof. For every $m \in M$ let $O(m)$ be an open neighborhood of m in X satisfying $O(m) \cap M = \{m\}$. Then

$$X = (X \setminus M) \cup (\bigcup_{m \in M} O(m))$$

is a covering of X by open sets. Since X is compact, there exists a finite subset $M_0 \subseteq M$ such that

$$X = (X \setminus M) \cup (\bigcup_{m \in M_0} O(m)).$$

Then

$$M = (\bigcup_{m \in M_0} O(m)) \cap M = \bigcup_{m \in M_0} (O(m) \cap M) = M_0.$$

□

Definition 2.2.15 (Product topology)
Let $(X_1, \mathfrak{T}_1)$ and $(X_2, \mathfrak{T}_2)$ be two topological spaces. The *product topology* $\mathfrak{T}$ on $X_1 \times X_2$ consists of all possible unions of sets of the form $U \times V$, where $U \in \mathfrak{T}_1$ and $V \in \mathfrak{T}_2$.

Remark 2.2.16 The product topology is the weakest topology on $X_1 \times X_2$, for which the projections $\mathrm{pr}_1 : X_1 \times X_2 \to X_1$ and $\mathrm{pr}_2 : X_1 \times X_2 \to X_2$ are continuous.

2.3 Topological groups

Definition 2.3.1 A group G, which is simultaneously a topological space, is called a *topological group*, if the maps $\cdot : G \times G \to G$, $(x, y) \mapsto xy$ and ${}^{-1} : G \to G$, $x \mapsto x^{-1}$ are continuous.

Remark 2.3.2 Let G be a topological group. Then the following statements are valid.

(1) For every open set $U \subseteq G$ and any element $g \in G$ the sets gU and Ug are open.

(2) For every neighborhood V of 1 in G, there exists a neighborhood U of 1 in G such that $UU^{-1} \subseteq V$.

Proof. (2) Since $1 \cdot 1 = 1$ and since the multiplication in G is continuous, there exist two neighborhoods of 1, say U_1, U_2 such that $U_1 U_2 \subseteq V$. For $U_3 = U_1 \cap U_2$ we have $U_3 U_3 \subseteq V$. For $U = U_3 \cap U_3^{-1}$ we finally have $UU^{-1} \subseteq V$. □

Definition 2.3.3 A subgroup H of a topological group G is called *discrete* in G if H is discrete as a subset of the topological space G.

Theorem 2.3.4 Every discrete subgroup H of a Hausdorff topological group G is closed.

Proof. Since H is discrete in G, there exists an open neighborhood V of 1 such that $V \cap H = \{1\}$. Then there exists an open neighborhood U of 1 such that $UU^{-1} \subseteq V$.

We show that $G \setminus H$ is open. Let $g \in G \setminus H$. We shall show that there exists an open neighborhood W of g which does not contain elements of H. Try $W = Ug$. Suppose it contains some $h \in H$. Then, since G is Hausdorff, we can find an open neighborhood $W_1 \subseteq W$ of g, which does not contain h. Suppose that W_1 contains some other $h_1 \in H$. Then $hg^{-1}, h_1 g^{-1} \in U$. Then $hh_1^{-1} = (hg^{-1})(h_1 g^{-1})^{-1} \in UU^{-1} \subseteq V$, hence $hh_1^{-1} = 1$, a contradiction. Thus, W_1 is an open neighborhood of g that does not contain elements of H. Hence, $G \setminus H$ is open. □

Remark 2.3.5 The variant of Theorem 2.3.4 for subspaces of topological spaces is not valid. Indeed, consider the interval $[0, 1]$ as a topological space with the topology that is induced by the canonical topology on $\mathbb{R}$. Then the topological space $[0, 1]$ is Hausdorff and compact. Die subset $\{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ of the topological space $[0, 1]$ is discrete but not closed.

Corollary 2.3.6 Every discrete subgroup H of a Hausdorff kompact topological group G is finite.

Proof. The proof follows straightforwardly from Theorem 2.3.4 and Lemma 2.2.14. □

Corollary 2.3.7 Every discrete subgroup of the orthogonal group $O(n)$ is finite.

Lemma 2.3.8 (Criterium of discreteness of a topological subgroup) Let G be a topological group with a first countable Hausdorff topology. Then the following holds:

A subgroup H of G is discrete in G if and only if from $h_n \to e$ (where $h_n \in H$ and e is a neutral element) follows that $h_n = e$ for all sufficiently large n.

2.4 First two definitions of a Fuchsian group

The group $\mathrm{SL}_2(\mathbb{R})$ can be considered as a subset of $\mathbb{R}^4$ by identifying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow (a, b, c, d)$$

Thus, $\mathrm{SL}_2(\mathbb{R})$ can be considered as a topological group (and even a metric space) with the topology induced from $\mathbb{R}^4$. Now we consider the canonical epimorphism

$$\psi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R}).$$

From now on, we consider $\mathrm{PSL}_2(\mathbb{R})$ as a topological group with the quotient topology determined by ψ.

Definition 2.4.1 (first definition of a Fuchsian group) A Fuchsian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

To understand the discreteness in $\mathrm{PSL}_2(\mathbb{R})$ better, we need the following two general statements.

Lemma 2.4.2 Let G_1 be a topological group, G_2 be a group and $\varphi : G_1 \to G_2$ an epimorphism. We consider G_2 as a topological group with respect to the quotient topology. Let H be a subgroup of G_2. Then the following statements are valid.

(a) If $\varphi^{-1}(H)$ is discrete in G_1, then H is discrete in G_2.

(b) Suppose additionally that G_1 is Hausdorff and that $\ker(\varphi)$ is finite.
If H is discrete in G_2, then $\varphi^{-1}(H)$ is discrete in G_1.

