O. BOGOPOLSKI

Institute of Mathematics
Siberian Branch of Russian Academy of Sciences
Novosibirsk, Russia

Düsseldorf University, Germany
Oleg_Bogopolski@yahoo.com

E. VENTURA

Dept. Mat. Apl. III
Universitat Politècnica de Catalunya
Manresa, Barcelona, Catalunya
enric.ventura@upc.edu

Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. From the main result in the paper, Theorem 1.2, we deduce the following two corollaries.

First, we show that there exists a computable constant $\mathcal{C} = \mathcal{C}(\delta, \sharp S)$ such that, for any endomorphism φ of H, if $\varphi(h)$ is conjugate to h for every element $h \in H$ of length up to $\mathcal{C}$, then φ is an inner automorphism. Second, we show a mixed (conjugate/non-conjugate) version of the classical Whitehead problem for tuples is solvable in torsion-free hyperbolic groups.

Keywords: Hyperbolic group; automorphism; Whitehead algorithm.

1. Introduction

Let G be a group and A be a subset of G. An endomorphism φ of G is called *pointwise inner on* A if the element $\varphi(g)$ is conjugate to g, for every $g \in A$. We call φ *pointwise inner* if it is pointwise inner on G. The group of all pointwise inner automorphisms of G is denoted by $\mathrm{Aut}_{\mathrm{pi}}(G)$. Clearly, $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}_{\mathrm{pi}}(G) \trianglelefteq \mathrm{Aut}(G)$.

There are groups admitting pointwise inner automorphisms which are not inner. For example, some finite groups (see [19]), some torsion-free nilpotent groups (see [20]), some nilpotent Lie groups (see [7]), and direct products of such groups with arbitrary groups. The fact that some nilpotent Lie groups admit such automorphisms was used in [7] to construct isospectral but not isometric Riemannian manifolds.

On the other hand, for free nilpotent groups (see [6]), for free groups (see [8, 11]), for nontrivial free products (see [18]), and for fundamental groups of closed surfaces of negative Euler characteristic (see [1]), all pointwise inner automorphisms are indeed inner. In the last paper, this property was used to show that surface groups satisfy a weak Magnus property.

One of the results in the present paper implies that torsion-free hyperbolic groups also fall into this last class of groups. In fact, we prove a stronger computational version of this fact: endomorphisms of torsion-free hyperbolic groups which are pointwise inner on a ball of a uniformly bounded (and computable) radius, are indeed inner automorphisms.

Theorem 1.1. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Then, there exists a computable constant $\mathcal{C}$ (depending only on δ and the cardinal $\sharp S$) such that, for every endomorphism φ of H, if $\varphi(g)$ is conjugate to g for every element g in the ball of radius $\mathcal{C}$, then φ is an inner automorphism.*

An immediate consequence of Theorem 1.1 is that one can algorithmically decide whether a given endomorphism of a torsion-free hyperbolic group (given by a finite presentation, and images of generators) is or is not an inner automorphism. This can also be easily deduced from the well-known fact that hyperbolic groups and their direct products are bi-automatic; an alternative proof can also be found in [4, Theorem A]. However we stress, that the purpose of the present paper is not the conjugacy problem for subsets of elements in hyperbolic groups.

Theorem 1.1 follows immediately from the main result of this paper (of a more technical nature):

Theorem 1.2. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Let $a_1, \ldots, a_n$ and $a_{1*}, \ldots, a_{n*}$ be elements of H such that a_{i*} is conjugate to a_i for every $i = 1, \ldots, n$. Then, there is a uniform conjugator for them if and only if $W(a_{1*}, \ldots, a_{n*})$ is conjugate to $W(a_1, \ldots, a_n)$ for every word W in n variables and length up to a computable constant depending only on $\delta, \sharp S$ and $\sum_{i=1}^{n} |a_i|$.*

Note that Theorem 1.1 was formulated in [2, Theorem 2]. Independently, Minasyan and Osin [17] proved a variant of Theorem 1.2, for relatively hyperbolic groups but without the statement on computability for the involved constant. Note also that our Theorem 1.1 and [17, Theorem 1.1] both imply that if H is a torsion-free hyperbolic group, then the groups $\mathrm{Inn}(H)$ and $\mathrm{Aut}_{\mathrm{pi}}(H)$ coincide.

Metaftsis and Sykiotis [14, 15] proved that, for any (relatively) hyperbolic group H, the group $\mathrm{Inn}(H)$ has finite index in $\mathrm{Aut}_{\mathrm{pi}}(H)$. Their proof is not constructive, it uses ultrafilters and ideas of Paulin on limits of group actions.

The constructive character of Theorem 1.2 suggests to replace there and in Theorem 1.1, the "ball" of words by a unique word. We raise the following problems.

Problem 1. Let n, s be natural numbers and let δ be a real nonnegative number. Does there exist a word $W_{n,\delta,s}$ in n variables, such that the following statement holds?

Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Let $a_1, \dots, a_n$ and $a_{1*}, \dots, a_{n*}$ be elements of H such that a_{i*} is conjugate to a_i for every $i = 1, \dots, n$. Then, there is a uniform conjugator for them if and only if $W(a_{1*}, \dots, a_{n*})$ is conjugate to $W(a_1, \dots, a_n)$ for $W = W_{n,\delta,\sharp S}$.

In the special case, where H is a free group and the elements $a_1, \dots, a_n$, generate a noncyclic subgroup this problem has a positive solution due to Lee (see [10, Theorem, p. 2]). Note that the paper of Lee is a development of the paper of Ivanov [9] on test words in free groups.

Problem 2. Let s be a natural number and δ be a real nonnegative number. Does there exist a test word $W_{\delta,s}$ in s variables, such that the following statement holds?

Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S and let φ be an endomorphism of H. Then φ is an inner automorphism if and only if $\varphi(W_{\delta,\sharp S})$ is conjugate to $W_{\delta,\sharp S}$.

Clearly, a positive solution to Problem 1 would imply a positive solution to Problem 2.

Now we describe another application of Theorem 1.2. Consider a list of elements in a finitely presented group G, organized in n blocks:

$$\begin{aligned} &u_{1,1}, \dots, u_{1,m_1}, \\ &u_{2,1}, \dots, u_{2,m_2}, \\ &\dots \\ &u_{n,1}, \dots, u_{n,m_n}. \end{aligned} \tag{1}$$

The *mixed Whitehead problem* consists in finding an algorithm to decide whether, given two such lists, there exists an automorphism of G sending the first list to the second up to conjugation, but asking for a uniform conjugator in every block (and possibly different from those in other blocks).

Note that in the case where each block consists of a single element (i.e. $m_i = 1$ for all $i = 1, \dots, n$), this is exactly asking whether there exists an automorphism of G sending the first list of elements to the second one up to conjugacy, with no restriction for the conjugators.

On the other hand, if there is only one block (i.e. $n = 1$), the mixed Whitehead problem is equivalent to asking whether there exists an automorphism of G sending the first list of elements exactly to the second.

We call these cases of the general problem the *first* and the *second Whitehead problem* for G, respectively.

For finitely generated free groups, the first Whitehead problem was solved by Whitehead in 1936 (see [21] or [12]), and the second Whitehead problem was solved by McCool in 1974 (see [13] or [12]). These two results have been recently generalized to arbitrary hyperbolic groups by Dahmani and Guirardel (see [5, Corollary 5]).

Using Theorem 1.2, we can extend Dahmani–Guirardel's result to solving the mixed Whitehead problem for hyperbolic groups, in the torsion-free case.

Theorem 1.3. *Let H be a torsion-free δ-hyperbolic group, with respect to a finite generating set S. The mixed Whitehead problem for H is solvable.*

The structure of the paper is as follows. In Sec. 2, we recall some definitions and basic facts on hyperbolic metric spaces and hyperbolic groups. Also, we prove there several statements (specially about norms and axes of elements, and about controlling cancelations in some products of elements) which will be used later. The main theorem will be proved in Secs. 3 to 5. In Sec. 3, we prove auxiliary statements, in Sec. 4, we consider the case $n = 2$ and in Sec. 5, we consider the general case. These three sections are sequential and the arguments in each one are helpful for the next one. In Sec. 6, we prove Theorem 1.3.

2. Hyperbolic Preliminaries

2.1. *Hyperbolic spaces*

Let $(\mathcal{X}, d)$ be a metric space.

If A, B are points or subsets of $\mathcal{X}$, the distance between them will be denoted by $d(A, B)$, or simply by $|AB|$ if there is no risk of confusion.

A *path* in $\mathcal{X}$ is a map $p\colon I \to \mathcal{X}$, where I is an interval of the real line (bounded or unbounded) or else the intersection of $\mathbb{Z}$ with such an interval. In the last case the path is called *discrete*. If $I = [a, b]$ then $p_- = p(a)$ and $p_+ = p(b)$ are called the *endpoints* of p. In that case we say that the path p is *bounded* and *passes from p_- to p_+*; otherwise, we use the terms *infinite path* and *bi-infinite path* with the obvious meaning. Sometimes we will identify a path with its image in $\mathcal{X}$.

We say that a path p is *geodesic* if $d(p(r), p(s)) = |r - s|$ for every $r, s \in I$. The space $(\mathcal{X}, d)$ is said to be a *geodesic* metric space if for every two points $A, B \in \mathcal{X}$ there is a geodesic from A to B (not necessarily unique). Such a geodesic is usually denoted $[AB]$.

By a *geodesic n-gon* $A_1A_2\cdots A_n$, where $n \geq 3$, we mean a cyclically ordered list of points $A_1, \ldots, A_n \in \mathcal{X}$ together with chosen geodesics $[A_1A_2], [A_2A_3], \ldots, [A_{n-1}A_n], [A_nA_1]$; each of these geodesics is called a *side* of the n-gon, and each A_i a *vertex*. A geodesic 3-gon is usually called a *geodesic triangle*, and a geodesic 4-gon a *geodesic rectangle*.

Definition 2.1. Let $(\mathcal{X}, d)$ be a geodesic metric space and δ be a nonnegative real number.

A geodesic triangle $A_1A_2A_3$ in $\mathcal{X}$ is called *δ-thin* if for any vertex A_i and any two points $X \in [A_i, A_j]$, $Y \in [A_i, A_k]$ with

$$|A_iX| = |A_iY| \leq \frac{1}{2}(|A_iA_j| + |A_iA_k| - |A_jA_k|),$$

we have $|XY| \leq \delta$. The space $\mathcal{X}$ is called *δ-hyperbolic* if every geodesic triangle in $\mathcal{X}$ is δ-thin.

Directly from this definition it follows that each side of a δ-thin triangle is contained in the δ-neighborhood of the union of the other two. By induction, one can easily extend this observation to n-gons.

Proposition 2.2. *If $A_1A_2\cdots A_n$ is a geodesic n-gon in a δ-hyperbolic geodesic space, then each side is contained in the $(n-2)$ δ-neighborhood of the union of all the others.*

The following result is straightforward and will be used later (it is known as the *rectangle inequality*).

Proposition 2.3 (see [3, Remark 1.21, Chap. III.H]). *Any 4-gon $ABCD$ in a δ-hyperbolic geodesic space $(\mathcal{X}, d)$ satisfies the following inequality*:

$$|AC| + |BD| \leq \max\{|BC| + |AD|, |AB| + |CD|\} + 2\delta.$$

Along the paper, we will need to use some approximations to the concept of geodesic. Here is a technical result and two standard notions.

Lemma 2.4. *Let $A_1, A_2, \ldots, A_n$ be $n \geq 3$ points in a δ-hyperbolic geodesic space satisfying the following conditions*:

(i) $|A_{i-1}A_{i+1}| \geq |A_{i-1}A_i| + |A_iA_{i+1}| - 2\delta$, *for every* $2 \leq i \leq n-1$,
(ii) $|A_{i-1}A_i| > (2n-3)\delta$, *for every* $3 \leq i \leq n-1$.

Then,

$$|A_1A_n| \geq \sum_{i=1}^{n-1} |A_iA_{i+1}| - (4n-10)\delta. \tag{2}$$

Proof. The proof goes by induction on n. Note that for $n = 3$, the result is obvious.

