# GRK Workshop, Ci-Fields 

Ultraproducts and transfer principles I

Zeynep Kısakürek

January 21, 2021

## Ultrafilter, non-principal

$I$ : an infinite set, $\wp(I)$ : the power set of $I$ An ultrafilter on $I$ is a collection $\mathcal{F}$ of infinite elements of $\wp(I)$ such that


For any $A \in \wp(I)$, either $A \in \mathcal{F}$ or $A^{c} \in \mathcal{F}$. In particular, $A \in \mathcal{F}$ and $A \subseteq B \subseteq I \Rightarrow B \in \mathcal{F}$

Remark (Literature-wise)
Anv proper collection of elements of $\delta(1)$ is a filter on I if it is closed under intersection and supersets. In particular, any ultrafilter is a filter which is maximal (wrt inclusion). The above ultrafilters are called non-principal.

## Ultrafilter, non-principal

$I$ : an infinite set, $\wp(I)$ : the power set of $I$
An ultrafilter on $I$ is a collection $\mathcal{F}$ of infinite elements of $\wp(I)$ such that
$\circledast I \in \mathcal{F}$

For any $A \in \wp(I)$, either $A \in \mathcal{F}$ or $A^{c} \in \mathcal{F}$. In particular,
$A \in \mathcal{F}$ and $A \subseteq B \subseteq I \Rightarrow B \in \mathcal{F}$
Remark (Literature-wise)
Any proper collection of elements of $8(1)$ is a filter on I if it is closed under intersection and supersets. In particular, any ultrafilter is a filter which is maximal (wrt inclusion). The above ultrafilters are called non-principal.

## Ultrafilter, non-principal

$I$ : an infinite set, $\wp(I)$ : the power set of $I$
An ultrafilter on $I$ is a collection $\mathcal{F}$ of infinite elements of $\wp(I)$ such that
$\circledast I \in \mathcal{F}$
$\circledast A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
For any $A \in \delta(I)$, either $A \in \mathcal{F}$ or $A^{c} \in \mathcal{F}$. In particular,


Remark (Literature-wise)
Any proper collection of elements of $8(1)$ is a filter on I if it is closed under intersection and supersets. In particular, any ultrafilter is a filter which is maximal (wrt inclusion). The above ultrafilters are called non-principal.

## Ultrafilter, non-principal

I: an infinite set, $\wp(I)$ : the power set of $I$
An ultrafilter on $I$ is a collection $\mathcal{F}$ of infinite elements of $\wp(I)$ such that
$\circledast I \in \mathcal{F}$
$\circledast A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
${ }^{*}$ For any $A \in \wp(I)$, either $A \in \mathcal{F}$ or $A^{c} \in \mathcal{F}$. In particular,

## Remark (Literature-wise)

Any proner collection of elements of $\rho(1)$ is a filter on I if it is closed under intersection and supersets. In particular, any ultrafilter is a filter which is maximal (wrt inclusion). The above ultrafilters are called non-principal.

## Ultrafilter, non-principal

$I$ : an infinite set, $\wp(I)$ : the power set of $I$
An ultrafilter on $I$ is a collection $\mathcal{F}$ of infinite elements of $\wp(I)$ such that
$\circledast I \in \mathcal{F}$
$\circledast A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
$\circledast$ For any $A \in \wp(I)$, either $A \in \mathcal{F}$ or $A^{c} \in \mathcal{F}$. In particular,
$\circledast \emptyset \notin \mathcal{F}$
$\circledast A \in \mathcal{F}$ and $A \subseteq B \subseteq I \Rightarrow B \in \mathcal{F}$
Remark (Literature-wise)
Any proper collection of elements of $\wp(I)$ is a filter on I if it is closed
under intersection and supersets. In particular, any ultrafilter is a filter
which is maximal (wrt inclusion). The above ultrafilters are called
non-principal.