Lemma 2.4.3 Let G_1 be a topological group, G_2 be a group. Let $\varphi : G_1 \to G_2$ be an epimorphism and G_2 is endowed by the quotient topology. Then the following statements are valid.

(1) If a subset $\mathcal{O} \subseteq G_1$ is open, then its image $\varphi(\mathcal{O})$ is open.

(2) If G_1 is first countable, then G_2 is also first countable.

Corollary 2.4.4 The topological group $\mathrm{PSL}_2(\mathbb{R})$ is Hausdorff and first countable.

Using this corollary and the discreteness criterium 2.3.8, we give the second (equivalent) definition of a Fuchsian group.

Definition 2.4.5 (second definition of a Fuchsian group) A subgroup $H \leqslant \mathrm{PSL}_2(\mathbb{R})$ is called Fuchsian if for any sequence $(h_n)_{n\in\mathbb{N}}$ of elements of H with $h_n \to 1$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$, we have $h_n = 1$.

2.5 Proper discontinuous actions of groups on metric spaces

Definition 2.5.1 Let X be a topological space and S a subset of X. A point $x \in X$ is called an *accumulation point* of S if every neighborhood of x contains a point of S different from x. The set of all accumulation points of S in X is denoted by $\mathbf{AP}_X(S)$.

Assumption. From now on we assume that (X, d) is a metric space and G is a group acting on X *by isometries*, i.e. for any element $g \in G$ and any two points $x_1, x_2 \in X$ we have

$$d(x_1, x_2) = d(g(x_1), g(x_2)).$$

An action (by isometries) of G on X is denoted by $G \curvearrowright X$.

Definition 2.5.2 An action $G \curvearrowright X$ is called *proper discontinuous* (abbreviated PDA), if for every point $x \in X$ there exists a neighborhood V of x such that

$$|\{g \in G \,|\, g(V) \cap V \neq \emptyset\}| < \infty. \tag{2.5.1}$$

Lemma 2.5.3 A group G acts on a metric space X properly discintinuously if and only if the folowing two conditions are satisfied.

(1) For any $x \in X$ the orbit $G(x)$ does not have an accumulation point in X.

(2) For any $x \in X$ the stabilizer $\mathrm{St}_G(x) := \{g \in G \,|\, g(x) = x\}$ is finite.

Proof. $\boxed{(1)\,\&(2) \Rightarrow \text{(PDA):}}$

Let $x \in X$ be an arbitrary point. It follows from (1) that there exists $\varepsilon > 0$ with

$$B_\varepsilon(x) \cap G(x) = \{x\}.$$

We claim that $V := B_{\varepsilon/2}(x)$ satisfies (2.5.1). Indeed, if $g \in G$ is an element satisfying

$$g\big(B_{\varepsilon/2}(x)\big) \cap B_{\varepsilon/2}(x) \neq \emptyset, \tag{2.5.2}$$

then $d(x, g(x)) < \varepsilon$, hence

$$g(x) \in B_\varepsilon(x) \cap G(x) = \{x\},$$

i.e. $g \in \mathrm{St}_G(x)$. By (2), this stabilizer is finite. Thus, there exists only finitely many $g \in G$ satisfying (2.5.2), and (PDA) is proved.

$\boxed{\neg(2) \Rightarrow \neg(\mathrm{PDA})}$ is evident. Now we prove

$\boxed{\neg(1) \Rightarrow \neg(\mathrm{PDA}):}$

By $\neg(1)$, there exists $x \in X$ such that $G(x)$ has an accumulation point y. Then there exist different $g_i \in G$, $i \in \mathbb{N}$, such that $g_i(x) \in B_{1/i}(y) \setminus \{y\}$. In particular,

$$g_i^{-1}\big(B_{1/i}(y)\big) \cap g_j^{-1}\big(B_{1/j}(y)\big) \neq \emptyset$$

for all i, j. Therefore, for $j \geqslant i$ we have

$$g_j g_i^{-1}\big(B_{1/i}(y)\big) \cap B_{1/i}(y) \neq \emptyset.$$

Let $\varepsilon > 0$. We take $i_0 \in \mathbb{N}$ such that $1/i_0 < \varepsilon$. Then

$$g_j g_{i_0}^{-1}\big(B_\varepsilon(y)\big) \cap B_\varepsilon(y) \neq \emptyset.$$

for all $j \geqslant i_0$. This implies $\neg$(PDA). □

Lemma 2.5.4 Let $z_0 \in \mathbb{H}$ and let K be a compact subset of $\mathbb{H}$. Then the set

$$M := \{T \in \mathrm{SL}_2(\mathbb{R}) \,|\, T(z_0) \in K\}$$

is compact.

Proof. It suffices to show that M (as a subset of $\mathbb{R}^4$) is closed and bounded.

1) We prove that M is closed. Consider the map

$$\psi : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{H}, \qquad T \mapsto T(z_0).$$

Then

$$M = \{T \in \mathrm{SL}_2(\mathbb{R}) \,|\, \psi(T) \in K\} = \psi^{-1}(K).$$

Since ψ is continuous and K is closed, M is closed as well.

2) We prove that M is bounded.

a) Since K is bounded, there exists a constat $C_1 > 0$ such that $|z| < C_1$ for all $z \in K$. Then we have

$$\left|\frac{az_0 + b}{cz_0 + d}\right| < C_1 \tag{2.5.3}$$

for all $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from M.

b) Since K is bounded, there exists a constant $C_2 > 0$ such that

$$\operatorname{Im}\Big(\frac{az_0+b}{cz_0+d}\Big) \geqslant C_2.$$

By (1.2.2), we have

$$\frac{\operatorname{Im}(z_0)}{|cz_0+d|^2} \geqslant C_2.$$

Therefore

$$|cz_0+d| \leqslant \sqrt{\frac{\operatorname{Im}(z_0)}{C_2}}. \tag{2.5.4}$$

This and (2.5.3) imply

$$|az_0+b| \leqslant C_1\sqrt{\frac{\operatorname{Im}(z_0)}{C_2}}. \tag{2.5.5}$$

From (2.5.4) and (2.5.5) we deduce that $|a|, |b|, |c|, |d|$ are bounded. □

Theorem 2.5.5 A subgroup G of $\mathrm{PSL}_2(\mathbb{R})$ is Fuchsian if and only if G acts properly discontinuously on $\mathbb{H}$.