Assume the result valid for n points and let us prove it for $n+1$. Let $A_1, A_2, \ldots, A_n, A_{n+1}$ be $n+1$ points satisfying condition (i) for $2 \leq i \leq n$, and condition (ii) for $3 \leq i \leq n$. Clearly then $A_1, A_2, \ldots, A_n$ satisfy the corresponding conditions and, by the inductive hypothesis, we have Eq. (2), so

$$|A_1A_n| \geq \sum_{i=1}^{n-1} |A_iA_{i+1}| - (4n-10)\delta \geq |A_1A_{n-1}| + |A_{n-1}A_n| - (4n-10)\delta.$$

From condition (i) with $i = n$, we have

$$|A_{n-1}A_{n+1}| \geq |A_{n-1}A_n| + |A_nA_{n+1}| - 2\delta. \tag{3}$$

Adding these two last inequalities and applying condition (ii) for $i = n$, we get

$$\begin{aligned}|A_1A_n| + |A_{n-1}A_{n+1}| &\geq |A_1A_{n-1}| + |A_nA_{n+1}| + 2|A_{n-1}A_n| - (4n-8)\delta \\ &> |A_1A_{n-1}| + |A_nA_{n+1}| + 2\delta.\end{aligned}$$

Therefore, the maximum in the rectangle inequality applied to $A_1A_{n-1}A_nA_{n+1}$ (see Proposition 2.3),

$$|A_1A_n| + |A_{n-1}A_{n+1}| \leq \max\{|A_1A_{n-1}| + |A_nA_{n+1}|, |A_1A_{n+1}| + |A_{n-1}A_n|\} + 2\delta,$$

is achieved in the second entry. Hence,

$$|A_1A_n| + |A_{n-1}A_{n+1}| \leq |A_1A_{n+1}| + |A_{n-1}A_n| + 2\delta. \tag{4}$$

On the other hand, from the induction hypothesis (2) and inequality (3), we have

$$\begin{aligned}|A_1A_n| + |A_{n-1}A_{n+1}| &\geq \left(\sum_{i=1}^{n-1}|A_iA_{i+1}| - (4n-10)\delta\right) + |A_{n-1}A_n| \\ &\quad + |A_nA_{n+1}| - 2\delta \\ &= \sum_{i=1}^{n}|A_iA_{i+1}| + |A_{n-1}A_n| - (4n-8)\delta.\end{aligned}$$

From this and inequality (4), we complete the proof:

$$|A_1A_{n+1}| \geq \sum_{i=1}^{n}|A_iA_{i+1}| - (4n-6)\delta = \sum_{i=1}^{n}|A_iA_{i+1}| - (4(n+1)-10)\delta. \qquad \square$$

Definition 2.5. Let $(\mathcal{X}, d)$ be a metric space and $p\colon I \to \mathcal{X}$ be a path. Let $k > 0$, $\lambda \geq 1$ and $\epsilon \geq 0$ be real numbers. The path p is said to be *k-local geodesic* if $d(p(r), p(s)) = |r - s|$ for all $r, s \in I$ with $|r - s| \leq k$. And it is said to be *(λ, ϵ)-quasi-geodesic* if, for all $r, s \in I$, we have

$$\frac{1}{\lambda}|r-s| - \epsilon \leq d(p(r), p(s)) \leq \lambda|r-s| + \epsilon.$$

Proposition 2.6 (see [3, Theorem 1.13(3), Chap. III.H]). *Let $\mathcal{X}$ be a δ-hyperbolic geodesic space and let $p\colon [a,b] \to \mathcal{X}$ be a k-local geodesic with $k > 8\delta$. Then, p is a (λ, ϵ)-quasi-geodesic, where $\lambda = \frac{k+4\delta}{k-4\delta}$ and $\epsilon = 2\delta$.*

The following proposition (without the statement on computability for R) is [3, Theorem 1.7, Chap. III.H]. The computability of R can be easily extracted from the proof there.

Proposition 2.7 (see [3, Theorem 1.7, Chap. III.H]). *If $\mathcal{X}$ is a δ-hyperbolic geodesic space, p is a bounded (λ, ϵ)-quasi-geodesic in $\mathcal{X}$ and c is a geodesic segment joining the endpoints of p, then* $\operatorname{im} c$ *and* $\operatorname{im} p$ *are contained in the R-neighborhood of each other, where $R = R(\delta, \lambda, \epsilon)$ is a computable function.*

The following lemma is a slight modification of [16, Lemma 4.2]; see Remark 2.9. It states that, under some conditions, a concatenation of several $(\bar{\lambda}, \bar{\epsilon})$-quasi-geodesics in a δ-hyperbolic geodesic space is (λ, ϵ)-quasi-geodesic, where the numbers λ, ϵ are computable and do not depend on the number of concatenated paths.

Lemma 2.8 (see [15, Lemma 4.2] and Remark 2.9 below). *Let $\mathcal{X}$ be a δ-hyperbolic geodesic space. For every $\bar{\lambda} \geq 1, \bar{\epsilon} \geq 0, K \geq 14\delta$, there exist computable constants $\lambda = \lambda(\delta, \bar{\lambda}, \bar{\epsilon}, K) \geq 1$ and $\epsilon = \epsilon(\delta, \bar{\lambda}, \bar{\epsilon}, K) \geq 0$ satisfying the statement below.*

Assume $N \in \mathbb{N}, X_i \in \mathcal{X}, i = 0, \ldots, N,$ and $q_i : [0, n_i] \to \mathcal{X}$, where $n_i \geq 1$ are $(\bar{\lambda}, \bar{\epsilon})$-quasi-geodesic paths in $\mathcal{X}$ from X_{i-1} to $X_i, i = 1, \ldots, N,$ and suppose that the following holds for all possible i:

(1) $|X_{i-1}X_i| \geq 12(K + \delta) + \bar{\epsilon} + 1,$
(2) $|X_{i-1}X_{i+1}| \geq |X_{i-1}X_i| + |X_iX_{i+1}| - 2K.$

Then the path q obtained by a consequent concatenation of $q_1, q_2, \ldots, q_N$ is (λ, ϵ)-quasi-geodesic.

Remark 2.9. (a) Our (λ, ϵ)-quasi-geodesics are $(\frac{1}{\lambda}, \epsilon)$-quasi-geodesics in the sense of [16].

(b) Using the condition $n_i \geq 1$, it is easy to check that for the path $q : [0, \sum_{i=1}^{N} n_i] \to \mathcal{X}$ from Lemma 2.8 and for all r, s from the domain of q the following holds:

$$d(q(r), q(s)) \leq (\bar{\lambda} + \bar{\epsilon})|r - s| + 2\bar{\epsilon}.$$

(c) In [16, Lemma 4.2], there is the condition $\|q_i\| \geq (C_1 + \bar{c})/\bar{\lambda}$, however, the proof uses only its consequence $\|[X_{i-1}, X_i]\| \geq C_1$, which we adapted as condition (1) above.

2.2. *Hyperbolic groups*

Let H be a group given, together with a finite generating set S.

The length of an element $g \in H$ (with respect to S), denoted $|g|$, is defined as the length of the shortest word in $S^{\pm 1}$ which equals g in H. This naturally turns H into a metric space; $|\cdot|$ is usually called the *word* metric.

Let $\Gamma(H, S)$ be the geometric realization of the right Cayley graph of H with respect to S. We will consider $\Gamma(H, S)$ as a metric space with the metric, induced by the word metric on H: $d(g_1, g_2) = |g_1^{-1}g_2|$. In particular, edges are isometric to the real interval $[0, 1]$. We highlight the fact that there is a notational incoherence in using $|AB|$ to denote the distance between points A and B in the Cayley graph $\Gamma(H, S)$, while $|a^{-1}b|$ is the distance between the elements a and b of H; however, there will be no confusion because we adopt the convention of using capital letters when thinking elements of H as vertices of the Cayley graph.

The *ball of radius* r around 1 in $\Gamma(H,S)$ is denoted $\mathcal{B}(r)$. The cardinality of any subset $M \subseteq H$ is denoted $\sharp M$. For brevity, the cardinality of the set $\mathcal{B}(r) \cap H$ is denoted by $\sharp\mathcal{B}(r)$. Clearly, an upper bound for $\sharp\mathcal{B}(r)$ is the number of elements in the similar ball for the free group with basis S, so $\sharp\mathcal{B}(r) \leq 2(2\sharp S - 1)^r$.

The group H is called *δ-hyperbolic with respect to S* if the corresponding metric space $\Gamma(H,S)$ is δ-hyperbolic. It is well-known that if a group is hyperbolic with respect to some finite generating set, then it is also hyperbolic with respect to any other finite generating set (with a possibly different δ). This allows to define hyperbolic groups: H is said to be *hyperbolic* if for some finite generating set S, and some real number $\delta \geq 0$, H is δ-hyperbolic with respect to S. It is also well-known that a finitely generated group is free if and only if it is 0-hyperbolic with respect to some finite generating set S.

Let us begin with some well-known results about hyperbolic groups that will be needed later. The first one reproduces [3, Proposition 3.20, Chap. III.**Γ**] plus the computability of the involved constant, which can be easily extracted from the proof there.

Proposition 2.10 (see [3, Proposition 3.20, Chap. III.Γ]). *Let H be a δ-hyperbolic group with respect to a finite generating set S. For every finite set of elements $h_1, \ldots, h_r \in H$ there exists an integer $n > 0$ such that $\langle h_1^n, \ldots, h_r^n \rangle$ is free (of rank r or less). Furthermore, the integer n is a computable function of δ, $\sharp S$ and $\sum_i^r |h_i|$.*

Theorem 2.11 (see [4, Proposition 2.3]). *Let H be a δ-hyperbolic group with respect to a finite generating set S. If $u, v \in H$ are conjugate, then the length of the shortest conjugator is bounded from above by a computable function of $\max\{|u|, |v|\}, \delta$ and $\sharp S$.*

Proposition 2.12 (see [3, Corollary 3.10(1), Chap. III.Γ]). *Let H be a δ-hyperbolic group with respect to a finite generating set S, and let $g \in H$ be an element of infinite order. Then the map $\mathbb{Z} \to H$ given by $n \mapsto g^n$ is a quasi-geodesic.*

The following lemma is well-known and can be easily deduced from Proposition 2.12.

Lemma 2.13. *Let H be a δ-hyperbolic group with respect to a finite generating set S, and let $g \in H$ be an element of infinite order. If g^p and g^q are conjugate then $p = \pm q$.*

Now, we provide an alternative proof for Proposition 2.12, in order to gain computability of the involved constants.

Lemma 2.14. *The constants λ and ϵ in Proposition 2.12 are computable functions depending only on $\delta, \sharp S$ and $|g|$.*

Proof. First, we make the following two easy observations:

(1) Let $k \geq 1$ be a natural number and suppose that the map $\mathbb{Z} \to H$ given by $n \mapsto g^{kn}$ is (λ', ϵ')-quasi-geodesic. Then the map $\mathbb{Z} \to H$ given by $n \mapsto g^n$ is (λ, ϵ)-quasi-geodesic, where λ, ϵ are computable constants depending on $\lambda', \epsilon', k, |g|$ (one can take $\lambda = k\lambda'$ and $\epsilon = k^2\lambda'|g| + k\epsilon' + k$). Thus, at any moment we can replace g by an appropriate power g^k.

(2) Let g_0 be a conjugate of g in H, say $g = h^{-1}g_0h$ for some $h \in H$, and suppose that the map $\mathbb{Z} \to H$, $n \mapsto g_0^n$, is (λ', ϵ')-quasi-geodesic. Then, the map $\mathbb{Z} \to H$, $n \mapsto g^n$, is (λ, ϵ)-quasi-geodesic, where $\lambda = \lambda'$ and $\epsilon = \epsilon' + 2|h|$. Thus, at any moment we can replace g by any conjugate $h^{-1}gh$.

Now, let us prove the result. Take an element $g \in H$ of infinite order. By Lemma 2.13, there must exists an exponent $1 \leq r \leq 1 + \sharp\mathcal{B}(8\delta)$ such that the shortest conjugate of g^r, say g_0, has length $|g_0| = k > 8\delta$ (note that both r and the corresponding conjugate are effectively computable by Theorem 2.11). Replacing g by g_0 and applying the previous two paragraphs, we may assume that $|g| = k > 8\delta$ and no conjugate of g is shorter than g itself.

Take a geodesic expression for g, say $g = s_1 \cdots s_k$ with $s_i \in S^{\pm 1}$, and consider the bi-infinite path $p_g \colon \mathbb{Z} \to H$ defined by the following rule: if $n \geq 0$ and $n = tk + r$, where $0 \leq r < k$, then $p_g(n) = g^t s_1 \cdots s_r$ and $p_g(-n) = g^{-t}s_k^{-1} \cdots s_{k-r+1}^{-1}$; this corresponds to the bi-infinite word $g^\infty = \cdots s_1 \cdots s_k s_1 \cdots s_k \cdots$. Clearly, any segment of length k is of the form $s_i \cdots s_k s_1 \cdots s_{i-1}$, i.e. a conjugate of g and hence geodesic. So, p_g is a k-locall geodesic and thus a $(8\delta + 1)$-local geodesic. Finally, by Proposition 2.6, p_g is a $(3, 2\delta)$-quasi-geodesic. Hence the map $n \mapsto g^n$ is a $(3k, 2\delta)$-quasi-geodesic. ☐

Combining Proposition 2.7 with Proposition 2.12 and Lemma 2.14, we obtain the following three corollaries.

Corollary 2.15. *Let H be a δ-hyperbolic group with respect to a finite generating set S, and let $g \in H$ be of infinite order. Then for any integers $i < j$, the set $\{g^i, g^{i+1}, \ldots, g^j\}$ and any geodesic segment $[g^i, g^j]$ lie in the μ-neighborhood of each other, where $\mu = \mu(\delta, \sharp S, |g|)$ is a computable function.*

Corollary 2.16. *Let H be a δ-hyperbolic group with respect to a finite generating set S, and let $g \in H$ be of infinite order. For any natural numbers s, t we have*

$$|g^{s+t}| \geq |g^s| + |g^t| - 2\mu,$$

where $\mu = \mu(\delta, \sharp S, |g|)$ is the constant from Corollary 2.15.

