Fuchsian Groups

(course outline)

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This course is devoted to Fuchsian groups, which play an important role in different areas of mathematics: in the theory of Riemann surfaces, in number theory (modular forms), in dense sphere packings, in dynamical systems, and in geometric group theory.

The goal of this course is to give a detailed introduction to Fuchsian groups and their generalisations. The course is addressed to PhD students and researchers at the Institute of Mathematics, University of Szczecin.

Prerequisites: Basic knowledge in analysis and group theory.

Part I. Hyperbolic geometry

Lecture 1. (8.11., 13:00-14:30)
The hyperbolic plane $\mathbb{H}$. The group of Möbius transformations of $\mathbb{H}$.
Geodesic lines in $\mathbb{H}$.

Lecture 2. (15.11., 11:30-13:00)
Some formulas for computing hyperbolic distances: cross-ratio formula,
Log-and-Hyp formulas.

Lecture 3. (15.11., 14:30-16:00)
The isometry group of $\mathbb{H}$. Hyperbolic area. Angles. Gauß-Bonnet formula.
Hyperbolic trigonometry. Hyperbolic Pythagoras formula.

Part II. Fuchsian groups

Lecture 4. (29.11., 11:30-13:00)
A classification of elements of $\mathrm{PSL}_2(\mathbb{R})$. Dynamic properties of elliptic, parabolic, and loxodromic elements.

Lecture 5. (29.11., 14:30-16:00)
Three equivalent definitions of Fuchsian groups.

Lecture 6. (6.12., 11:30-13:00)
Limit points of Fuchsian groups. Some algebraic properties of Fuchsian groups.

Lecture 7. (6.12., 14:30-16:00)
Elementary Fuchsian groups.

Lecture 8. (13.12., 11:30-13:00)
Jorgensen inequality. A discreteness criterion for subgroups of $\mathrm{PSL}_2(\mathbb{R})$.

Part III. Fundamental domains of Fuchsian groups

Lecture 9. (13.12., 14:30-16:00)
Definition and some properties of a fundamental domain of a Fuchsian group.

Lecture 10. (20.12., 11:30-13:00)
Dirichlet domains.

Lecture 11. (20.12., 14:30-16:00)
Some theorems about the limit set $\Lambda(G)$ of a Fuchsian group G.

Lecture 12. (10.1., 11:30-13:00)
Generators of Fuchsian groups arising from Dirichlet domains. Poincaré theorem.

Part IV. Arithmetic Fuchsian groups

Lecture 13. (10.1., 14:30-16:00)
Quaternion algebras and quaternion Fuchsian groups.

Lecture 14. (17.1., 11:30-13:00)
Arithmetic Fuchsian groups.

Part V. Hyperbolic groups

Lecture 15. (17.1., 14:30-16:00)
Equivalent definitions and some properties of hyperbolic groups.

Lecture 16. (24.1., 11:30-13:00)
Small cancellations groups as a source of hyperbolicity.

Lecture 17. (24.1., 14:30-16:00)
Rips construction.

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Tutorials: **15** hours (10:00-11:30 on 11.11., 18.11., 25.11., 2.12., 9.12., 16.12., 6.1., 13.1.)

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The book [1] below gives a nice introduction to Fuchsian groups. The book [2] is more advanced and contains more information. The book [3] is a handbook for geometric group theorists; Chapter III of this book is devoted to hyperbolic groups. The book [4] is a handbook for combinatorial group theorists; Chapter V is devoted to small cancellation groups. The short paper [5] is important.

Literatur

[1] Svetlana Katok, *Fuchsian groups*, The university of Chicago press, Chicago, 1992.

[2] Alan F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, No. **91**, Springer, 1983.

[3] Martin Bridson, André Haeffliger, *Metric spaces of non-positive curvature*, Springer, 1999.

[4] Roger C. Lyndon, Pauln Schupp, *Combinatorial group theory*, Springer, Berlin, 1977.

[5] Eliath Rips, *Subgroups of small cancellation groups*, Bull. London Math. Soc., **14** (1982), 45-47.