Proof. Let $\psi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ be the canonical epimorphism. Then (tutorial)

- G is discrete in $\mathrm{PSL}_2(\mathbb{R})$ if and only if $\psi^{-1}(G)$ discrete in $\mathrm{SL}_2(\mathbb{R})$.
- G acts on $\mathbb{H}$ totally discontinuously if and only if $\psi^{-1}(G)$ acts on $\mathbb{H}$ totally discontinuously.

Therefore it suffices to prove the following:

A subgroup G of $\mathrm{SL}_2(\mathbb{R})$ is Fuchsian if and only if G acts properly discontinuously on $\mathbb{H}$.

1) Suppose that G is discrete in $\mathrm{SL}_2(\mathbb{R})$.

To the contrary, suppose that the action $G \curvearrowright \mathbb{H}$ is not proper discontinuous. Then there exists $z_0 \in \mathbb{H}$ such that the set

$$G_0 := \{g \in G \,|\, g(B_1(z_0)) \cap B_1(z_0) \neq \emptyset\}$$

is infinite. We set $K := \overline{B_2(z_0)}$. Then $g(z_0) \in K$ for all $g \in G_0$. Therefore

$$G_0 \subseteq \{g \in \mathrm{SL}_2(\mathbb{R}) \,|\, g(z_0) \in K\} \cap G.$$

- The set $\{g \in \mathrm{SL}_2(\mathbb{R}) \,|\, g(z_0) \in K\}$ is compact (see Lemma 2.5.4).
- Since G is discrete in $\mathrm{SL}_2(\mathbb{R})$, G is closed (see Lemma 2.3.4).
- The intersection of a compact set and a discrete closed set is finite (see Lemma 2.2.14).

Then G_0 is finite. A contradiction.

2) Suppose G is not discrete in $\mathrm{SL}_2(\mathbb{R})$. Then there exists an infinite sequence $(g_k)_{k\in\mathbb{N}}$ of different and nontrivial elements of G such thatz $g_k \to e$. Each element from $\mathrm{SL}_2(\mathbb{R})$ fixes at most one point of $\mathbb{H}$. Let $z_0 \in \mathbb{H}$ be a point, which is not fixed by any g_k. Then

- $g_k(z_0) \neq z_0$ for all k,
- $g_k(z_0) \to z_0$.

Thus, z_0 is an accumulation point of $G(z_0)$. By Lemma 2.5.3 the action $G \curvearrowright \mathbb{H}$ is not proper discontinuous. $\square$

Corollary 2.5.6 (Criterium for Fuchsian groups) A subgroup $G \leqslant \mathrm{PSL}_2(\mathbb{R})$ is Fuchsian if and only if for every point $z \in \mathbb{H}$ the orbit $G(z)$ does not have accumulation points in $\mathbb{H}$.

Proof. ($\Rightarrow$) follows from Theorem 2.5.5 and Lemma 2.5.3.
($\Leftarrow$) Suppose that G is not discrete. As in the second part of the proof of Theorem 2.5.5, there exists a point $z_0 \in \mathbb{H}$ such that the orbit $G(z_0)$ has an accumulation point in $\mathbb{H}$. A contradiction. $\square$

Definition 2.5.7 The *limit set* of a subgroup $G \leqslant \mathrm{PSL}_2(\mathbb{R})$ is the set

$$\Lambda(G) = \bigcup_{z\in\mathbb{H}} \mathbf{AP}_{\widehat{\mathbb{H}}}(G(z)).$$

Thus, this is the set of all accumulation points (in $\widehat{\mathbb{H}}$) of all orbits $G(z)$, $z \in \mathbb{H}$.

Lemma 2.5.8 If G is a Fuchsian group, then

(1) $\Lambda(G) \subseteq \mathbb{R} \cup \{\infty\}$,

(2) $G(\Lambda(G)) = \Lambda(G)$.

Proof. (1) follows from Corollary 2.5.6, (2) from Definition 2.5.7. $\square$

Examples.

1) For $G = \langle A \rangle$ with $A = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$ we have $\Lambda(G) = \{0, \infty\}$.

2) For $G = \mathrm{PSL}_2(\mathbb{Z})$ we have $\Lambda(G) = \mathbb{R} \cup \{\infty\}$.

Corollary 2.5.9 For every Fuchsian group G the set of fixpoints of all its elliptic elements does not have accumulation points in $\mathbb{H}$. With other words, the set

$$\{z \in \mathbb{H} \,|\, g(z) = z \text{ for some nontrivial } g \in G\}$$

does not have accumulation points in $\mathbb{H}$.