Proof. Consider the points $A = 1$, $B = g^s$ and $C = g^{s+t}$ and choose geodesics $[AB]$, $[BC]$ and $[AC]$. By Corollary 2.15, there exists $D \in [AC]$ such that $|BD| \leq \mu$. Then,

$$|AC| = |AD| + |DC| \geq (|AB| - |BD|) + (|CB| - |BD|) \geq |AB| + |BC| - 2\mu. \qquad \square$$

The next result in this subsection is about torsion-free hyperbolic groups. It uses the following well-known result.

Proposition 2.17. *Let H be a torsion-free δ-hyperbolic group. Then, centralizers of nontrivial elements are infinite cyclic. In particular, extraction of roots is unique in H (i.e. $g_1^r = g_2^r$ implies $g_1 = g_2$). Furthermore, if for $1 \neq g \in H, g^p$ and g^q are conjugate then $p = q$.*

Proof. Cyclicity of centralizers is proven in [3, pp. 462–463].

Suppose $g_1^r = g_2^r$. Then both g_1 and g_2 belong to the infinite cyclic group $C_H(g_1^r)$ and so, $g_1 = g_2$.

Finally, suppose that $g^p = h^{-1}g^q h$; by Lemma 2.13, $p = \epsilon q$ where $\epsilon = \pm 1$. Extracting roots, $h^{-1}gh = g^\epsilon$. Thus, h^2 commutes with g so both are powers of a common element, say $z \in H$. But h also commutes with z so they are both powers of a common y, and so is g too. Hence, $h^{-1}gh = g$ and $\epsilon = 1$. Thus, $p = q$. □

This proposition allows to use rational exponents in the notation, when working in torsion-free δ-hyperbolic groups (with $g^{1/s}$ meaning the unique element x such that $x^s = g$, assuming it exists). For example, it is easy to see that in such a group, every element commuting with $g^r \neq 1$ must be a rational power of g.

Corollary 2.18. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. There exists a computable function $f : \mathbb{N}^2 \to \mathbb{N}$ such that, for any two elements $g, v \in H$ and for any nonnegative integers p, q, the following holds*

$$|g^p v g^q| > |g^{p+q}| - f(|g|, |v|).$$

Proof. Let $\mu = \mu(|g|)$ be the computable constant given in Corollary 2.15: for any two integers $i < j$, the set $\{g^i, g^{i+1}, \ldots, g^j\}$ is contained in the μ-neighborhood of any geodesic with endpoints g^i and g^j. Let $N = \sharp\mathcal{B}(2\delta + 2\mu + |v|)$ and $M = 2(N+1)(\mu+1)$.

Given $p, q \geq 0$, consider the points $A = 1$, $B = g^p$, $C = g^p v$, and $D = g^p v g^q$, and choose geodesics $[AB]$, $[AC]$, $[CD]$ and $[DA]$ (see Fig. 1). Let P be the point in $[CD]$ at distance $\ell = \frac{1}{2}(|AC| + |CD| - |AD|)$ from C.

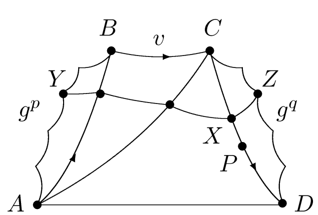

Fig. 1.

If $\ell < M$ then

$$|g^p v g^q| = |AD| = |AC| + |CD| - 2\ell \geq \big(|g^p| - |v|\big) + |g^q| - 2\ell$$
$$> |g^{p+q}| - |v| - 2M.$$

Otherwise, if $\ell \geq M$, we will prove that g and v commute and so, $|g^p v g^q| = |g^{p+q} v| \geq |g^{p+q}| - |v|$, concluding the proof.

So, assume $\ell \geq M$ and let us prove that g and v commute.

Let X be an arbitrary point on $[CD]$ with $|CX| \leq \ell$. Then X is at distance at most δ from the side $[AC]$ of the geodesic triangle ACD. But this side is in the $(\delta + |v|)$-neighborhood of the side $[AB]$ of the geodesic triangle ABC. And, by Corollary 2.15, this last one is in the μ-neighborhood of the set $\{1, g, \ldots, g^p\}$. Hence, there is a point of the form $Y = g^{p_0}$, $0 \leq p_0 \leq p$, such that $|XY| \leq 2\delta + \mu + |v|$. Similarly, X is in the μ-neighborhood of $\{C, Cg, \ldots, Cg^q\}$, i.e. there exists a point of the form $Z = Cg^{q_0} = g^p v g^{q_0}$, $0 \leq q_0 \leq q$, such that $|XZ| \leq \mu$. Thus, $|g^{p-p_0} v g^{q_0}| = |YZ| \leq |YX| + |XZ| \leq 2\delta + 2\mu + |v|$.

Now, let $X_1, \ldots, X_{N+1}$ be points on $[CD]$, such that $|CX_i| = 2i(\mu + 1)$ (the existence of all these points is ensured by our assumption $\ell \geq M$). The previous paragraph gives us points $Y_i = g^{p_i}$ and $Z_i = g^p v g^{q_i}$, with $0 \leq p_i \leq p$ and $0 \leq q_i \leq q$, such that $|X_i Y_i| \leq 2\delta + \mu + |v|$ and $|X_i Z_i| \leq \mu$; thus, $|g^{p-p_i} v g^{q_i}| \leq 2\delta + 2\mu + |v|$, for all $i = 1, \ldots, N+1$. Furthermore, note that $q_i \neq q_j$ whenever $i \neq j$ (otherwise, $Z_i = Z_j$ and $|X_i X_j| \leq |X_i Z_i| + |Z_j X_j| \leq 2\mu$, a contradiction).

This way we have obtained $N+1$ elements $g^{p-p_i} v g^{q_i}$ all of them in the ball $\mathcal{B}(2\delta + 2\mu + |v|)$, which has cardinal N. Thus, there must be at least one coincidence, $g^{p-p_i} v g^{q_i} = g^{p-p_j} v g^{q_j}$, for $i \neq j$. Hence, $v g^{q_j - q_i} v^{-1} = g^{p_j - p_i}$. Since $q_i \neq q_j$, Proposition 2.17 implies that $q_j - q_i = p_j - p_i$ and, extracting roots, $vgv^{-1} = g$. This means that g commutes with v, completing the proof. □

The following lemma is a slight modification of [16, Lemma 4.3]. Since the computability of the constant k_0 is not mentioned there, we reproduce the proof here with emphasis on the computational aspect.

Lemma 2.19 (see [16, Lemma 4.3]). *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S, and let a, b be two elements, such that $b \notin C_H(a)$. Then there is a computable integer $k_0 = k_0(|a|, |b|, \delta, \sharp S) > 0$, such that for every $k > k_0$, the element ab^k is root-free.*

Proof. Let w_1, w_2 be shortest words in the alphabet $S^{\pm 1}$ representing a and b respectively. By Proposition 2.12 and Lemma 2.14, there exist computable constants $\bar{\lambda}, \bar{\epsilon}'$ depending on δ, $\sharp S$, and $|b|$, such that for any natural n any path in $\Gamma(H, S)$ labeled by w_2^n is $(\bar{\lambda}, \bar{\epsilon}')$-quasi-geodesic. Then for any natural n any path in $\Gamma(H, S)$ labeled by $w_1 w_2^n$ is $(\bar{\lambda}, \bar{\epsilon})$-quasi-geodesic, where $\bar{\epsilon} = \bar{\epsilon}' + 2|a|$. Define two real constants:

$$K = \mu(|b|) + \frac{1}{2} f(|b|,|a|) + \frac{3}{2}|a| + 14\delta,$$

$$n_0 = \frac{\bar{\lambda}\,(12(K+\delta) + 2\bar{\epsilon} + 1) - |a|}{|b|},$$

where μ and f are the functions from Corollaries 2.15 and 2.18.

We claim that there exist computable constants $\lambda, \epsilon \in \mathbb{R}$ depending on δ, $\sharp S$, $|a|$, and $|b|$, such that for every natural $n > n_0$, the infinite path in $\Gamma(H,S)$ starting at 1 and labeled by $(w_1 w_2^n)(w_1 w_2^n)\dots$ is (λ,ϵ)-quasi-geodesic.

We prove this with the help of Lemma 2.8, where we set $X_i = (ab^n)^i$ and define q_i to be the path in $\Gamma(H,S)$ from X_{i-1} to X_i with label $w_1 w_2^n$. Since the paths q_i are $(\bar{\lambda},\bar{\epsilon})$-quasi-geodesic, it suffices to verify Conditions (1) and (2) of that lemma. Condition (1) is trivially satisfied (we use $n > n_0$, $|a| = |w_1|$, and $|b| = |w_2|$):

$$|X_{i-1}X_i| = |ab^n| \geq (\bar{\lambda})^{-1}(n|w_2| + |w_1|) - \bar{\epsilon} \geq 12(K+\delta) + \bar{\epsilon} + 1.$$

We check condition (2) with the help of Corollaries 2.18 and 2.16:

$$\begin{aligned} |X_{i-1}X_{i+1}| = |(ab^n)^2| &\geq |b^n a b^n| - |a| \\ &> |b^{2n}| - f(|b|,|a|) - |a| \\ &\geq 2|b^n| - 2\mu(|b|) - f(|b|,|a|) - |a| \\ &\geq 2|ab^n| - 2\mu(|b|) - f(|b|,|a|) - 3|a| \\ &\geq 2|ab^n| - 2K \\ &= |X_{i-1}X_i| + |X_i X_{i+1}| - 2K. \end{aligned}$$

Now we fix a natural $n > n_0$. Let $x \in C_H(ab^n)$. We take an arbitrary natural number l and consider the geodesic quadrangle $Y_1Y_2Y_3Y_4$ in $\Gamma(H,S)$ with $Y_1 = 1$, $Y_2 = x$, $Y_3 = x(ab^n)^l$, $Y_4 = x(ab^n)^l x^{-1} = (ab^n)^l$, and (λ,ϵ)-quasi-geodesic paths p from Y_2 to Y_3 and q from Y_1 to Y_4, both labeled by $(w_1w_2^n)^l$.

By Proposition 2.7, there is a constant $\nu = \nu(\delta,\lambda,\epsilon)$, such that p and the geodesic segment $[Y_2,Y_3]$ are contained in the ν-neighborhood of each other, and also q and $[Y_1,Y_4]$ are contained in the ν-neighborhood of each other.

Observe that for sufficiently large l, the sides $[Y_2,Y_3]$ and $[Y_1,Y_4]$ are much longer than the sides $[Y_1,Y_2]$ and $[Y_3,Y_4]$. Then the δ-hyperbolicity of H implies that some sufficiently long subsegments of $[Y_2,Y_3]$ and $[Y_1,Y_4]$ are contained in the 2δ-neighborhood of each other.

Thus, for sufficiently large l, there is a subpath r of p labeled by w_2^n and there are points u,v on q such that $d(r_-,u) \leq 2\nu + 2\delta$, and $d(r_+,v) \leq 2\nu + 2\delta$. Then

$$\begin{aligned} d(u,v) &\geq d(r_-,r_+) - d(r_-,u) - d(r_+,v) \\ &\geq |b^n| - (4\nu + 4\delta) \\ &\geq (\bar{\lambda})^{-1} n|b| - \bar{\epsilon}' - (4\nu + 4\delta). \end{aligned}$$

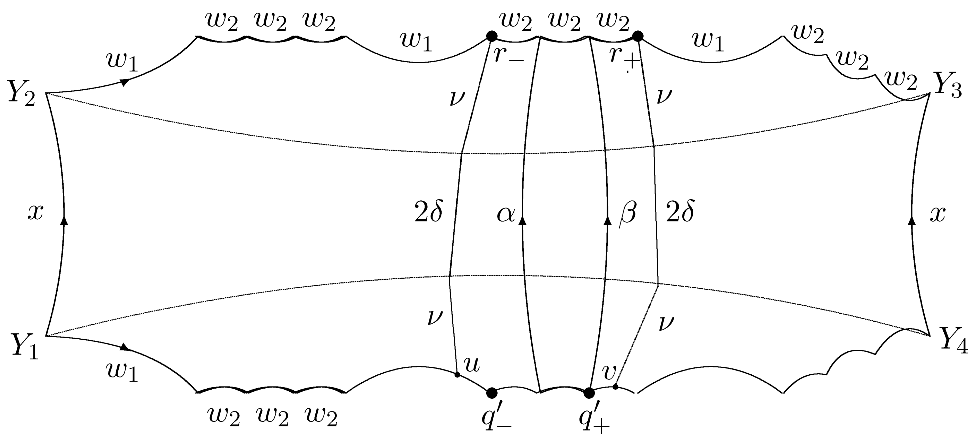

Fig. 2. Case $n = l = 3$.

