Around Ax-Kochen/Ershov transfer principle

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GRK Workshop on C_i -fields

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Long story short...

Definition

Given integers $i \ge 0$ and $d \ge 1$, a field K is called

- $C_i(d)$ if every homogeneous polynomial of degree d with coefficients in K and $n > d^i$ variables has a non-trivial solution in K.
- C_i if it is $C_i(d)$ for every $d \ge 1$.

So far in this workshop we have proved:

- K is algebraically closed if and only if K is C_0 .
- If K is C_1 , then K has trivial Brauer group.
- Finite fields are C_1 .
- If K is C_i and L|K is a finite extension, then L is C_i .
- If K is C_i , then K(t) is C_{i+1} .
- If K is C_i , then K((t)) is C_{i+1} .
- ▶ $\mathbb{F}_p(t)$ and $\mathbb{F}_p((t))$ are C_2 , and $\mathbb{C}((t))$ is C_1 (so in particular, it has trivial Brauer group).
- \mathbb{Q}_p is $C_2(3)$.

Long story short...

Knowing that the fields $\mathbb{F}_p((t))$ and \mathbb{Q}_p share many properties, and that $\mathbb{F}_p((t))$ was C_2 , Artin conjectured that \mathbb{Q}_p was also C_2 . However, (as for \mathbb{Q} and \mathbb{R}) \mathbb{Q}_p is not C_i for any i.

However, Ax and Kochen showed that, to a certain extend, Artin had the right intuition.

Theorem (Ax-Kochen)

Fix d > 0. Then, there is a finite set E_d of prime numbers such that for every $p \notin X_d$, \mathbb{Q}_p is $C_2(d)$.

This is the situation:

- ▶ Since \mathbb{Q}_p is not C_i for every $i \ge 0$: for every $i \ge 0$ and every prime p, there is d = d(i, p) such that \mathbb{Q}_p is not $C_i(d)$.
- ▶ By the previous theorem, for every $d \ge 1$, there is $N = N(d) \ge 1$ such that if p > N then \mathbb{Q}_p is $C_2(d)$ (and hence $C_i(d)$ for every $i \ge 2$).

How similar are $\mathbb{F}_p((t))$ and \mathbb{Q}_p ?

Clearly we have that

$$\mathbb{F}_p((t)) \not\cong \mathbb{Q}_p$$

but what kind of relation could one establish between these two fields?



- ▶ both fields are complete (and hence henselian)
- ▶ both fields have residue field \mathbb{F}_p
- ▶ both fields have value group \mathbb{Z}

The key is to forget the previous question and look at the classes $(\mathbb{F}_p((t))_p)_{p>0}$ and $(\mathbb{Q}_p)_{p>0}$ asymptotically!

The transfer principle

Theorem (Ax-Kochen/Ershov)

Let φ be a first order sentence in the language of valued fields. Then there is a finite set of prime numbers E_{φ} such that for all $p \notin E_{\varphi}$

 φ holds in \mathbb{Q}_p if and only if φ holds in $\mathbb{F}_p((t))$.

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 φ holds in \mathbb{Q}_p if and only if φ holds in $\mathbb{F}_p((t))$.

 $\mathbb{Q}_p \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \models \varphi$

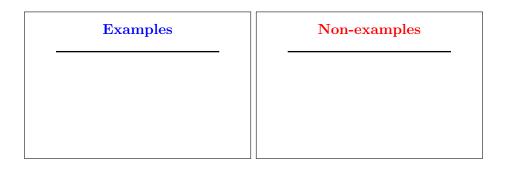
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Informally, a first-order formula in the language of rings \mathcal{L}_{ring} is a formal expression build up using

- variables x_1, x_2, x, y, z etc.
- \blacktriangleright boolean connectives $\land,\lor,\neg,\rightarrow,\leftrightarrow$
- ▶ quantifiers \forall, \exists
- the equality symbol =
- ▶ the formal symbols of the language of rings $\mathcal{L}_{ring} = \{+, -, \cdot, 1, 0\}$
- ▶ and parenthesis symbols (for convenience),

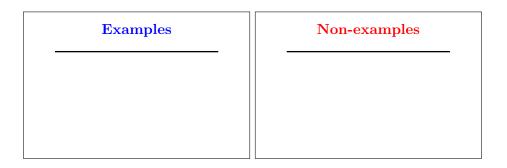
following "natural" rules of construction.