Proof. The proof follows from Theorem 2.5.5 and Definition 2.5.2. $\square$

2.6 Some algebraic properties of Fuchsian groups

Recall our agreements:

$$\frac{a\cdot 0+b}{c\cdot 0+d} := \begin{cases} b/d, & \text{if } d\neq 0 \\ \infty, & \text{if } d=0 \end{cases}, \qquad \frac{a\infty+b}{c\infty+d} := \begin{cases} a/c, & \text{if } c\neq 0 \\ \infty, & \text{if } c=0 \end{cases},$$

The centralizer of an element a of G is the subgroup

$$C_G(a) := \{g\in G \mid ga=ag\}.$$

The normalizer of a subgroup H of G is the subgroup

$$N_G(H) := \{g\in G \mid gH=Hg\}.$$

Lemma 2.6.1 Let T, S be two nontrivial elements of $\text{Möb}_{\mathbb{R}}$. If $TS=ST$, then

$$S\Big(\widehat{\text{Fix}}(T)\Big) = \widehat{\text{Fix}}(T).$$

Proof. Let $z\in\widehat{\text{Fix}}(T)$. Then $S(z)=ST(z)=TS(z)$, hence $S(z)\in\widehat{\text{Fix}}(T)$. Thus, $S\Big(\widehat{\text{Fix}}(T)\Big)\subseteq\widehat{\text{Fix}}(T)$. Similarly, $TS^{-1}=S^{-1}T$ implies $S^{-1}\Big(\widehat{\text{Fix}}(T)\Big)\subseteq\widehat{\text{Fix}}(T)$. □

Theorem 2.6.2 Let T, S be two nontrivial elements of $\text{Möb}_{\mathbb{R}}$. Then

$$TS=ST \Leftrightarrow \widehat{\text{Fix}}(T)=\widehat{\text{Fix}}(S).$$

In particular, the types of S and T are coincide.

Proof. Consider three cases.

1) Let T be parabolic. W.l.o.g. (and using conjugation), we may assume $T: z\mapsto z+1$. In particular, $\widehat{\text{Fix}}(T)=\{\infty\}$.

- First suppose that $TS=ST$. Then, by Lemma 2.6.1, we have $S(\infty)=\infty$, hence $S: z\mapsto az+b$. Furthermore, from $ST=TS$, we deduce that $a=1$, so $S: z\mapsto z+b$. Thus, $\widehat{\text{Fix}}(S)=\{\infty\}$.
- Now suppose that $\widehat{\text{Fix}}(T)=\widehat{\text{Fix}}(S)$. Then $\widehat{\text{Fix}}(S)=\{\infty\}$. Then $S: z\mapsto az+b$, $a\neq 0$. If $a\neq 1$, then we have an additional fixed point. Thus, $S: z\mapsto z+b$ that implies $TS=ST$.

2) Let T be hyperbolic. W.l.o.g. (and using conjugation), we may assume $T: z\mapsto kz$, $k>0$, $k\neq 1$. In particular, $\widehat{\text{Fix}}(T)=\{0,\infty\}$.

- First suppose that $TS=ST$. Then, by Lemma 2.6.1, we have $S(\{0,\infty\})=\{0,\infty\}$. The case $S(0)=\infty$ and $S(\infty)=0$ is impossible, otherwise $S: z\mapsto -\mu/z$ for some $\mu>0$, and hence $TS\neq ST$. Thus, $S(0)=0$ and $S(\infty)=\infty$, i.e. $\widehat{\text{Fix}}(S)=\widehat{\text{Fix}}(T)$.
- Now suppose that $\widehat{\text{Fix}}(S)=\widehat{\text{Fix}}(T)$. Then $\widehat{\text{Fix}}(S)=\{0,\infty\}$. Hence $S: z\mapsto\mu z$ for some $\mu>0$. Then $ST=TS$.

3) Let T be elliptic. W.l.o.g. (and using conjugation), we may assume $T : z \mapsto \frac{\cos\theta \cdot z + \sin\theta}{-\sin\theta \cdot z + \cos\theta}$ for some θ. In particular, $\widehat{\mathrm{Fix}}(T) = \{i\}$.

- First suppose that $TS = ST$. Then, by Lemma 2.6.1, we have $i \in \widehat{\mathrm{Fix}}(S)$. Then $\{i\} = \widehat{\mathrm{Fix}}(S)$ by the classification Lemma 2.1.2.
- Now suppose that $\widehat{\mathrm{Fix}}(S) = \widehat{\mathrm{Fix}}(T)$. Then $\widehat{\mathrm{Fix}}(S) = \{i\}$. Hence, $z \mapsto \frac{\cos\varphi \cdot z + \sin\varphi}{-\sin\varphi \cdot z + \cos\varphi}$ for some φ. Then $ST = TS$.

□

Corollary 2.6.3 The centralizer in $\text{Möb}_{\mathbb{R}}$ of a hyperbolic, parabolic, or elliptic element consists of **id** and all hyperbolic, parabolic, or elliptic elements, respectively, which have the same fixed points.

Corollary 2.6.4 Two hyperbolic elements commute if and only if they have the same axes.

Corollary 2.6.5 Let A, B, C three nontrivial Möbius transformations. If $AB = BA$ and $BC = CB$, then $AC = CA$.

Lemma 2.6.6 Any discrete subgroup of $(\mathbb{R}, +)$ is isomorphic to $\mathbb{Z}$. Any discrete subgroup of $S^1 = ([0, 2\pi], +_{\text{ mod } 2\pi})$ is isomorphic to $\mathbb{Z}_n$ for some finite n.

Theorem 2.6.7 Let G be a Fuchsian group such that all nontrivial elements of G have the same fixed points. Then G is cyclic. Moreover, if G contains a hyperbolic or a parabolic element, then $G \cong \mathbb{Z}$, and if G contains an elliptic element, then $G \cong \mathbb{Z}_n$ for some n.

Proof. Let $g_0 \in G \setminus \{1\}$ be a fixed element and $g \in G \setminus \{1\}$ an arbitrary. Consider three cases.

1) g_0 is hyperbolic. After conjugation, we may assume $g_0 : z \mapsto kz$ for some $k > 0$. Then $\widehat{\mathrm{Fix}}(g) = \mathrm{Fix}(g_0) = \{0, \infty\}$. Therefore $g : z \mapsto \mu z$ for some $\mu > 0$, and we have

$$G \leqslant \{z \mapsto \lambda z \,|\, \lambda > 0\} \cong (\mathbb{R}_+, \cdot) \cong (\mathbb{R}, +).$$

By Lemma 2.6.6, $G \cong \mathbb{Z}$.