We can compute $n_1 > n_0$ such that for all $n \geq n_1$

$$d(u, v) \geq (2\bar{\lambda})^{-1} n|b| + 2 \max\{|a|, |b|\}.$$

Then the subpath of q with endpoints u, v contains a subpath q' labeled by w_2^t, where

$$t = \lfloor (2\bar{\lambda})^{-1} n \rfloor. \tag{†}$$

Since every side of a geodesic quadrangle in $\Gamma(H, S)$ lies in the 2δ-neighborhood of the union of its other sides, we have

$$[u, v] \subseteq \mathcal{O}_{2\delta}([r_-, r_+] \cup [r_-, u] \cup [r_+, v]) \subseteq \mathcal{O}_{2\nu+4\delta}([r_-, r_+]),$$

hence (by Proposition 2.7)

$$q' \subseteq \mathcal{O}_\nu([u, v]) \subseteq \mathcal{O}_{3\nu+4\delta}([r_-, r_+]) \subseteq \mathcal{O}_{4\nu+4\delta}(r).$$

Consider the vertices $a_0 = q'_-, a_1, \ldots, a_t = q'_+$ of the path q' such that the subpaths between a_{i-1} and a_i are labeled by w_2 for every $1 \leq i \leq t$ (they are called *phase* vertices). Then each of them is at distance at most $4\nu + 4\delta + |w_2|$ from a phase vertex of r. Therefore, if

$$t > \sharp\mathcal{B}(4\nu + 4\delta + |b|), \tag{‡}$$

there will be two paths α and β connecting two different phase vertices of q' with some phase vertices of r having the same word written on them. Let z be the element of H represented by this word. Then $b^m = z b^{m'} z^{-1}$ for some nonzero m, m'. Since H is a torsion free hyperbolic group, we have $z \in C_H(b)$.

By construction, $x = (ab^n)^s b^k z b^{k'} (ab^n)^{s'}$ for some s, s', k, k'. We have

$$b^k z b^{k'} = (ab^n)^{-s} x (ab^n)^{-s'} \in C_H(b) \cap C_H(ab^n) = 1$$

and so $x = (ab^n)^{s+s'}$. This shows that (ab^n) is root-free for all $n \geq n_1$ as soon as we can provide condition (‡). By (†), this condition will be fulfilled for every natural

$n \geq n_2$, where

$$n_2 = 2\bar{\lambda}\left(\sharp\mathcal{B}(4\nu + 4\delta + |b|) + 1\right).$$

So, we can set $k_0 = \lceil \max\{n_1, n_2\} \rceil$. □

2.3. *Controlling cancelation*

Definition 2.20. Let H be a δ-hyperbolic group with respect to a finite generating set S. For elements $g, u, v \in H$ and a real number $c > 0$, we write $g = u \underset{c}{\cdot} v$ if $g = uv$ and $\frac{1}{2}(|u| + |v| - |uv|) < c$. Also, we write $g = u \underset{c}{\cdot} v \underset{c}{\cdot} w$ if $g = uvw$ and $uv = u \underset{c}{\cdot} v$, and $vw = v \underset{c}{\cdot} w$.

The definition of $u \underset{c}{\cdot} v$ is equivalent to $|uv| > |u| + |v| - 2c$. So, if H is a free group, $u \underset{c}{\cdot} v$ means precisely that the maximal terminal segment of u and the maximal initial segment of v which can be canceled in the product uv both have length smaller than c.

Lemma 2.21. *Let H be a δ-hyperbolic group with respect to a finite generating set S. If $c \in \mathbb{R}$ and $u, v, w \in H$ are such that $uvw = u \underset{c}{\cdot} v \underset{c}{\cdot} w$ and $|v| > 2c + \delta$, then*

$$|uvw| > |u| + |v| + |w| - (4c + 2\delta).$$

Proof. Connect the points $A = 1$, $B = u$, $C = uv$ and $D = uvw$ by geodesic segments and consider the geodesic rectangle $ABCD$. By assumption, $|BC| > 2c+\delta$. From $u \underset{c}{\cdot} v$ and $v \underset{c}{\cdot} w$, we deduce

$$|AC| > |AB| + |BC| - 2c > |AB| + \delta$$

and

$$|BD| > |BC| + |CD| - 2c > |CD| + \delta,$$

respectively. From this and the rectangle inequality (Proposition 2.3), we deduce

$$(|AB| + |BC| - 2c) + (|BC| + |CD| - 2c) < |AC| + |BD| \leq |BC| + |AD| + 2\delta,$$

which implies

$$|u| + |v| + |w| - (4c + 2\delta) = |AB| + |BC| + |CD| - (4c + 2\delta) < |AD| = |uvw|.$$

□

Next, we give some results about controlling cancelation that will be used later. Note that the important point in the following lemma is the constant c being independent from k.

Lemma 2.22. *Let H be a δ-hyperbolic group with respect to a finite generating set S, and let $w, b \in H$ with $b \neq 1$. For every integer $k \geq 0$ and every $z \in H$,* *there exists* $x \in H$ *and* $0 \le l \le k$, *such that* $z^{-1}wb^kz = x^{-1} \underset{c}{\cdot} b^{k-l}wb^l \underset{c}{\cdot} x$, *where* $c = 3\delta + \mu(|b|) + |w| + 1$ (*and* μ *is the computable function given in Corollary* 2.15).

Proof. Fix $k \ge 0$ and $z \in H$, and let $0 \le l \le k$ and $x \in H$ be such that $z^{-1}wb^kz = x^{-1}b^{k-l}wb^lx$, with the shortest possible length for x; we will prove that these l and x satisfy the conclusion of the lemma. Suppose they do not, i.e. suppose that either $x^{-1}b^{k-l}wb^l = x^{-1} \underset{c}{\cdot} b^{k-l}wb^l$ or $b^{k-l}wb^lx = b^{k-l}wb^l \underset{c}{\cdot} x$ is not true, and let us find a contradiction. We consider only the case where the first of these expressions fails, i.e. $|x^{-1}b^{k-l}wb^l| \le |x^{-1}| + |b^{k-l}wb^l| - 2c$; the second case can be treated analogously.

Consider the points $A = 1$, $B = x^{-1}$, $C = x^{-1}b^{k-l}$, $D = x^{-1}b^{k-l}w$, $E = x^{-1}b^{k-l}wb^l$ and $F = x^{-1}b^{k-l}wb^lx$, and connect them by geodesic segments, forming a 6-gon. In terms of the geodesic triangle ABE, our assumption says $\frac{1}{2}(|AB| + |BE| - |AE|) \ge c$. By δ-hyperbolicity of H, there exist points $X_1 \in [AB]$ and $X_2 \in [BE]$ such that $|BX_1| = |BX_2| = c$ and $|X_1X_2| \le \delta$. And, by Proposition 2.2 applied to the rectangle $BCDE$, there exists a point $X_3 \in [BC] \cup [CD] \cup [DE]$ such that $|X_2X_3| \le 2\delta$.

Case 1. $X_3 \in [BC]$ (see Fig. 3). Since $C = Bb^{k-l}$, Corollary 2.15 implies that there exists an element $X_4 = Bb^s$ for some $0 \le s \le k - l$, such that $|X_3X_4| \le \mu(|b|)$. Hence, $|X_1X_4| \le |X_1X_2| + |X_2X_3| + |X_3X_4| \le 3\delta + \mu(|b|) < c$ and $z^{-1}wb^kz = X_4b^{k-l-s}wb^{l+s}X_4^{-1}$.

Case 2. $X_3 \in [CD]$. In this case, take $X_4 = C$ and we have $|X_1X_4| \le |X_1X_2| + |X_2X_3| + |X_3X_4| \le 3\delta + |w| < c$ as well. Similarly, $z^{-1}wb^kz = X_4wb^kX_4^{-1}$.

Case 3. $X_3 \in [DE]$. Since $E = Db^l$, Corollary 2.15 implies again that there exist an element $X_4 = Db^s$ for some $0 \le s \le l$, such that $|X_3X_4| \le \mu(|b|)$. Like in Case 1, we have $|X_1X_4| < c$ and $z^{-1}wb^kz = X_4b^{k-s}wb^sX_4^{-1}$.

In any case, we have found an element $X_4 \in H$ and a decomposition of $z^{-1}wb^kz$ of the form $z^{-1}wb^kz = X_4b^{k-s}wb^sX_4^{-1}$, with $0 \le s \le k$ and $|X_1X_4| < c$. Since $|X_1B| = c$, we have

$$|X_4| = |AX_4| \le |AX_1| + |X_1X_4| < |AX_1| + |X_1B| = |AB| = |x|,$$

contradicting the minimality of $|x|$. □

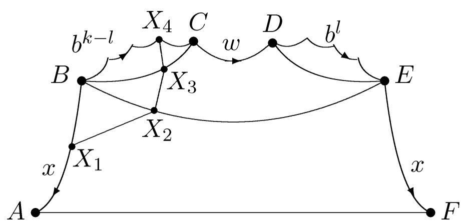

Fig. 3.

The previous lemma in the particular case of $w=1$ says that, for every $b,z\in H$ and every $k\geq 0$, there exists $x\in H$ such that $z^{-1}b^kz = x^{-1}\underset{c}{\cdot} b^k \underset{c}{\cdot} x$ (where c is a computable function depending only on δ and $|b|$). In the following result we present a technical improvement (which will be crucial later) showing that, in fact, one can choose a uniform x valid for every k.

Lemma 2.23. *Let H be a δ-hyperbolic group with respect to a finite generating set S, and let $z,b\in H$. There exists an element $x\in H$ such that, for every integer k, we have $z^{-1}b^kz = x^{-1}\underset{c}{\cdot} b^k \underset{c}{\cdot} x$, where $c=\delta+\mu(|b|)$.*

Proof. Let x^{-1} be one of the shortest elements in the set $\mathcal{G}=\{z^{-1}b^n \mid n\in\mathbb{Z}\}$. Clearly, $z^{-1}b^kz = x^{-1}b^kx$ for every $k\in\mathbb{Z}$. We show that $z^{-1}b^kz = x^{-1}\underset{c}{\cdot} b^k \underset{c}{\cdot} x$. Fix $k\in\mathbb{Z}$ and denote $A=1$, $B=x^{-1}$, and $C=x^{-1}b^k$. We choose geodesic segments $[AB]$, $[BC]$ and $[AC]$ and consider the points $X\in[BA]$, $Y\in[BC]$ such that $|BX|=|BY|=\frac{1}{2}(|BA|+|BC|-|AC|)$. By δ-hyperbolicity we have $|XY|\leq\delta$. By Corollary 2.14, the point $Y\in[BC]$ lies at distance at most $\mu(|b|)$ from a point $D\in\mathcal{G}$. By the choice of x^{-1}, we have $|AB|\leq|AD|$ and so

$$|AX|+|XB|=|AB|\leq|AD|\leq|AX|+|XY|+|YD|\leq|AX|+\delta+\mu(|b|).$$

Hence $|XB|\leq c$, i.e. $\frac{1}{2}(|x^{-1}|+|b^k|-|x^{-1}b^k|)\leq c$ and hence, $x^{-1}b^k = x^{-1}\underset{c}{\cdot} b^k$. Inverting the last element, and changing k by $-k$, we have $b^kx = b^k\underset{c}{\cdot} x$. Thus, $x^{-1}b^kx = x^{-1}\underset{c}{\cdot} b^k \underset{c}{\cdot} x$. □

2.4. *The norm and the axis of an element*

Definition 2.24. Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S, and let $g\in H$. The *norm of* g, denoted $\|g\|$, is defined as

$$\min\{d(x,gx)\mid x\in\Gamma(H,S)\}.$$

The *axis of* g, denoted $\mathcal{A}_g$, is the set of points $x\in\Gamma(H,S)$ where this minimum is achieved,

$$\mathcal{A}_g=\{x\in\Gamma(H,S)|d(x,gx)=\|g\|\}.$$

The following facts are easy to see:

(1) $\mathcal{A}_g\cap H$ is nonempty, in particular

$$\|g\|=\min\{|x^{-1}gx\|x\in H\}.$$

Moreover, $\mathcal{A}_g$ lies in the 1-neighborhood of $\mathcal{A}_g\cap H$;

(2) $\|g\|$ is a nonnegative integer satisfying $0\leq\|g\|\leq|g|$. Moreover, $\|g\|=0$ iff $g=1$;

(3) $\mathcal{A}_g$ is $C_H(g)$-invariant: for every $x\in\mathcal{A}_g$ and $h\in C_H(g)$ we have $hx\in\mathcal{A}_g$;

(4) for any $x\in\mathcal{A}_g$, any geodesic segment $[x,gx]$ also lies in $\mathcal{A}_g$;

(5) for any $h \in H$ we have $\|hgh^{-1}\| = \|g\|$ and $\mathcal{A}_{hgh^{-1}} = h\mathcal{A}_g$;
(6) for any $g \in H$ and any $x \in \Gamma(H, S)$, we have $d(x, gx) \leq \|g\| + 2d(x, \mathcal{A}_g)$.