The slogan: "given a finite sequence of symbols φ build up from the symbols above, if after replacing (the free occurrences of) variables by elements in any ring A, we obtain a statement which is true or false in A, then φ is an \mathcal{L}_{ring} -formula".

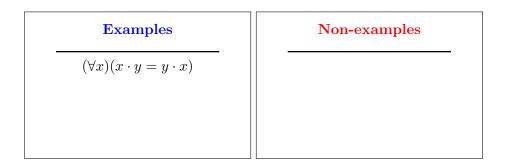


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 $(\forall x)(x \cdot y = y \cdot x)$

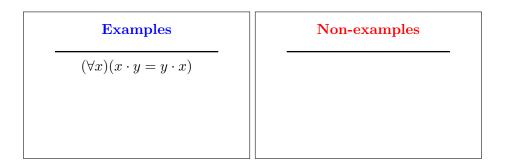


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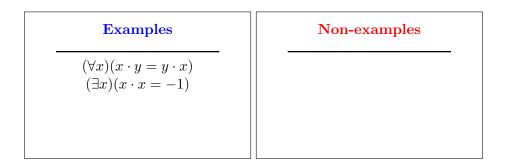


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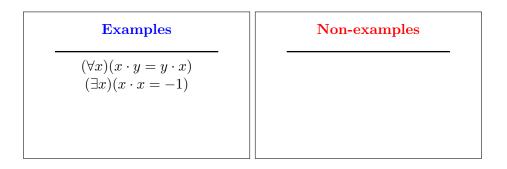
$$(\exists x)(x \cdot x = -1)$$



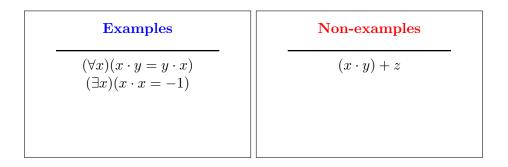
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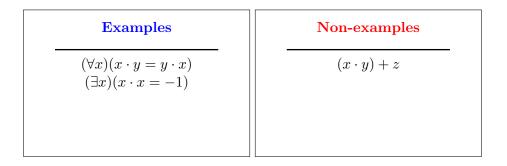
 $(x \cdot y) + z$



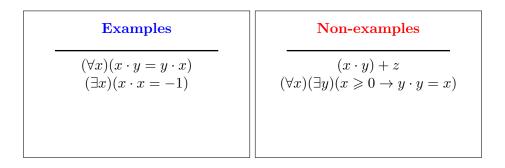
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$$(\forall x)(\exists y)(x \ge 0 \to y \cdot y = x)$$

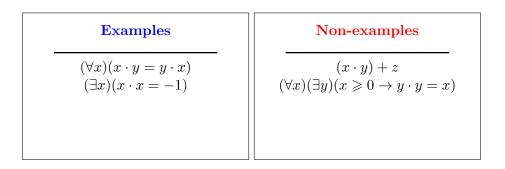


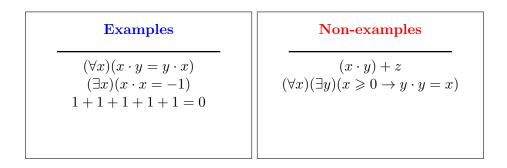
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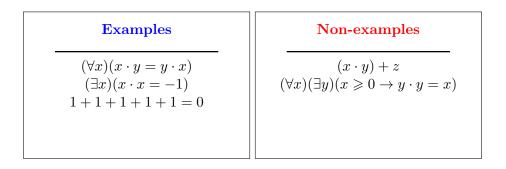
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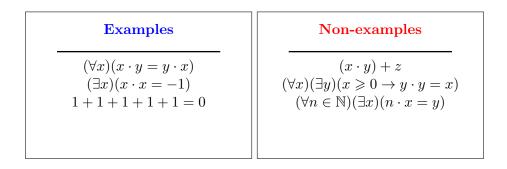
1 + 1 + 1 + 1 + 1 = 0



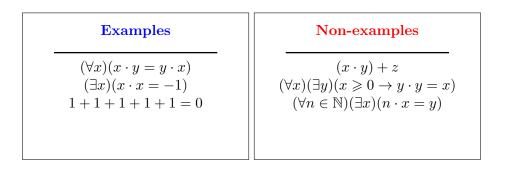


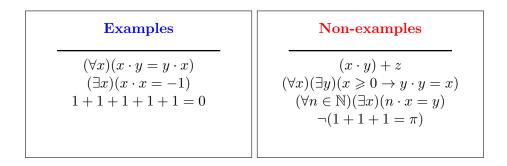
 $(\forall n \in \mathbb{N})(\exists x)(n \cdot x = y)$