2) g_0 is parabolic. After conjugation, we may assume $g_0 : z \mapsto z + 1$ for some $k > 0$. Then $\widehat{\mathrm{Fix}}(g) = \mathrm{Fix}(g_0) = \{\infty\}$. Therefore $g : z \mapsto z + b$ for some $b \in \mathbb{R}$, and we have

$$G \leqslant \{z \mapsto z + \lambda \,|\, \lambda \in \mathbb{R}\} \cong (\mathbb{R}, +).$$

By Lemma 2.6.6, $G \cong \mathbb{Z}$.

3) g_0 is elliptic. After conjugation, we may assume $g_0 : z \mapsto \frac{\cos\theta \cdot z + \sin\theta}{-\sin\theta \cdot z + \cos\theta}$ for some θ. Then $\widehat{\mathrm{Fix}}(g) = \mathrm{Fix}(g_0) = \{i\}$. Therefore

$$G \leqslant \left\{ z \mapsto \frac{\cos\varphi \cdot z + \sin\varphi}{-\sin\varphi \cdot z + \cos\varphi} \,\middle|\, 0 \leqslant \varphi < 2\pi \right\} \cong S^1.$$

By Lemma 2.6.6, $G \cong \mathbb{Z}_n$.

□

Theorem 2.6.8 (1) Every abelian Fuchsian group is cyclic.

(2) If G is Fuchsian and $g \in G \setminus \{1\}$, then the centralizer $C_G(g)$ is cyclic.

Proof. Statement (1) follows from Theorems 2.6.2 and 2.6.7. Statement (2) follows from Corollary 2.6.5 and statement (1).

Theorem 2.6.9 Let G be a Fuchsian group. If G is noncyclic, then $N = N_{\mathrm{PSL}_2(\mathbb{R})}(G)$ is also a Fuchsian group.

Proof. Suppose that N is not discrete. Then there exists a sequence $(T_i)_{i \in \mathbb{N}}$, $T_i \to 1$, where all T_i are different and nontrivial. Then for any $g \in G$ we have

$$T_i^{-1} g T_i \to g.$$

Since G is discrete, there exists $i_0 = i_0(g)$ such that for any $i \geqslant i_0$ we have

$$T_i^{-1} g T_i = g.$$

Let g_1, g_2 be two nontrivial elements of G. Then there exists T_k from the above sequence such that $T_k^{-1} g_1 T_k = g_1$ and $T_k^{-1} g_2 T_k = g_2$. By Corollary 2.6.5, g_1 and g_2 are commute, hence G is abelian. By Theorem 2.6.8, G is cyclic. □

2.7 Elementary Fuchsian groups

Definition 2.7.1 A subgroup $G \leqslant \mathrm{PSL}_2(\mathbb{R})$ is called *elementary* if there exists a point $z \in \widehat{\mathbb{H}}$ such that the orbit $G(z)$ is finite.

Remark 2.7.2 Since $\mathbb{H}$ and $\mathbb{R} \cup \{\infty\}$ are $\mathrm{PSL}_2(\mathbb{R})$-invariant, we have

$$G(z) \subseteq \mathbb{H} \text{ or } G(z) \subseteq \mathbb{R} \cup \{\infty\}.$$

Theorem 2.7.3 Suppose that G is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ such that all nontrivial elements of G are elliptic. Then all elements of G have a common fixed point in $\widehat{\mathbb{H}}$. In particular, G is abelian and elementary.

Beweis. Let A be an element in $G \setminus \{e\}$. After an appropriate conjugation, we have

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary element of G. We have $\det(B) = 1$. Then we have

$$\begin{aligned} \operatorname{Tr}(ABA^{-1}B^{-1}) &= 2ad\cos^2(\theta) + (a^2+b^2+c^2+d^2)\sin^2(\theta) - 2bc\cos^2(\theta) \\ &= 2\cos^2(\theta) + (a^2+b^2+c^2+d^2)\sin^2(\theta) \\ &= 2 + (a^2+b^2+c^2+d^2-2)\sin^2(\theta) \\ &= 2 + ((a-d)^2 + (b+c)^2)\sin^2(\theta) \geqslant 2. \end{aligned}$$

Therefore the commutator $[A, B]$ is either trivial or elliptic. But since it is not elliptic by assumption, we have $[A, B] = 1$. Therefore G is abelian. By Theorem 2.6.2, all nontrivial elements of G have the same fixed points in $\widehat{\mathbb{H}}$. In particular, G is elementary. E □

Corollary 2.7.4 Every Fuchsian group G, whose nontrivial elements are elliptic is a finite cyclic group.

Proof. By Theorem 2.7.3, G is abelian. By Theorem 2.6.2, the fixed points of nontrivial elements of G coincide. By Theorem 2.6.8, G is a finite cyclic group. $\square$

Theorem 2.7.5 Every elementary Fuchsian group G is isomorphic to $\mathbb{Z}$, $\mathbb{Z}_n$ (for some finite n), or D_∞ (the infinite dihedral group). Moreover, the following holds:

(1) If $G \cong \mathbb{Z}$, then G is conjugate to the group $\langle z \mapsto kz \rangle$ for some $k > 0$, or to the group $\langle z \mapsto z + k \rangle$ for some $k > 0$.

(2) If $G \cong \mathbb{Z}_n$, then G is conjugate to the group

$$\left\langle z \mapsto \frac{\cos \frac{2\pi}{n} \cdot z + \sin \frac{2\pi}{n}}{-\sin \frac{2\pi}{n} \cdot z + \cos \frac{2\pi}{n}} \right\rangle$$

for some $n \in \mathbb{N}$.