Lemma 2.25. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. For any $1 \neq g \in H$, there exists a computable integer $r = r(|g|) \geq 1$ such that*

$$\bigcup_{k=1}^{\infty} \mathcal{A}_{g^k} \subseteq \langle g \rangle \mathcal{B}(r).$$

Proof. By Property (1), $\bigcup_{k=1}^{\infty} \mathcal{A}_{g^k}$ lies in the 1-neighborhood of $\bigcup_{k=1}^{\infty} \mathcal{A}_{g^k} \cap H$. The strategy now is to see that this last set lies at bounded (in terms of $|g|$) distance from the centralizer $C_H(g)$; and then, we will see that $C_H(g)$ lies at bounded distance from $\langle g \rangle$.

Take an arbitrary $z \in \cup_{k=1}^{\infty} \mathcal{A}_{g^k} \cap H$. By Properties (1)–(2), there is $k \geq 1$ such that $|z^{-1} g^k z|$ is minimal among the lengths of all conjugates of g^k (in particular, $|z^{-1} g^k z| \leq |g^k|$). By Corollary 2.23, there exists $x \in H$ such that $z^{-1} g^k z = x^{-1} \underset{c}{\cdot} g^k \underset{c}{\cdot} x$, where the constant $c = c(|g|)$ is computable and independent from k. Thus, we have $|x^{-1} \underset{c}{\cdot} g^k \underset{c}{\cdot} x| \leq |g^k|$. Let us consider two cases.

Case 1: $|g^k| > 2c + \delta$. By Lemma 2.21, $|x^{-1} \underset{c}{\cdot} g^k \underset{c}{\cdot} x| > 2|x| + |g^k| - (4c + 2\delta)$. Therefore $|x| < 2c + \delta$. Moreover, $z \in C_H(g)x$.
Case 2: $|g^k| \leq 2c + \delta$. From $|z^{-1} g^k z| \leq |g^k|$ and Theorem 2.11, we conclude that there exists $y \in H$ such that $z^{-1} g^k z = y^{-1} g^k y$ and the length of y is bounded by a computable constant, depending only on $|g^k|$ (i.e. on $2c+\delta$). Moreover, $z \in C_H(g)y$.

In both cases, z lies at bounded (in terms of $|g|$) distance from $C_H(g)$.

It remains to prove that $C_H(g)$ is at bounded distance from $\langle g \rangle$. Let $z \in C_H(g)$. By Lemma 2.13, there exists a (computable) natural number $s \leq \sharp\mathcal{B}(4\delta)$, such that g^s is not conjugate into the ball $\mathcal{B}(4\delta)$. In this situation, in [3, Proof of Corollary 3.10, Chap. III.F] shows that the distance from z to the set $\langle g^s \rangle$ is at most $2|g^s| + 4\delta$. Hence, the distance from z to $\langle g \rangle$ is bounded by a computable constant depending only on $\delta, \sharp S$ and $|g|$. □

From this lemma, it is easy to deduce the following corollaries.

Corollary 2.26. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. For any $1 \neq g \in H$ and any integer $k \neq 0$, there exists an element $x \in \mathcal{A}_{g^k} \cap H$ of length at most $r(|g|)$.*

Corollary 2.27. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. For any $1 \neq g \in H$ and any integer $k \neq 0$, we have $\|g^k\| \geq |g^k| - 2r(|g|)$.*

Proof. Take the element x from Corollary 2.26. Then $\|g^k\| = d(x, g^k x) = |x^{-1}g^k x| \geq |g^k| - 2|x| \geq |g^k| - 2r(|g|)$. $\square$

Corollary 2.28. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. For any $1 \neq g \in H$ and any $C > 0$, there exists a computable integer $k_0 = k_0(|g|, C)$ such that for any $k > k_0$, we have $\|g^k\| > C$.*

Proof. Using Corollary 2.27 and Proposition 2.12 complemented with Lemma 2.14, we deduce $\|g^k\| \geq |g^k| - 2r(|g|) \geq \frac{1}{\lambda}k - \epsilon - 2r(|g|)$ for every $k > 0$, where λ, ϵ and r are computable functions of $|g|$. Now, the result follows easily. $\square$

Corollary 2.29. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. There exist computable functions $f_1 : \mathbb{N} \to \mathbb{N}$ and $f_2 : \mathbb{N} \to \mathbb{N}$ such that, for every $1 \neq g \in H$ and every natural numbers $s, t > 0$, we have*

$$\|g^{s+t}\| - f_1(|g|) \leq \|g^s\| + \|g^t\| \leq \|g^{s+t}\| + f_2(|g|).$$

Proof. Take $f_1(n) = 4r(n)$ and the first inequality follows from Corollary 2.27:

$$\|g^{s+t}\| \leq |g^{s+t}| \leq |g^s| + |g^t| \leq \|g^s\| + \|g^t\| + 4r(|g|).$$

And taking $f_2(n) = 2r(n) + 2\mu(n)$, the second inequality follows from Corollaries 2.27 and 2.16:

$$\|g^s\| + \|g^t\| \leq |g^s| + |g^t| \leq |g^{s+t}| + 2\mu(|g|) \leq \|g^{s+t}\| + 2r(|g|) + 2\mu(|g|). \quad \square$$

Next, we will state several lemmas about distances to axes.

Lemma 2.30. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Let $1 \neq g \in H$, let A be a point in $\Gamma(H, S)$, and let B be a point in $\mathcal{A}_g$ at minimal distance from A. Then, for every geodesic segment $[BC] \subset \mathcal{A}_g$, we have*

$$|AC| \geq |AB| + |BC| - 2\delta.$$

Proof. Consider a given geodesic segment $[BC]$ contained in $\mathcal{A}_g$, and choose geodesic segments $[AB]$ and $[AC]$. Let $X \in [BA]$ and $Y \in [BC]$ be points such that $|BX| = |BY| = \frac{1}{2}(|BA| + |BC| - |AC|)$. Then $|XY| \leq \delta$. Since the point Y also lies on $\mathcal{A}_g$, we have that $|AB| \leq |AY|$. Therefore, $|XB| \leq |XY| \leq \delta$. Thus,

$$|AC| = |AB| + |BC| - 2|BX| \geq |AB| + |BC| - 2\delta. \quad \square$$

Lemma 2.31. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Let $g \in H$, and let k be an integer number such that $\|g^k\| > 5\delta$. Let A be an element of H, and $n \geq 0$ be such that $d(A, g^k A) = \|g^k\| + n$. Then, $A = g^t v$ for some $t \in \mathbb{Z}$ and $v \in H$ with $|v| \leq \frac{n}{2} + 3\delta + r(|g|)$, where r is the function introduced in Lemma 2.25.*

Proof. By the hypothesis, $g \neq 1$. Let B be a point in $\mathcal{A}_{g^k}$ at minimal distance from A. Let $C = g^k B$ and $D = g^k A$. Since $C \in \mathcal{A}_g$ is at minimal distance from D (the same as $|AB|$), Lemma 2.30 tells us that

$$|AC| \geq |AB| + |BC| - 2\delta$$

and

$$|DB| \geq |CD| + |BC| - 2\delta.$$

Moreover, $|BC| = \|g^k\| > 5\delta$. Therefore, by Lemma 2.4 applied to points A, B, C, D, we deduce

$$\begin{aligned} |AD| &\geq |AB| + |BC| + |CD| - 6\delta \\ &= 2|AB| + \|g^k\| - 6\delta. \end{aligned}$$

Hence, $|AB| \leq \frac{n}{2} + 3\delta$. By Lemma 2.25, B lies at distance at most $r(|g|)$ from $\langle g \rangle$. Hence, A lies at distance at most $\frac{n}{2} + 3\delta + r(|g|)$ from $\langle g \rangle$. This completes the proof. □

Lemma 2.32. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S, and let $g \in H$ with $\|g\| > 5\delta$. Then the middle point of any geodesic segment $[A, gA]$, where A is a point of $\Gamma(H, S)$, lies in the (5δ)-neighborhood of the axis $\mathcal{A}_g$.*

Proof. By the hypothesis, $g \neq 1$. Let B be a point in $\mathcal{A}_g$ at minimal distance from A. Let $C = gB$ and $D = gA$. Exactly like in the previous lemma, we obtain

$$2|AB| + |BC| \leq |AD| + 6\delta. \tag{5}$$

Now, take geodesic segments $[AD]$ and $[BC]$, and let M and N be their middle points, respectively. Clearly, $N \in \mathcal{A}_g$. In order to estimate the distance $|NM|$, we consider the geodesic rectangle $AMDN$. By the rectangle inequality, we have

$$\begin{aligned} |NM| + |AD| &\leq \max\{|AM| + |DN|, |DM| + |AN|\} + 2\delta \\ &= \max\left\{\frac{1}{2}|AD| + |DN|, \frac{1}{2}|AD| + |AN|\right\} + 2\delta. \end{aligned}$$

But $|AN| \leq |AB| + |BN| = |AB| + \frac{1}{2}|BC|$. Therefore from (5), we have $|AN| \leq \frac{1}{2}|AD| + 3\delta$. Analogously, $|DN| \leq \frac{1}{2}|AD| + 3\delta$. From all this we deduce

$$|NM| + |AD| \leq \frac{1}{2}|AD| + \frac{1}{2}|AD| + 3\delta + 2\delta.$$

Thus, $|NM| \leq 5\delta$. □

Proposition 2.33. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S, and let $g, h \in H$ with $\|g\| > 15\delta, \|h\| > 15\delta$ and $\|gh\| > 5\delta$. Then the distance between the axes $\mathcal{A}_g$ and $\mathcal{A}_h$ is at most*

$$\max\left\{15\delta,\ \frac{1}{2}(\|gh\| - \|g\| - \|h\|) + 18\delta\right\}.$$

Proof. By the hypotheses, g, h and gh are all nontrivial. Let $d = d(\mathcal{A}_g, \mathcal{A}_h)$, and let $X \in \mathcal{A}_h$ and $Y \in \mathcal{A}_g$ be such that $|XY| = d$. If $d \leq 15\delta$, we are done so, let us assume $d > 15\delta$.

Consider the points $A_1 = X$, $A_2 = Y$, $A_3 = gY$, $A_4 = gX$, $A_5 = ghX$, $A_6 = ghY$, $A_7 = ghgY$, $A_8 = ghgX$, and $A_9 = ghghX$. By Lemma 2.30 and doing the appropriate translation, we have $|A_{i-1}A_{i+1}| \geq |A_{i-1}A_i| + |A_iA_{i+1}| - 2\delta$ for every $i = 2, \ldots, 8$. Moreover, $|A_{i-1}A_i|$ equals either d, or $\|g\|$, or $\|h\|$ which are all bigger than 15δ. So, Lemma 2.4 tells us that

$$\begin{aligned} d(A_1, A_9) = d(X, (gh)^2X) \geq\ & d(X, Y) + d(Y, gY) + d(gY, gX) + d(gX, ghX) \\ & + d(ghX, ghY) + d(ghY, ghgY) + d(ghgY, ghgX) \\ & + d(ghgX, ghghX) - 26\delta \\ =\ & 2(d + \|g\| + d + \|h\|) - 26\delta. \end{aligned}$$

On the other hand,

$$\begin{aligned} d(A_1, A_5) = d(X, ghX) &\leq d(X, Y) + d(Y, gY) + d(gY, gX) + d(gX, ghX) \\ &= d + \|g\| + d + \|h\|. \end{aligned}$$

Let now $[A_1A_5]$ be a geodesic segment, and consider its translation $(gh)[A_1A_5]$, say $[A_5A_9]$. Let M be the middle point of $[A_1A_5]$ and $M' = ghM$ be the middle point of $[A_5A_9]$. Since $\frac{1}{2}d(A_1, A_5) = d(A_1, M) = d(M, A_5) = d(M', A_9)$, using the previous inequalities we have

$$\begin{aligned} d(M, M') &\geq d(A_1, A_9) - d(A_1, M) - d(M', A_9) \\ &= d(A_1, A_9) - d(A_1, A_5) \\ &\geq 2d + \|g\| + \|h\| - 26\delta. \end{aligned}$$

Finally, by Lemma 2.32, M lies at distance at most 5δ from the axis $\mathcal{A}_{gh}$. Therefore, $d(M, M') = d(M, ghM) \leq 10\delta + \|gh\|$. Hence $d \leq \frac{1}{2}(\|gh\| - \|g\| - \|h\|) + 18\delta$. □

3. Auxiliary Statements

In this section we prove statements which will be used in Sec. 4. In particular we prove Proposition 3.3 which will be used in the proof of Theorem 4.5.

Let us start with the following lemma, which considers the situation where the product of conjugates of two powers of a given element equals the product of these powers, and analyzes how the involved conjugators must look like.