## Ultrafilter, non-principal

$I$ : an infinite set, $\wp(I)$ : the power set of $I$
An ultrafilter on $I$ is a collection $\mathcal{F}$ of infinite elements of $\wp(I)$ such that
$\circledast I \in \mathcal{F}$
$\circledast A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
$*$ For any $A \in \wp(I)$, either $A \in \mathcal{F}$ or $A^{c} \in \mathcal{F}$. In particular,
$\circledast \emptyset \notin \mathcal{F}$
$\circledast A \in \mathcal{F}$ and $A \subseteq B \subseteq I \Rightarrow B \in \mathcal{F}$

## Remark (Literature-wise)

Any proper collection of elements of $\wp(I)$ is a filter on I if it is closed under intersection and supersets. In particular, any ultrafilter is a filter which is maximal (wrt inclusion). The above ultrafilters are called non-principal.

The language of rings:

$$
\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}
$$

$$
\begin{aligned}
& \text { two binary function symbols } \mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\} \\
& \text { a unary function symbol } \mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\} \\
& \text { two constant symbols } \mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}
\end{aligned}
$$

## Definition (quite informal)

## A lanouase $\mathcal{C}$ is a set of function relation and constant symbols <br> An $\mathcal{L}$-structure can be defined as a triple ( $M, \mathcal{L}, I$ ) consisting of a non-empty domain $M$, language $\mathcal{L}$ and an interpretation function $I$.

The language of rings:

$$
\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}
$$

$\circledast$ two binary function symbols $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$

$$
\begin{aligned}
& \text { a unary function symbol } \mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\} \\
& \text { two constant symbols } \mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}
\end{aligned}
$$

## Definition (quite informal)

## A lanowase $r$ is a set of function relation and constant symbols <br> An $\mathcal{L}$-structure can be defined as a triple ( $M, \mathcal{L}, I$ ) consisting of a non-empty domain $M$, language $\mathcal{L}$ and an interpretation function $I$.

The language of rings:

$$
\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}
$$

$\circledast$ two binary function symbols $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$
$\circledast$ a unary function symbol $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$
two constant symbols $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$

## Definition (quite informal)

## A lanouase $\mathcal{C}$ is a set of function relation and constant symbols An $\mathcal{L}$-structure can be defined as a triple ( $M, \mathcal{L}, I$ ) consisting of a non-empty domain $M$, language $\mathcal{L}$ and an interpretation function $I$.

## Interlude on Pablo's talk

The language of rings:

$$
\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}
$$

$\circledast$ two binary function symbols $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$
$\circledast$ a unary function symbol $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$

* two constant symbols $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$


## Definition (quite informal)

## A lanouage $r$ is a set of function relation and constant symbols An $\mathcal{L}$-structure can be defined as a triple ( $M, \mathcal{L}, I$ ) consisting of a non-empty domain $M$, language $\mathcal{L}$ and an interpretation function $I$.

## Interlude on Pablo's talk

The language of rings:

$$
\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}
$$

$\circledast$ two binary function symbols $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$
$\circledast$ a unary function symbol $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$
$\circledast$ two constant symbols $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$

Definition (quite informal)
A language $\mathcal{L}$ is a set of function, relation and constant symbols.
An $\mathcal{L}$-structure can be defined as a triple ( $M, \mathcal{L}, I$ ) consisting of a
non-empty domain $M$, language $\mathcal{L}$ and an interpretation function $I$.

## Interlude on Pablo's talk

The language of rings:

$$
\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}
$$

$\circledast$ two binary function symbols $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$
$\circledast$ a unary function symbol $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$
$\circledast$ two constant symbols $\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$

Definition (quite informal)
A language $\mathcal{L}$ is a set of function, relation and constant symbols.
An $\mathcal{L}$-structure can be defined as a triple ( $M, \mathcal{L}, I$ ) consisting of a non-empty domain $M$, language $\mathcal{L}$ and an interpretation function $I$.