(3) If $G \cong D_\infty$, then G is conjugate to the group $H_k = \langle \theta_k, \psi \rangle$ for some $k > 0$, where $\theta_k : z \mapsto kz$ and $\psi : z \mapsto -\frac{1}{z}$.

Proof. Let $\mathcal{O}$ be a finite orbit of G in $\widehat{\mathbb{H}}$.
Case 1. Let $|\mathcal{O}| = 1$.
Then $\mathcal{O} = \{\alpha\}$ for some $\alpha \in \widehat{\mathrm{Fix}}(G)$.

Fall 1.1. Sei $\alpha \in \mathbb{H}$.

Then all elements of $G \setminus \{e\}$ are elliptic. By Corollary 2.7.4, G is a finite cyclic group.

Fall 1.2. Let $\alpha \in \mathbb{R} \cup \{\infty\}$.

Then G has no elliptic elements. There are three subcases:

(a) G contains both, hyperbolic and parabolic elements.
After an appropriate conjugation G contains an element $g : z \mapsto \lambda z$, where $\lambda > 0$. We have $\alpha \in \widehat{\mathrm{Fix}}(G) \subseteq \widehat{\mathrm{Fix}}(g) = \{0, \infty\}$. Therefore $\alpha = 0$ or $\alpha = \infty$.
If $\alpha = 0$, we consider $\psi G \psi^{-1}$ instead of G (recall that $\psi : z \to -\frac{1}{z}$). Then
• $\widehat{\mathrm{Fix}}(\psi G \psi^{-1}) = \psi\big(\widehat{\mathrm{Fix}}(G)\big) = \psi(\alpha) = \{\infty\}$.
• $g = \psi g^{-1} \psi^{-1} \in \psi G \psi^{-1}$.
Therefore we may assume that $\alpha = \infty$. If necessary, we also can replace g by g^{-1} and assume that $\lambda > 1$. Let h be a parabolic element from G. From $\alpha \in \widehat{\mathrm{Fix}}(G) \subseteq \widehat{\mathrm{Fix}}(h)$ we deduce $\{\infty\} = \widehat{\mathrm{Fix}}(h)$. Then $h : z \to z + b$ for some $b \in \mathbb{R}$. We have

$$g^{-n} h g^n(z) = z + \lambda^{-n} b.$$

Since $\lambda > 0$, we have

$$g^{-n} h g^n \to \mathrm{id}.$$

that contradicts the discreteness of G.

(b) $G \setminus \{e\}$ contains only parabolic elements.
Then each of them has α as a single fixed point. Then, by Theorem 2.6.7, we have $G \cong \mathbb{Z}$.

(c) $G \setminus \{e\}$ contains only hyperbolic elements.
As above, we may assume that G contains $g : z \mapsto \lambda z$ $(\lambda \neq 1)$. and $\alpha = \infty$. If $\langle g \rangle = G$, then we are done.

If $\langle g \rangle \neq G$, then we consider some $h \in G \setminus \langle g \rangle$. Since $\alpha \in \widehat{\mathrm{Fix}}(h)$, we have $h : z \mapsto az + b$ for some $a \neq 0$ and $b \in \mathbb{R}$. Then $ghg^{-1}h^{-1} : z \mapsto z + (\lambda - 1)b$. If $b \neq 0$, then this is a parabolic element that contradicts the assumption. If $b = 0$, then $h : z \mapsto az$. Thus, all elements of G have the form $z \mapsto kz$, $k \in \mathbb{R}_+$. Then, up to an isomorphism, G is a discrete subgroup of $(\mathbb{R}_+, \cdot)$. Then $G \cong \mathbb{Z}$.

Case 2. Let $|\mathcal{O}| = 2$, say $\mathcal{O} = \{\alpha_1, \alpha_2\}$.

Fall 2.1. $\mathcal{O} \subseteq \mathbb{H}$.

Then all elements of $G \setminus \{e\}$ are elliptic. By Corollary 2.7.4, G is a finite cyclic group.

Fall 2.2. $\mathcal{O} \subseteq \mathbb{R} \cup \{\infty\}$.

Then G does not have parabolic elements. (Indeed, a parabolic element has only one finite orbit in $\mathbb{R} \cup \{\infty\}$, and this orbit contains only one point.) We consider three cases:

(a) $G \setminus \{e\}$ contains only hyperbolic elements. Then α_1 and α_2 are their common fixed points and both, $\{\alpha_1\}$ and $\{\alpha_2\}$ are orbits of G. Hence, $\mathcal{O}$ is not an orbit of G. A contradiction.

(b) $G \setminus \{e\}$ contains only elliptic elements. Then, by Corollary 2.7.4, G is a finite cyclic group.

(c) $G \setminus \{e\}$ contains both, elliptic and hyperbolic elements. Then all elliptic elements have the order 2 and permute α_1, α_2. After an appropriate conjugation, we may assume that $\alpha_1 = 0$ and $\alpha_2 = \infty$. Then all hyperbolic elements in G have the form

$$g_k : z \mapsto kz, \;\; k > 0, k \neq 1,$$

and all elliptic elements in G have the form

$$e_\lambda : z \mapsto -\frac{\lambda}{z}, \;\; \lambda > 0.$$

Let G_0 be the subgroup of G consisting of id and all hyperbolic elements of G. Then G_0 has index 2 in G. The second coset of G_0 in G consists of elliptic elements. Let e_λ be one of them. We conjugate G by

$$q : z \mapsto \sqrt{\lambda} z.$$

Then $qG_0q^{-1} = G_0$ and $qe_\lambda q^{-1} = e_1 = \psi$. Therefore

$$qGq^{-1} = G_0 \cup \psi G_0.$$

Since G_0 is discrete, $G_0 = \langle g_k \rangle$ for some $k > 0$, $k \neq 1$. Hence $qGq^{-1} = H_k$.