Lemma 3.1. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. There exists a computable function $\hbar : \mathbb{N} \to \mathbb{R}^+$ with the following* *property: for any three elements* $b, x, y \in H$ *and any two positive integers* s, t*, which satisfy* $\|b^s\|, \|b^t\| > 15\delta, \|b^{s+t}\| > 5\delta$ *and*

$$(x \cdot b^s \cdot x^{-1})(y \cdot b^t \cdot y^{-1}) = b^{s+t}, \tag{6}$$

there exist integers n_1, n_2, n_3, n_4 *and elements* $v_x, v_y \in H$ *of length at most* $\hbar(|b|)$ *such that*

$$x = b^{n_1} v_x b^{n_2} \quad \textit{and} \quad y = b^{n_3} v_y b^{n_4}.$$

Proof. Let b, x, y and s, t be as in the statement (in particular, $b \neq 1$). Consider the axes $\mathcal{A}_{xb^sx^{-1}} = x\mathcal{A}_{b^s}$ and $\mathcal{A}_{yb^ty^{-1}} = y\mathcal{A}_{b^t}$. By Proposition 2.33 applied to the elements xb^sx^{-1} and yb^ty^{-1} (note that $\|xb^sx^{-1}\| = \|b^s\| > 15\delta$, $\|yb^ty^{-1}\| = \|b^t\| > 15\delta$ and $\|(xb^sx^{-1})(yb^ty^{-1})\| = \|b^{s+t}\| > 5\delta$ by hypothesis), the distance between $x\mathcal{A}_{b^s}$ and $y\mathcal{A}_{b^t}$ is at most

$$\max\left\{15\delta,\, \frac{1}{2}(\|b^{s+t}\| - \|b^s\| - \|b^t\|) + 18\delta\right\}.$$

By Corollary 2.29, this value does not exceed $\frac{1}{2}f_1(|b|) + 18\delta$, an upper bound which is independent from s and t.

Now, take an element $Q \in y\mathcal{A}_{b^t} \cap H$ such that $d(Q, x\mathcal{A}_{b^s}) \leq \frac{1}{2}f_1(|b|) + 18\delta + 1$, and set $P = (yb^ty^{-1})^{-1}Q$. In particular, $P \in y\mathcal{A}_{b^t} \cap H$ and $d(P, Q) = \|yb^ty^{-1}\| = \|b^t\|$. Then we have

$$\begin{aligned}
d(P, b^{s+t}P) &= d(P, (xb^sx^{-1})(yb^ty^{-1})P) = d(P, (xb^sx^{-1})Q) \\
&\leq d(P, Q) + d(Q, (xb^sx^{-1})Q) \\
&\leq d(P, Q) + 2d(Q, \mathcal{A}_{xb^sx^{-1}}) + \|b^s\| \\
&\leq \|b^t\| + \|b^s\| + f_1(|b|) + 36\delta + 2 \\
&\leq \|b^{s+t}\| + f_1(|b|) + f_2(|b|) + 36\delta + 2,
\end{aligned}$$

where the last inequality uses Corollary 2.29 again. Next, apply Lemma 2.31 to conclude that $P = b^{n_3}v_1$ for some $n_3 \in \mathbb{Z}$ and $v_1 \in H$ with $|v_1| \leq \frac{1}{2}f_1(|b|) + \frac{1}{2}f_2(|b|) + r(|b|) + 21\delta + 1$. And since $P \in y\mathcal{A}_{b^t} \cap H$, we deduce from Lemma 2.25 that $y^{-1}P = b^{-n_4}v_2$, for some $n_4 \in \mathbb{Z}$ and $v_2 \in H$ with $|v_2| \leq r(|b|)$. Hence,

$$y = b^{n_3} v_y b^{n_4},$$

where $v_y = v_1 v_2^{-1}$ has length bounded by

$$|v_y| = |v_1 v_2^{-1}| \leq |v_1| + |v_2| \leq \frac{1}{2}f_1(|b|) + \frac{1}{2}f_2(|b|) + 2r(|b|) + 21\delta + 1.$$

Finally, inverting and replacing b to b^{-1} in Eq. (6), we obtain again the same equation with x and y interchanged. So, the same argument shows that

$$x = b^{n_1} v_x b^{n_2},$$

for some $n_1, n_2 \in \mathbb{Z}$ and some $v_x \in H$ with the same upper bound for its length.

Hence, the function $\hbar(n) = \frac{1}{2}f_1(n) + \frac{1}{2}f_2(n) + 2r(n) + 21\delta + 1$ satisfies the statement of the lemma. □

Corollary 3.2. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. There exists a computable function $\hbar : \mathbb{N} \to \mathbb{R}^+$ with the following property: if $b, x_1, x_2, x_3 \in H$ and $0 \neq m_1, m_2, m_3 \in \mathbb{Z}$ are such that $\|b^{m_1}\|, \|b^{m_2}\|, \|b^{m_3}\| > 15\delta$, $x_1x_2x_3 = 1$, $m_1+m_2+m_3 = 0$, and $x_1b^{m_1}x_2b^{m_2}x_3b^{m_3} = 1$, then each of the x_i can be written in the form $b^{n_1}ub^{n_2}vb^{n_3}$, where $n_1, n_2, n_3 \in \mathbb{Z}$, and both u, v have length at most $\hbar(|b|)$.*

Proof. Inverting the last equation and cyclically permuting if necessary, we may assume that $m_1 > 0$ and $m_2 > 0$. Now, Lemma 3.1 gives the conclusion. □

Proposition 3.3. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Then, for any $g \in H$, there is a computable constant $C = C(|g|) > 0$ with the following property: for every $a, b \in \langle g \rangle$ with $\|a\|, \|b\|, \|ab^{\pm 1}\| > 15\delta$, and every conjugate b_* of b, if ab_*^s is conjugate to ab^s for every $s = -C, \ldots, C$, then $b_* = b$.*

Proof. Let $a = g^n$ and $b = g^m$ (with $n, m \neq 0$ and $n \neq \pm m$), and let $b_* = x^{-1}bx$ for some $x \in H$ (which can always be multiplied on the left by a power of b).

We may assume $n, m > 0$. Indeed, if $n < 0$, we replace g by g^{-1}, and n by $-n$, and m by $-m$; the statement does not change and we get $n > 0$. If then $m < 0$, we replace b by $b^{-1} = g^{-m}$ and b_* by b_*^{-1}; again the statement does not change and we get $m > 0$.

So, let us assume $n, m > 0$, $\|a\|, \|b\|, \|ab^{\pm 1}\| > 15\delta$, and ab_*^s being conjugate to ab^s for every $s = -C, \ldots, C$, where C is yet to be determined.

Taking $C \geq 1$, we have ab_*^{-1} conjugate to ab^{-1}, that is, $g^n \cdot x^{-1}g^{-m}x = h^{-1}g^{n-m}h$ for some $h \in H$. Rewrite this last equation into the following two forms

$$xh^{-1}g^{m-n}hx^{-1} \cdot xg^nx^{-1} = g^m, \tag{7}$$

$$h^{-1}g^{n-m}h \cdot x^{-1}g^mx = g^n. \tag{8}$$

If $m > n$, then from Eq. (7) and Lemma 3.1, we get

$$x = g^pvg^q$$

for some $p, q \in \mathbb{Z}$ and $v \in H$ with $|v| \leq \hbar(|g|)$. Otherwise, $m < n$ and then from Eq. (8) and Lemma 3.1, we get the same expression for x. Replacing x by $g^{-p}x$, we can assume $p = 0$, i.e. $x = vg^q$. And now, replacing b_* by $g^qb_*g^{-q}$, which does not affect neither the hypothesis nor the conclusion of the proposition (recall that both a and b are powers of g), we may assume that $x = v$, $|v| \leq \hbar(|g|)$.

Let us impose that ab_*^s and $ab^s = g^{n+sm}$ are conjugate, for some positive value of s. By Lemma 2.23, there exists $z_s \in H$ such that

$$g^n \cdot x^{-1}g^{sm}x = ab_*^s = z_s^{-1} \underset{c}{\cdot} g^{n+sm} \underset{c}{\cdot} z_s, \tag{9}$$

where the constant c depends only on $|g|$, δ and $\sharp S$. By Proposition 2.12 and Lemma 2.14, we can compute a constant C_0 such that $|g^{n+sm}| > 2c+\delta$, for every $s \geq C_0$. Taking at least this value for C, and using Lemma 2.21 and Corollary 2.16, we deduce that

$$|g^n| + |g^{sm}| + 2|x| \geq |ab_*^s| > |g^{n+sm}| + 2|z_s| - (4c+2\delta) \geq |g^n| \\ + |g^{sm}| - 2\mu + 2|z_s| - (4c+2\delta),$$

where $\mu = \mu(|g|)$ is the computable function from Corollary 2.15. Hence, $|z_s| \leq \hbar(|g|) + \mu(|g|) + 2c + \delta$.

Finally, take $C = C_0 + \sharp\mathcal{B}\big(\hbar(|g|) + \mu(|g|) + 2c + \delta\big)$. Having ab_*^s conjugate to ab^s for every $s = -C, \ldots, C$, we obtain elements z_s, $s = C_0, \ldots, C$, all of them in the ball $\mathcal{B}\Big(\hbar(|g|) + \mu(|g|) + 2c + \delta\Big)$ by the previous paragraph.

Hence, there must be a repetition, i.e. there exist $C_0 < s_1 < s_2 < C$ such that $z_{s_1} = z_{s_2}$ (denote it by z). We have

$$ab_*^{s_1} = z^{-1} g^{n+s_1 m} z \tag{10}$$

and

$$ab_*^{s_2} = z^{-1} g^{n+s_2 m} z,$$

from which we deduce

$$b_*^{s_2 - s_1} = z^{-1} g^{m(s_2 - s_1)} z.$$

This implies $b_* = z^{-1} g^m z$, and then (10) implies $a = z^{-1} g^n z$. Since $a = g^n$, the element z commutes with g and so, again from (10), $b_* = b$. □

4. The Main Theorem for Two Words

The following lemma is a preliminary step in proving the main result for the case of two words (Theorem 4.5). Note that Eqs. (11) and (12) in its formulation have the following common form: the product of certain conjugates of two elements equals the product of these two elements.

Lemma 4.1. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S, and let $b, w \in H$. There exists a computable constant $M = M(|b|, |w|)$ such that the following holds: if b_* is conjugate to b (say $b_* = h^{-1}bh$), and wb_*^k is conjugate to wb^k for every $k = 1, \ldots, M$, then there exists an element $d \in H$ and integers m, s, t, such that $s + t > 0$ and*

$$(d \cdot b^s \cdot d^{-1})(dw \cdot b^t \cdot w^{-1} d^{-1}) = b^{s+t}, \tag{11}$$

$$(d^{-1} h \cdot w \cdot h^{-1} d)(d^{-1} \cdot b^m \cdot d) = wb^m. \tag{12}$$

Proof. The result is obvious if $b = 1$. Let us assume $b \neq 1$.

If we prove the statement for a particular conjugator h, then we immediately have the same result for an arbitrary other, just replacing h to $b^q h$ and d to $b^q d$ (for q rational). So, we can choose our favorite h.

By Lemma 2.23, there exists a conjugator $h \in H$ such that for any integer $k \geq 0$, we have $b_*^k = h^{-1} \underset{c}{\cdot} b^k \underset{c}{\cdot} h$, where $c = \delta + \mu(|b|)$. Let us show the result for this particular h. Since this expression remains valid while enlarging the constant c, we shall consider it with $c = 3\delta + \mu(|b|) + |w| + 1$ in order to match with other calculations below. Thus,

$$wb_*^k = w\big(h^{-1} \underset{c}{\cdot} b^k \underset{c}{\cdot} h\big), \tag{13}$$

for every $k \geq 0$. Suppose that wb_*^k is conjugate to wb^k for every $k = 1, \ldots, M$, where M is still to be determined. Then, by Lemma 2.22, for each of these k's, there exist an element $e_k \in H$ and an integer l_k, such that $0 \leq l_k \leq k$ and

$$wb_*^k = e_k^{-1} \underset{c}{\cdot} (b^{k-l_k} w b^{l_k}) \underset{c}{\cdot} e_k. \tag{14}$$

We will show, that for large enough M, we will have $e_i = e_j$ for some pair of integers $i \neq j$. By Corollary 2.18, Proposition 2.12 and Lemma 2.14, there exists a computable constant $k_0 = k_0(|b|, |w|) > 0$ such that both $|b^{k-l_k} w b^{l_k}|$ and $|b^k|$ are bigger than $2c + \delta$ for all $k \geq k_0$.

We introduce the following notation: for two sequences of elements $u_k \in H$ and $v_k \in H$ (where k runs through a subset of $\mathbb{N}$) we write $u_k \approx v_k$ if $|u_k^{-1} v_k|$ is bounded from above by a computable function, depending on δ, $\sharp S$, w, and b only (so, in particular, not depending on k). The function will be clear from the context. Similarly, we write $|u_k| \approx |v_k|$ if $\| u_k| - |v_k\|$ is bounded from above by a computable function, depending on the same arguments.