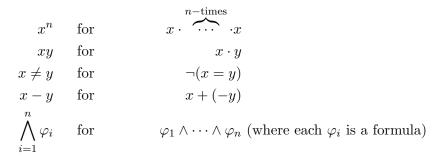
 $\neg(1+1+1=\pi)$





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We will from now on abuse of notation, and write expressions like



For example,

$$(\forall x)(x^2 + y^2 = 1 \to xy \neq 0)$$

is an abbreviation for the \mathcal{L}_{ring} -formula

$$(\forall x)((x \cdot x + y \cdot y = 1) \to \neg (x \cdot y = 0))$$

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What is a first order formula in the language of... valued fields?

We basically play the same game as for \mathcal{L}_{ring} but we add one new formal symbol: a binary relation VF(x, y) which we interpret in any valued field (K, v) as

$$VF(x,y) \Leftrightarrow v(x) \leqslant v(y).$$

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We denote this language \mathcal{L}_{VF} . The following are examples of \mathcal{L}_{VF} -formulas

- VF(1, x)
- $\blacktriangleright \ \mathrm{VF}(x^2+1,xy-1) \to x \neq 1$
- $\blacktriangleright \ (\forall x)(\forall y)(\mathrm{VF}(1,x)\wedge \mathrm{VF}(1,y)\rightarrow \mathrm{VF}(1,x+y))$
- ▶ every \mathcal{L}_{ring} -formula!

An \mathcal{L} -sentence (where \mathcal{L} is either \mathcal{L}_{ring} or \mathcal{L}_{VF}) is an \mathcal{L} -formula which has no free variables.

In particular, if φ is an \mathcal{L}_{ring} -sentence, then for any ring A it either holds in A or not. Similarly, if φ is an \mathcal{L}_{VF} -sentence and (K, v) is a valued field then φ either holds in K or not.

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Sentences

Examples:

- $\blacktriangleright \ (\exists x)(x^2=-1)$
- $\blacktriangleright \ (\forall x)(\forall y)(xy=yx)$

$$\chi_p \coloneqq 1 + \underbrace{ \ddots }_{p}^p + 1 = 0$$

$$\blacktriangleright \ (\forall y_0) \cdots (\forall y_m) (\forall x) (\forall z) (\neg (\bigwedge_{i=0}^m y_i = 0) \rightarrow (\sum_{i=0} y_i x^i = \sum_{i=0} y_i x^i \rightarrow x = z)).$$

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A trivial instance of the transfer principle (in order to get used to it)

Theorem (Ax-Kochen/Ershov)

Let φ be a \mathcal{L}_{VF} -sentence in the language of valued fields. Then there is a finite set of prime numbers E_{φ} such that for all $p \notin E_{\varphi}$

 $\mathbb{Q}_p \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \models \varphi$

Given a prime number q, consider the sentence $\chi_q := 1 + \underbrace{\cdots}_{q}^{q} + 1 = 0.$

Clearly, if p is a prime number bigger than q, we have both $\mathbb{F}_p((t)) \not\models \chi_q$ and $\mathbb{Q}_p \not\models \chi_q$ so setting $E_{\chi_q} = \{q' \in \mathbb{P} : q' \leq q\}$ we have that for all $p \notin E_{\chi_q}$

$$\mathbb{Q}_p \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \models \varphi$$

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A less trivial application

Theorem (Ax-Kochen/Ershov)

Let φ be a \mathcal{L}_{VF} -sentence in the language of valued fields. Then there is a finite set of prime numbers E_{φ} such that for all $p \notin E_{\varphi}$

 $\mathbb{Q}_p \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \models \varphi$

Proposition

The property "having characteristic 0" is not expressible by a \mathcal{L}_{VF} -sentence.

How to express being $C_i(d)$ in the language of rings?

For integers d > 0, $i \ge 0$ and $n > d^i$, say that a field K satisfies the property $C_i(d, n)$ if every homogeneous polynomial of degree d in n variables with coefficients in K has a non-trivial root in K. Clearly, K is $C_i(d)$ if it satisfies $C_i(d, n)$ for every $n > d^i$.

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Being $C_i(d, n)$ is expressible by an \mathcal{L}_{ring} -sentence!

How to express being $C_i(d)$ in the language of rings?