Case 3. Let $|\mathcal{O}| \geqslant 3$.

Then $\mathcal{O} \subseteq \mathbb{R} \cup \{\infty\}$ and $G \setminus \{e\}$ contains only elliptic elements. By Corollary 2.7.4, G is a finite cyclic group. $\square$

2.8 Jorgensen inequality

Lemma 2.8.1 Let $S, T \in \mathrm{PSL}_2(\mathbb{R})$. We set $S_0 := S$, and $S_{r+1} = S_r T S_r^{-1}$ for $r \geqslant 0$. If there is $n \geqslant 1$ such that $S_n = T$, then $\langle S, T\rangle$ is an elementary group and $S_2 = T$.

Beweis. For $T = \mathrm{Id}$ the statement is evident. Let $T \neq \mathrm{Id}$.

Case 1. Let $|\widehat{\mathrm{Fix}}(T)| = 1$.

Then $\widehat{\mathrm{Fix}}(T) = \{\alpha\}$ for some $\alpha \in \widehat{\mathbb{H}}$.

Since S_{r+1} is conjugate to T, we have

$$|\widehat{\mathrm{Fix}}(S_{r+1})| = 1 \quad \text{for all } r \geqslant 0. \tag{2.8.1}$$

Moreover,

$$S_{r+1} \circ S_r(\alpha) = S_r \circ T \circ S_r^{-1} \circ S_r(\alpha) = S_r(\alpha).$$

This implies

$$\widehat{\mathrm{Fix}}(S_{r+1}) = \{S_r(\alpha)\}. \tag{2.8.2}$$

By assumption $S_n = T$, therefore $\widehat{\mathrm{Fix}}(S_n) = \widehat{\mathrm{Fix}}(T) = \{\alpha\}$. With the help of (2.8.1) and (2.8.2), we consequently obtain

$$\begin{aligned} \widehat{\mathrm{Fix}}(S_{n-1}) \quad &= \{\alpha\}, \\ \ldots \\ \widehat{\mathrm{Fix}}(S_1) \quad &= \{\alpha\}, \\ \widehat{\mathrm{Fix}}(S) \quad &\supseteq \{\alpha\}. \end{aligned}$$

Therefore we have

1) $\alpha \in \widehat{\mathrm{Fix}}\langle S, T\rangle$. Hence, $\langle S, T\rangle$ is elementary.

2) $\widehat{\mathrm{Fix}}(S_1) = \{\alpha\} = \widehat{\mathrm{Fix}}(T)$. Hence, $S_1 T = T S_1$ (see Theorem 2.6.2), i.e. $S_2 = T$.

Case 2. Let $|\widehat{\mathrm{Fix}}(T)| = 2$.

Then $\widehat{\mathrm{Fix}}(T) = \{\alpha, \beta\}$ for some $\alpha, \beta \in \widehat{\mathbb{H}}$. Since S_{r+1} is conjugate to T, we have

$$|\widehat{\mathrm{Fix}}(S_{r+1})| = 2, \text{ and } S_{r+1} \text{ is hyperbolic for all } r \geqslant 0. \tag{2.8.3}$$

Moreover,

$$S_{r+1} \circ S_r(\{\alpha, \beta\}) = S_r \circ T \circ S_r^{-1} \circ S_r(\{\alpha, \beta\}) = S_r(\{\alpha, \beta\}).$$

Each hyperbolic element has a unique invariant 2-elements set and this set coincides with the foxed point set. Therefore

$$\widehat{\mathrm{Fix}}(S_{r+1}) = \{S_r(\alpha, \beta)\}. \tag{2.8.4}$$

By assumption $S_n = T$, therefore $\widehat{\mathrm{Fix}}(S_n) = \widehat{\mathrm{Fix}}(T) = \{\alpha\}$. With the help of (2.8.1) and (2.8.2), we consequently obtain

$$\begin{aligned}
\widehat{\mathrm{Fix}}(S_{n-1}) &= \{\alpha, \beta\}, \\
\dots \\
\widehat{\mathrm{Fix}}(S_1) &= \{\alpha, \beta\}, \\
\widehat{\mathrm{Fix}}(S) &= \{\alpha, \beta\}.
\end{aligned}$$

Therefore we have

1) $\{\alpha, \beta\} \supseteq \widehat{\mathrm{Fix}}\langle S, T\rangle$. Hence, $\langle S, T\rangle$ is elementary.

2) $\widehat{\mathrm{Fix}}(S_1) = \{\alpha, \beta\} = \widehat{\mathrm{Fix}}(S_2)$. Hence, $S_1 S_2 = S_2 S_1$ (see Theorem 2.6.2) that implies $S_1 T = T S_1$, i.e. $S_2 = T$.

Theorem 2.8.2 (Jorgensen inequality) Let $T, S \in \mathrm{PSL}_2(\mathbb{R})$ such that $\langle T, S\rangle$ is a nonelementary Fuchsian group. Then

$$|\mathrm{Tr}^2(T) - 4| + |\mathrm{Tr}(TST^{-1}S^{-1}) - 2| \geqslant 1. \tag{2.8.5}$$

Proof. We set $S_0 := S$, and $S_{r+1} = S_r T S_r^{-1}$ for $r \geqslant 0$ as in Lemma 2.8.1. We will show that if the inequality (2.8.5) is not valid, then $S_n = T$ for some $n \in \mathbb{N}$. By Lemma 2.8.1 this would imply that the group $\langle S, T\rangle$ is elementary, a contradiction.

Case 1. Suppose that T is parabolic.