Take $k \geq k_0$. Then from (13) and (14), and with the help of Lemma 2.21, we deduce

$$|wb_*^k| \approx 2|h| + |b^k|$$

and

$$|wb_*^k| \approx 2|e_k| + |b^{k-l_k} w b^{l_k}| \approx 2|e_k| + |b^k|,$$

where the last approximation is due to Corollaries 2.16 and 2.18. Therefore $|e_k| \approx |h|$.

Now we will prove that $e_k^{-1} \approx h^{-1}$. For that, we realize the right-hand side of (13) in the Cayley graph $\Gamma(H, S)$ as the path starting at 1 and consisting of 4 consecutive geodesics with labels equal in H to the elements w, h^{-1}, b^k, and h. Analogously, we realize the right-hand side of (14) as the path starting at 1 and consisting of 3 consecutive geodesics with labels equal in H to the elements e_k^{-1}, $b^{k-l_k} w b^{l_k}$ and e_k (see Fig. 4).

Both paths are (λ, ϵ)-quasi-geodesics connecting 1 and $C = wb_*^k$, where λ and ϵ are computable and depending only on c. We choose a geodesic $[1, C]$ and denote $X = wh^{-1}b^k$, $Y = e_k^{-1} b^{k-l_k} w b^{l_k}$.

By Proposition 2.7, these quasigeodesics are both at bounded distance $R = R(\delta, c)$ from the segment $[1, C]$. Therefore there are points $A, B \in [1, C]$, such that $|XA| \leq R$ and $|YB| \leq R$. In our notations we can write $|XA| \approx 0$ and $|YB| \approx 0$.

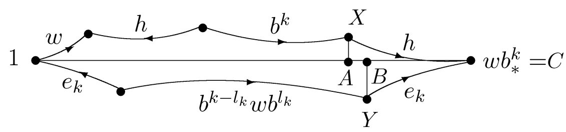

Fig. 4.

Therefore $|AC| \approx |XC| = |h|$ and $|BC| \approx |YC| = |e_k|$. Since $|h| \approx |e_k|$, we have $|AC| \approx |BC|$ and so $|AB| \approx 0$. Hence, $|he_k^{-1}| = |XY| \leq |XA| + |AB| + |BY| \approx 0$. This means that $e_k^{-1} \approx h^{-1}$ and so, e_k^{-1} lies in the ball with center h^{-1} and radius depending only on $|b|$ and $|w|$.

Let M be $1 + k_0$ plus the number of elements in this ball. There must exist $k_0 \leq k_1 < k_2 \leq M$ such that $e_{k_1} = e_{k_2}$. Denote this element by e and, rewriting Eq. (14) for these two special values of k,

$$wb_*^{k_1} = e^{-1}(b^{k_1 - l_{k_1}} wb^{l_{k_1}})e \tag{15}$$

and

$$wb_*^{k_2} = e^{-1}(b^{k_2 - l_{k_2}} wb^{l_{k_2}})e,$$

we get

$$b_*^{k_2 - k_1} = e^{-1}(b^{-l_{k_1}} w^{-1} b^{k_2 - k_1 + l_{k_1} - l_{k_2}} wb^{l_{k_2}})e.$$

Let $s = k_2 - k_1 + l_{k_1} - l_{k_2}$ and $t = l_{k_2} - l_{k_1}$ (so $s + t > 0$). Recalling that $b_*^{k_2 - k_1} = h^{-1}b^{k_2 - k_1}h$, we can rewrite the previous equation as

$$he^{-1}b^{-l_{k_1}} w^{-1} b^s wb^t b^{l_{k_1}} eh^{-1} = b^{s+t}.$$

Setting $d = he^{-1}b^{-l_{k_1}}w^{-1}$, we deduce $(db^s d^{-1}) \cdot (dwb^t w^{-1} d^{-1}) = b^{s+t}$, which is Eq. (11). And using Eq. (15), the definition of d and $b_*^{k_1} = h^{-1}b^{k_1}h$, we obtain $(d^{-1}hwh^{-1}d) \cdot (d^{-1}b^{k_1}d) = wb^{k_1}$, which is Eq. (12) with $m = k_1$. □

Now, using (11) and (12) and distinguishing the cases $st \neq 0$ or $st = 0$, we will obtain more information about relations between w, b and h.

Proposition 4.2. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S and let b, w, d be elements of H satisfying Eq.* (11). *Suppose additionally that $\|b^k\| > 15\delta$ for all $k > 0$, and that $st \neq 0$. Then, there exist integers p, q, r, and elements $u, v \in H$ of length at most $\hbar(|b|)$, such that*

$$w = b^p ub^r vb^q.$$

Proof. This follows directly from Corollary 3.2. □

Proposition 4.3. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S and let b, w, d, h be elements of H satisfying Eqs.* (11) *and* (12) *with $s+t>0$. Suppose additionally that $st=0$. Then $h=b^p w^q$ for some rational numbers p,q.*

Proof. Let us distinguish two cases.
Case 1. $s=0$. In this case, Eq. (11) says that dw commutes with b. So, $dw=b^p$ for some rational p. Plugging this into Eq. (12), we obtain $hwh^{-1}=b^{p+m}wb^{-p-m}$. Hence, $b^{-p-m}h$ commutes with w and the result follows.
Case 2. $t=0$. In this case, Eq. (11) says that d commutes with b. So, $d=b^p$ for some rational p. Plugging this into Eq. (12), we obtain $b^{-p}hwh^{-1}b^p=w$. Hence, $b^{-p}h$ commutes with w and the result follows. □

Next, we need to obtain some extra information by applying Lemma 4.1 to sufficiently many different elements w. To achieve this goal, given a pair of elements $a,b\in H$, we consider the finite set

$$\mathcal{W}=\{(a^ib)^{2j}\mid 1\leq i\leq 1+N,\ 1\leq j\leq 1+3N^2\}\subseteq\langle a,b\rangle\leq H,$$

where

$$N=N(|b|)=\sharp\mathcal{B}\left(\hbar(|b|)\right),$$

and $\hbar$ is the function from Lemma 3.1. Let us systematically apply Lemma 4.1 to every $w\in\mathcal{W}$.

Lemma 4.4. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Let $a,b\in H$ be elements generating a free subgroup of rank 2, and with $\|b^k\|>15\delta$ for all $k>0$. Suppose that for every $w\in\mathcal{W}$, there exists a conjugate b_* of b such that the elements w,b,b_* satisfy the hypothesis of Lemma 4.1 (i.e. wb_*^k is conjugate to wb^k, for every integer $k=1,\dots,M(|b|,|w|)$). Then, for at least one such $w\in\mathcal{W}$, the conclusion of Lemma 4.1 holds with $st=0$.*

Proof. Under the hypothesis of the lemma, suppose that we have Eqs. (11) and (12) with $st\neq 0$ for every $w\in\mathcal{W}$, and let us find a contradiction.

Write $\mathcal{W}=\bigsqcup_{i=1}^{1+N}\mathcal{W}_i$, where $\mathcal{W}_i=\{(a^ib)^{2j}\mid 1\leq j\leq 1+3N^2\}$, and fix a value for $i\in\{1,\dots,N+1\}$.

By Proposition 4.2, for every $w\in\mathcal{W}_i$, there exist integers p,q,r, and elements $u,v\in H$ of length at most $\hbar(|b|)$ such that

$$b^pwb^q=ub^rv \tag{16}$$

(of course, these integers and elements depend on w). Since $\sharp\mathcal{W}_i=1+3N^2>3(\sharp\mathcal{B}(\hbar(|b|)))^2$ (because $\langle a,b\rangle$ is free of rank 2) and the lengths of u and v are at most $\hbar(|b|)$, there must exist four diferent elements of $\mathcal{W}_i$ with the same u and v. That is, there exists $w_1=(a^ib)^{\sigma}$, $w_2=(a^ib)^{\tau}$, $w_3=(a^ib)^{\sigma'}$, and $w_4=(a^ib)^{\tau'}$

(where the exponents $0 < \sigma < \tau < \sigma' < \tau'$ all differ at least 2 from each other) such that

$$b^{p_1} w_1 b^{q_1} = u b^{r_1} v, \qquad b^{p_2} w_2 b^{q_2} = u b^{r_2} v,$$
$$b^{p_3} w_3 b^{q_3} = u b^{r_3} v, \qquad b^{p_4} w_4 b^{q_4} = u b^{r_4} v.$$

Combining these equations, we get

$$\begin{aligned} b^{p_2} w_2 b^{q_2 - q_1} w_1^{-1} b^{-p_1} &= u b^{r_2 - r_1} u^{-1}, \\ b^{p_4} w_4 b^{q_4 - q_3} w_3^{-1} b^{-p_3} &= u b^{r_4 - r_3} u^{-1}. \end{aligned} \tag{17}$$

Hence, the left-hand sides of these two equations commute. Let us rewrite them in the form

$$x = b^{\alpha}(a^i b)^{\tau} b^{\beta} (a^i b)^{-\sigma} b^{\gamma},$$
$$x' = b^{\alpha'}(a^i b)^{\tau'} b^{\beta'} (a^i b)^{-\sigma'} b^{\gamma'},$$

where $0 < \sigma < \tau$ and $0 < \sigma' < \tau'$ all differ at least 2 from each other (and we have no specific information about the integers $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$). The key point here is that this commutativity relation between x and x' happens inside the free group $\langle a, b\rangle$.

Consider now the monomorphism $\langle a, b\rangle \to \langle a, b\rangle$ given by $a \mapsto a^i b$, $b \mapsto b$. Since x and x' both lie in its image, and commute, their preimages, namely $y = b^{\alpha} a^{\tau} b^{\beta} a^{-\sigma} b^{\gamma}$ and $y' = b^{\alpha'} a^{\tau'} b^{\beta'} a^{-\sigma'} b^{\gamma'}$, must also commute.

Suppose $\beta\beta' \neq 0$. Then, y is not a proper power in $\langle a, b\rangle$ (in fact, its cyclic reduction is either $a^{\tau} b^{\beta} a^{-\sigma} b^{\alpha+\gamma}$ with $\alpha + \gamma \neq 0$ or $a^{\tau-\sigma} b^{\beta}$, which are clearly not proper powers). Similarly, y' is not a proper power either. Then the commutativity of y and y' forces $y = y'^{\pm 1}$, which is obviously not the case. Hence, $\beta\beta' = 0$. Without loss of generality, we can assume $\beta = 0$.

Let us go back to Eq. (17) which particularized to this special case, reads

$$b^{\alpha}(a^i b)^{\tau} b^0 (a^i b)^{-\sigma} b^{\gamma} = u b^{\theta} u^{-1},$$

where $\theta = r_2 - r_1$, that is,

$$b^{\alpha}(a^i b)^{\rho} b^{\gamma} = u b^{\theta} u^{-1}, \tag{18}$$

where $\rho = \tau - \sigma \geq 2$. Recall that all these arguments were started for a fixed value of i and that the corresponding element u (which depends on the chosen i) has length at most $\hbar(|b|)$.

Finally, it is time to move $i = 1, \ldots, 1 + N$. Since $1 + N > \sharp\mathcal{B}(\hbar(|b|))$, there must exist two indices $1 \leq i_1 < i_2 \leq 1 + N$ giving the same u. Equation (18) in these two special cases is

$$b^{\alpha}(a^{i_1} b)^{\rho} b^{\gamma} = u b^{\theta} u^{-1}$$

and

$$b^{\bar{\alpha}}(a^{i_2} b)^{\bar{\rho}} b^{\bar{\gamma}} = u b^{\bar{\theta}} u^{-1},$$

where $\rho, \bar{\rho} \geq 2$ and $1 \leq i_1 < i_2$. Again, $z = b^{\alpha}(a^{i_1} b)^{\rho} b^{\gamma}$ and $\bar{z} = b^{\bar{\alpha}}(a^{i_2} b)^{\bar{\rho}} b^{\bar{\gamma}}$ commute. Since i_1, i_2, ρ and $\bar{\rho}$ are all positive, this implies that some positive power of z equals some positive power of $\bar{z}$. But it is straightforward to see that (after all possible reductions) the first a-syllable of any positive power of z is a^{i_1} (here we use $\rho \geq 2$); similarly the first a-syllable of any positive power of $\bar{z}$ is a^{i_2}. Since $i_1 \neq i_2$, this is a contradiction and the proof is completed. □

Now, we can already prove the main Theorem 1.2, in the special case $n = 2$.

Theorem 4.5. *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S, and consider four elements $a, b, a_*, b_* \in H$ such that a_* is conjugate to a, and b_* is conjugate to b. There exists a computable constant L (only depending on $|a|, |b|, \delta$ and $\sharp S$), such that if $(a_*^i b_*^l)^j b_*^k$ is also conjugate to $(a^i b^l)^j b^k$ for every $i, j, k, l = -L, \ldots, L$ then there exists a uniform conjugator $g \in H$ with $a_* = g^{-1}ag$ and $b_* = g^{-1}bg$ (i.e. (a_*, b_*) is conjugate to (a, b)).*

Proof. The conclusion is obvious if a or b is trivial. So, let us assume $a \neq 1$ and $b \neq 1$. Note that $\langle a \rangle = \langle b \rangle$ is allowed.