Being $C_i(d, n)$ is expressible by an \mathcal{L}_{ring} -sentence! Indeed, set

$$\blacktriangleright x = (x_1, \ldots, x_n),$$

▶ let $I \subseteq \mathbb{N}^d$ be the set of tuples such that the sum of its coordinates is equal to d, so for $i = (i_1, \ldots, i_d) \in I$

$$\sum_{j=1}^{d} i_j = d,$$

• for
$$i \in I$$
, let $x^i = \prod_{j=1}^n x_j^{i_j}$

▶ let N be the cardinality of I and $s: I \to \{1, ..., N\}$ be a bijection.

Then, let $\varphi(d, i, n)$ be the \mathcal{L}_{ring} -sentence

$$(\forall y_1)\cdots(\forall y_N)(\exists x_1)\cdots(\exists x_n)(\neg \bigwedge_{j=1}^N y_i = 0 \to (\neg \bigwedge_{j=1}^n x_i = 0 \land \sum_{i \in I} y_{s(i)}x^i = 0))$$

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Applying the transfer principle

We have that for every (K, v)

$$(K,v) \models \varphi(d,i,n) \Leftrightarrow (K,v) \text{ is } C_i(d,n) .$$

We can apply the transfer principle to the \mathcal{L}_{ring} -sentence $\varphi(d, 2, n)$ and obtain that there is a finite set of primes E = E(d, 2, n) such that for all $p \notin E$

$$\mathbb{Q}_p$$
 is $C_2(d,n) \Leftrightarrow \mathbb{Q}_p \models \varphi(d,2,n) \Leftrightarrow \mathbb{F}_p((t)) \models \varphi(d,2,n) \Leftrightarrow \mathbb{F}_p((t))$ is $C_2(d,n)$.

Since $\mathbb{F}_p((t))$ is C_2 , we have in particular that $\mathbb{F}_p((t))$ is $C_2(d, n)$, and therefore, $\mathbb{Q}_p \models C_2(d, n)$ for all primes $p \notin E$.

But how to show that \mathbb{Q}_p is actually $C_2(d)$ for all but finite many primes? Here we use simple trick:

K is
$$C_2(d, n)$$
 for all $n > d^2 \Leftrightarrow K$ is $C_2(d, d^2 + 1)$.

Applying the transfer principle

K is $C_2(d, n)$ for all $n > d^2 \Leftrightarrow K$ is $C_2(d, d^2 + 1)$.

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Applying the transfer principle

We apply the transfer principle to the $\mathcal{L}_{\text{ring}}$ -sentence $\varphi = \varphi(d, 2, d^2 + 1)$ and obtain that there is a finite set of primes E = E(d) such that for all $p \notin E$

$$\mathbb{Q}_p$$
 is $C_2(d) \Leftrightarrow \mathbb{Q}_p \models \varphi \Leftrightarrow \mathbb{F}_p((t)) \models \varphi \Leftrightarrow \mathbb{F}_p((t))$ is $C_2(d)$.

Since $\mathbb{F}_p((t))$ is C_2 , we have in particular that $\mathbb{F}_p((t))$ is $C_2(d)$, and therefore, $\mathbb{Q}_p \models C_2(d)$ for all primes $p \notin E$.

Another application

Is there perhaps a similar trick in order to express the property C_2 (resp. C_i for $i \ge 2$) as a first order sentence in $\mathcal{L}_{\text{ring}}$ or \mathcal{L}_{VF} ?

No. Suppose for a contradiction it was an let ψ be an \mathcal{L}_{VF} -sentence such that for K either \mathbb{Q}_p or $\mathbb{F}_p((t))$

$$K \text{ is } C_2 \Leftrightarrow K \models \psi.$$

Then by the transfer principle where would be a finite set of primes E_{ψ} such that for every $p \notin E_{\psi}$

$$\mathbb{Q}_p \models \psi \Leftrightarrow \mathbb{F}_p((t)) \models \psi$$

But we know that \mathbb{Q}_p is not C_2 (resp. not C_i for every $i \ge 0$), so $\mathbb{Q}_p \nvDash \psi$. But then this implies that there are primes p for which $\mathbb{F}_p((t)) \nvDash \psi$, and hence $\mathbb{F}_p((t))$ is not C_2 , a contradiction. Hence the property C_i is not expressible by an \mathcal{L}_{VF} -sentence. Note: the property C_i is of course an infinite conjunction of \mathcal{L}_{ring} -sentences, namely the sentences $C_i(d, d^i + 1)$. Many thanks for your attention.