Since the trace is invariant with respect to conjugation of matrices, we may assume that

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Suppose that the inequality (2.8.5) is not valid. A straightforward calculation shows that $|c| < 1$. Let

$$S_r = \begin{bmatrix} a_r & b_r \\ c_r & d_r \end{bmatrix}.$$

From $S_{r+1} = S_r \circ T \circ S_r^{-1}$ we obtain

$$\begin{bmatrix} a_{r+1} & b_{r+1} \\ c_{r+1} & d_{r+1} \end{bmatrix} = \begin{bmatrix} a_r & b_r \\ c_r & d_r \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} d_r & -b_r \\ -c_r & a_r \end{bmatrix} = \begin{bmatrix} 1 - a_r c_r & a_r^2 \\ -c_r^2 & 1 + a_r c_r \end{bmatrix}$$

By induktion we obtain $c_r = -c^{2^r}$. Since $|c| < 1$, we have

$$c_r \to 0.$$

The equality $a_{r+1} = 1 - a_r c_r$ and $|c| < 1$ imply that

$$|a_{r+1}| \leqslant 1 + |a_r c_r| \leqslant 1 + |a_r| \leqslant \dots \leqslant (r+1) + |a|.$$

Therefore $|a_r c_r| \leqslant (r + |a|)|c_r| \leqslant (r + |a|)|c|^{2^r} \to 0$. It follows

$$a_{r+1} = 1 - a_r c_r \to 1.$$

Thus, $S_{r+1} \to T$. Since $\langle S, T\rangle$ is discrete, there exists n such that $S_n = T$. Then by Lemma 2.8.1, the group $\langle S, T\rangle$ is elementary. A contradiction.

Case 2. Suppose that T is hyperbolic.

After an appropriate conjugation, we may assume that

$$T = \begin{bmatrix} u & 0 \\ 0 & 1/u \end{bmatrix}, \quad S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We define

$$\mu := |\mathrm{Tr}^2(T) - 4| + |\mathrm{Tr}(TST^{-1}S^{-1}) - 2|.$$

Suppose that the inequality (2.8.5) is not valid. Then

$$\mu = (1 + |bc|)\Big|u - \frac{1}{u}\Big|^2 < 1.$$

In particular,

$$\Big|\frac{1}{u} - u\Big| \leqslant \mu^{\frac{1}{2}}. \tag{2.8.6}$$

From $S_{r+1} = S_r \circ T \circ S_r^{-1}$ follows

$$\begin{bmatrix} a_{r+1} & b_{r+1} \\ c_{r+1} & d_{r+1} \end{bmatrix} = \begin{bmatrix} a_r d_r u - \dfrac{b_r c_r}{u} & a_r b_r\Big(\dfrac{1}{u} - u\Big) \\ c_r d_r\Big(u - \dfrac{1}{u}\Big) & \dfrac{a_r d_r}{u} - b_r c_r u \end{bmatrix}.$$

Then

$$b_{r+1}c_{r+1} = -b_r c_r (1 + b_r c_r)\Big(u - \frac{1}{u}\Big)^2.$$

Claim. We have $|b_r c_r| \leqslant \mu^r |bc|$.

Proof. We proceed the inductive step (where we use $\mu < 1$):

$$|b_{r+1}c_{r+1}| \leqslant \mu^r|bc| \cdot (1 + \mu^r|bc|) \cdot \Big|u - \frac{1}{u}\Big|^2 \leqslant \mu^r|bc| \cdot (1 + |bc|) \cdot \Big|u - \frac{1}{u}\Big|^2 = \mu^{r+1}|bc|.$$

□

This claim and $\mu < 1$ imply the following statements.

(1) $b_{r+1}c_{r+1} \to 0$,

(2) $a_r d_r = 1 + b_r c_r \to 1$,

(3) $a_{r+1} \to u$ and $d_{r+1} \to \dfrac{1}{u}$.

Then for all sufficiently large r we have $\dfrac{a_r}{|u|} \leqslant \mu^{-\frac{1}{3}}$ (recall that $\mu < 1$). This and (2.8.6) imply

$$\frac{|b_{r+1}|}{|u^{r+1}|} = \frac{|b_r|}{|u^r|} \cdot \frac{|a_r|}{|u|} \cdot \frac{|\frac{1}{u} - u|}{1} \leqslant \frac{|b_r|}{|u^r|}\mu^{-\frac{1}{3}}\mu^{\frac{1}{2}} = \frac{|b_r|}{|u^r|}\mu^{\frac{1}{6}}.$$

Since $\mu < 1$, we deduce

(4) $\dfrac{|b_r|}{|u^r|} \to 0$. Analogously, we deduce

(5) $|c_r| \cdot |u^r| \to 0$.

Then

$$T^{-r} S_{2r} T^r = \begin{bmatrix} a_{2r} & \dfrac{b_{2r}}{u^{2r}} \\ c_{2r} u^{2r} & d_{2r} \end{bmatrix} \to T.$$

Since $\langle S, T \rangle$ is discrete, $T^{-r} S_{2r} T^r = T$ for all sufficiently large r. Then $S_{2r} = T$ for all sufficiently large r. We get a contradiction as in Case 1.

Case 3. Suppose that T is elliptic.

By classification Lemma 2.1.2, the eigenvalues λ_1, λ_2 of T satisfy $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_1 = \bar{\lambda}_2$, $\lambda_1 \lambda_2 = 1$. Hence, $\lambda_1 = e^{i\varphi}$ for some $\varphi \in (0, \pi) \cup (\pi, 2\pi)$. Thus, after an appropriate conjugation in $\mathrm{PSL}_2(\mathbb{C})$, we may assume that

$$T = \begin{bmatrix} u & 0 \\ 0 & 1/u \end{bmatrix} \quad (u = e^{i\varphi}, 0 < \varphi < \pi).$$

The remaining proof follows repeats that in Case 2. $\square$