Suppose that $(a_*^i b_*^l)^j b_*^k$ is conjugate to $(a^i b^l)^j b^k$ for every $i, j, k, l = -L, \ldots, L$, where L is still to be determined. We shall prove the result imposing several times that L is big enough, in a constructive way. At the end, collecting together all these requirements, we shall propose a valid value for L.

Since H is torsion-free, every nontrivial element has infinite cyclic centralizer (see Proposition 2.17). Let a_1, b_1 be generators of $C_H(a)$ and $C_H(b)$. Inverting a_1 or a_2 if necessary, we may assume that $a = a_1^p$ and $b = b_1^q$ for positive p and q. By Corollary 2.28, there exists a computable natural number r_0 such that for every $r \geq r_0$, $\|a_1^r\| > 15\delta$ and $\|b_1^r\| > 15\delta$. So, after replacing a, b, a_*, b_* by $a^{r_0}, b^{r_0}, a_*^{r_0}, b_*^{r_0}$, we can assume that $\|a^r\| > 15\delta$ and $\|b^r\| > 15\delta$ for every $r \neq 0$. Moreover, if a, b generate a cyclic group, then after the above replacement either $a = b$ or $\|ab^{-1}\| > 15\delta$. Analogously, either $a = b^{-1}$, or $\|ab\| > 15\delta$.

For every word w on a and b, let us denote by w_* the corresponding word on a_* and b_*. Now, observe that we can uniformly conjugate a_* and b_* by any element of H (and abuse notation denoting the result a_* and b_* again), and both the hypothesis and conclusion of the theorem does not change. In particular, for any chosen word of the form $w = (a^i b^l)^j b^k$ (with $i, j, k, l = -L, \ldots, L$), we can assume that $w_* = w$ (of course, with an underlying a_* and b_* now depending on w); when doing this, we say that we *center the notation on w*. Note that centering notation does not change a, b, therefore the constant L is not affected.

Let us distinguish two cases.

Case 1. *$\langle a, b \rangle$ is a cyclic group, say $\langle g \rangle$.* Centering the notation on a, we may assume that $a_* = a$. If $a = b^\epsilon$, where $\epsilon = \pm 1$, then we use that $ab_*^{-\epsilon}$ is conjugate to $ab^{-\epsilon} = 1$ and deduce immediately that $b_* = b$. Now, assume that $a \neq b^{\pm 1}$, and so $\|ab^{\pm 1}\| > 15\delta$. Part of our hypothesis says that $a_* b_*^l = ab_*^l$ is conjugate to ab^l for every $l = -L, \ldots, L$. Hence, taking L bigger than or equal to the constant $C = C(|g|)$ from Proposition 3.3, we obtain $b_* = b$. This concludes the proof in this case.

Case 2. $\langle a, b\rangle$ *is not cyclic.* By Proposition 2.10, there exists a sufficiently big and computable natural number p such that $\langle a^p, b^p\rangle$ is a free subgroup of H of rank 2. Note that, multiplying the constant by p, and using the uniqueness of root extraction in H, the result follows from the same result applied to the elements a^p, b^p and a_*^p, b_*^p. So, after replacing a, b, a_*, b_* by a^p, b^p, a_*^p, b_*^p, we can assume that $F_2 \simeq \langle a, b\rangle \leq H$.

With these gained assumptions, let us show that any constant

$$L \geq \max\{2 + 6N^2, \max_{w\in\mathcal{W}} M(|b|, |w|)\},$$

works for our purposes, where the number N and the set $\mathcal{W}$ are defined before Lemma 4.4, and the function M is defined in Lemma 4.1.

Part of our hypothesis says that, for every $w = (a^i b)^{2j} \in \mathcal{W}$, $w_* b_*^k = (a_*^i b_*)^{2j} b_*^k$ is conjugate to wb^k for every $k = 1, \ldots, M(|b|, |w|)$.

Fix $w \in \mathcal{W}$. Centering the notation on this w, we have that wb_*^k $(= w_* b_*^k)$ is conjugate to wb^k for every $k = 1, \ldots, M(|b|, |w|)$. That is, w satisfies the hypothesis of Lemma 4.1 (with the corresponding value of b_*). And this happens for every $w \in \mathcal{W}$. Thus, Lemma 4.4 ensures us that the conclusion of Lemma 4.1 holds with $st = 0$ for at least one $w_0 = (a^{i_0} b)^{2j_0} \in \mathcal{W}$, $1 \leq i_0 \leq 1 + N$, $1 \leq j_0 \leq 1 + 3N^2$ (note that Lemma 4.4 can be applied because we previously gained the assumptions $\|b^r\| > 15\delta$ for every $r \neq 0$, and $F_2 \simeq \langle a, b\rangle \leq H$). For the rest of the proof, let us center the notation on this particular w_0.

Using Proposition 4.3, we conclude that every conjugator from b to b_* (say $b_* = h^{-1}bh$) is of the form $h = b^p w_0^q$ for some rational numbers p, q. Hence, $w_0^{-q} b w_0^q = b_*$. Then,

$$((w_0^{-q} a w_0^q)^{i_0} b_*)^{2j_0} = w_0^{-q} (a^{i_0} b)^{2j_0} w_0^q = w_0^{-q} w_0 w_0^q = w_0 = w_{0*} = (a_*^{i_0} b_*)^{2j_0}.$$

Extracting roots twice, we conclude that $w_0^{-q} a w_0^q = a_*$. Thus, w_0^q is a uniform right conjugator from (a, b) to (a_*, b_*). This concludes the proof for this second case. □

5. Main Theorem for Several Words

Finally, we extend the result to arbitrary tuples of words, thus proving the main result of the paper.

Proof of Theorem 1.2. The implication to the right is obvious (without any bound on the length of W).

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$, and assume that $W(a_{1*}, \ldots, a_{n*})$ is conjugate to $W(a_1, \ldots, a_n)$ for every word W in n variables and length up to a constant yet to be determined. As above, we shall prove the result assuming several times this constant to be big enough, in a constructive way. The reader can collect together all these requirements, and find out a valid explicit value (which will depend only on δ, $\sharp S$ and $\sum_{i=1}^{n}|a_i|$). Decreasing n if necessary, we may assume that all a_i are nontrivial. If $n=1$, there is nothing to prove, so assume $n\geq 2$.

Suppose the elements $a_1,\ldots,a_n$ generate a cyclic group, say $\langle a_1,\ldots,a_n\rangle\leq\langle g\rangle\leq H$, with g root-free. Applying Theorem 4.5 to every pair a_1,a_j, we get a computable constant such that if $W(a_{1*},a_{j*})$ is conjugate to $W(a_1,a_j)$ for every word W of length up to this constant, then a_1 and a_j admit a common conjugator, say x_j. Taking the maximum of these constants over all $j=2,\ldots,n$, we are done because $x_j^{-1}a_1x_j=a_{1*}$ and $x_j^{-1}a_jx_j=a_{j*}$ for $j=2,\ldots,n$ imply that $x_2x_j^{-1}\in C_H(a_1)=\langle g\rangle$, and hence $x_2^{-1}a_jx_2=x_j^{-1}(x_jx_2^{-1}a_jx_2x_j^{-1})x_j=x_j^{-1}a_jx_j=a_{j*}$ for $j=2,\ldots,n$; thus, x_2 becomes a common conjugator.

So, we are reduced to the case where two elements of $\mathcal{A}$, say a_1 and a_2, generate a noncyclic group. In this case, by Proposition 2.10, there is a big enough computable m such that $\langle a_1^m,a_2^m\rangle$ is a free group of rank 2. Replacing a_1,a_2 by a_1^m,a_2^m and a_{1*},a_{2*} by a_{1*}^m,a_{2*}^m, and multiplying the computable constant by m, we may assume that $\langle a_1,a_2\rangle$ is free of rank 2.

By Theorem 4.5 (and taking the constant appropriately big), a_1 and a_2 admit a common conjugator. So, conjugating the whole tuple $a_{1*},\ldots,a_{n*}$ accordingly, we may assume that $a_{1*}=a_1$ and $a_{2*}=a_2$. We will prove that $a_{j*}=a_j$ for every $j=3,\ldots n$ as well.

By Lemma 2.19 twice, there exists a big enough computable $k\geq 2$ such that the elements $a_1a_2^k$ and $a_2(a_1a_2^k)^k$ are root-free (and form a new basis for $\langle a_1,a_2\rangle$). Replacing a_1 by $a_1a_2^k$ and a_{1*} by $a_{1*}a_{2*}^k$, and a_2 by $a_2(a_1a_2^k)^k$ and a_{2*} by $a_{2*}(a_{1*}a_{2*}^k)^k$, and updating the constant, we may assume that both a_1 and a_2 are root-free in H.

For every $j\geq 3$, let us apply Theorem 4.5 to the pairs (a_1,a_j) and $(a_{1*}=a_1,a_{j*})$; we obtain $x_j\in C_H(a_1)=\langle a_1\rangle$ such that $a_{j*}=x_j^{-1}a_jx_j$. Analogously, playing with the pair of indices $2,j$, we get $y_j\in C_H(a_2)=\langle a_2\rangle$ such that $a_{j*}=y_j^{-1}a_jy_j$. In particular, $x_j=a_1^{p_j}$ and $y_j=a_2^{q_j}$ for some integers p_j,q_j. Furthermore, $x_jy_j^{-1}\in C_H(a_j)$, that is, $a_1^{p_j}a_2^{-q_j}=a_j^{r_j}$ for some rational r_j. Note that if $p_jq_j=0$, then $a_{j*}=a_j$ as we want.

Again by Lemma 2.19, there are big enough computable $k',k''\geq 2$ such that $b_1=a_1a_2^{k'}$ and $b_2=a_2(a_1a_2^{k'})^{k''}$ are again root-free in H. Arguing like in the previous paragraph with these new elements, we deduce a similar conclusion: for each $j=3,\ldots,n$, either $a_{j*}=a_j$, or $b_1^{p'_j}b_2^{-q'_j}=a_j^{r'_j}$ for some nonzero integers p'_j,q'_j and some rational r'_j.

Thus, for each $j=3,\ldots,n$, we either have (1) $a_{j*}=a_j$, or (2) $a_1^{p_j}a_2^{-q_j}=a_j^{r_j}$ and $b_1^{p'_j}b_2^{-q'_j}=a_j^{r'_j}$ for some nonzero integers p_j,q_j,p'_j,q'_j and some rationals r_j,r'_j. But this last possibility would imply that the elements $a_1^{p_j}a_2^{-q_j}$ and $b_1^{p'_j}b_2^{-q'_j}=(a_1a_2^{k'})^{p'_j}(a_2(a_1a_2^{k'})^{k''})^{-q'_j}$ commute in the free group $\langle a_1,a_2\rangle$, which is not the case, taking into account that $p_jq_jp'_jq'_jk'k''\neq 0$. Therefore, $a_{j*}=a_j$ for each $j=1,\ldots,n$ and the proof is complete. □

6. Proof of Theorem 1.3

Let $(u_{i,j})$ and $(v_{i,j})$, where $i = 1, \ldots, n$, $j = 1, \ldots, m_i$, be two lists of elements in H, divided in n blocks, see (1). We shall decide, whether there is an automorphism of H sending the first list to the second up to conjugation, so that the conjugators in every block are the same.

For every $i = 1, \ldots, n$, we compute the constant C_i (depending only on δ, $\sharp S$ and $\sum_{j=1}^{m_i} |v_{i,j}|$) given in Theorem 1.2. By this theorem, for every $\alpha \in \mathrm{Aut}(H)$ and every $i = 1, \ldots, n$, the following statements are equivalent:

(1) the elements $v_{i,j}$ and $\alpha(u_{i,j})$, $j = 1, \ldots, m_i$, are conjugated by the same conjugator;

(2) the elements $W(v_{i,1}, \ldots, v_{i,m_i})$ and $W(\alpha(u_{i,1}), \ldots, \alpha(u_{i,m_i}))$ are conjugated for every word W in m_i variables and length up to C_i.

Now, let us enlarge each block of u's and v's with all the elements of the form $W(u_{i,1}, \ldots, u_{i,m_i})$ and $W(v_{i,1}, \ldots, v_{i,m_i})$, respectively, where W runs over the set of all words in m_i variables and length less than or equal to C_i. Our problem is now equivalent to deciding whether there exists an automorphism $\alpha \in \mathrm{Aut}(H)$ sending $W(u_{i,1}, \ldots, u_{i,m_i})$ to a conjugate of $W(v_{i,1}, \ldots, v_{i,m_i})$ for every i and for every W of length less than or equal to C_i. This is decidable by Dahmani–Guirardel's recent solution to the first Whitehead problem for hyperbolic groups (see [5, Corollary 5]). □

Acknowledgments

The first named author thanks the MPIM at Bonn for its support and excellent working conditions during the fall 2008, while this research was finished. The second named author gratefully acknowledges partial support from the MEC (Spain) and the EFRD (EC) through projects number MTM2006-13544 and MTM2008-01550. We are grateful to an anonymous referee for helpful comments.

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