## Ultraproduct Construction via Ultrafilters

## Setting

$I$ : an infinite index set with an ultrafilter $\mathcal{F}$ on it
$\left(\mathcal{M}_{i}\right)_{i \in I}$ : a family of $\mathcal{L}$-structures


Definition
Consider the Cartesian product $\prod M_{i}$ as the set of choice functions

$$
\left\{g: 1 \rightarrow \cup M_{i}: \forall i \in 1, g(i) \in M_{i}\right\}
$$

Define $\sim_{\mathcal{F}}$ on $\prod M_{i}$ by


## Ultraproduct Construction via Ultrafilters

## Setting

$I$ : an infinite index set with an ultrafilter $\mathcal{F}$ on it
$\left(\mathcal{M}_{i}\right)_{i \in I}$ : a family of $\mathcal{L}$-structures
$\mathcal{L}=\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$
$\rightsquigarrow\left(\mathcal{M}_{i}\right)_{i \in 1}$ :family of rings


Definition
Consider the Cartesian product $\prod M_{i}$ as the set of choice functions

$$
\left\{g: 1 \rightarrow \cup M_{i}: \forall i \in 1, g(i)=M_{i}\right\}
$$

Define $\sim_{\mathcal{F}}$ on $\prod M_{i}$ by


## Ultraproduct Construction via Ultrafilters

## Setting

$I$ : an infinite index set with an ultrafilter $\mathcal{F}$ on it
$\left(\mathcal{M}_{i}\right)_{i \in I}$ : a family of $\mathcal{L}$-structures
$\mathcal{L}=\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\}$
$\rightsquigarrow\left(\mathcal{M}_{i}\right)_{i \in 1}$ :family of rings
$\mathcal{L}=\mathcal{L}_{\text {ag }}=\{+,-, 0\}$
$\rightsquigarrow\left(\mathcal{M}_{i}\right)_{i \in 1}$ :family of abelian gps

Definition
Consider the Cartesian product $\prod M_{i}$ as the set of choice functions

$$
\left\{g: 1 \rightarrow \cup M_{i}: \forall i \in 1, g(i) \in M_{i}\right\}
$$

Define $\sim_{\mathcal{F}}$ on $\prod M_{i}$ by

## Ultraproduct Construction via Ultrafilters

## Setting

$I$ : an infinite index set with an ultrafilter $\mathcal{F}$ on it
$\left(\mathcal{M}_{i}\right)_{i \in I}$ : a family of $\mathcal{L}$-structures

$$
\begin{aligned}
& \mathcal{L}=\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\} \\
& \rightsquigarrow\left(\mathcal{M}_{i}\right)_{i \in I}: \text { family of rings } \\
& \mathcal{L}=\mathcal{L}_{\text {ag }}=\{+,-, 0\} \\
& \rightsquigarrow\left(\mathcal{M}_{i}\right)_{i \in 1}: \text { family of abelian gps }
\end{aligned}
$$

Definition
Consider the Cartesian product $\prod M_{i}$ as the set of choice functions

$$
\left\{g: I \rightarrow \cup M_{i}: \forall i \in I, g(i) \in M_{i}\right\}
$$

Define $\sim_{\mathcal{F}}$ on $\prod M_{i}$ by

$$
g \sim_{\mathcal{F}} h \Leftrightarrow\{i \in I: g(i)=h(i)\} \in \mathcal{F}
$$

## Ultraproduct Construction via Ultrafilters

Definition
With $\prod M_{i}, \mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,
The ultraproduct $\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure, is defined as follows The domain $\mathcal{L}=\prod M_{i} / \sim_{F}$ is the set of equivalence classes of $\sim_{F}$ in $\prod M_{i}$, denote the eq. cl. by $[g]$ or $[g(i): i \in I]$ $\forall$ function symbol $f \in \mathcal{L}$, define $f^{\mathcal{M}}$ by

$$
f^{\mathcal{M}}\left(\left[g_{1}\right], \ldots,\left[g_{n}\right]\right)=\left[f^{\mathcal{M}_{i}}\left(g_{1}(i), \ldots, g_{n}(i)\right): i \in I\right]
$$

$\forall$ relation symbol $R \in \mathcal{L}$, define $R^{\mathcal{M}}$ by

$\forall$ constant symbol $c \in \mathcal{L}$, define $c^{\mathcal{M}}$ by $c^{\mathcal{M}}=\left[c^{\mathcal{M}_{i}}: i \in I\right]$

## Ultraproduct Construction via Ultrafilters

Definition
With $\Pi M_{i}, \mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,
The ultraproduct $\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure, is defined as follows


## Ultraproduct Construction via Ultrafilters

Definition
With $\prod M_{i}, \mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,
The ultraproduct $\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure, is defined as follows
$*$ The domain $\mathcal{L}=\prod M_{i} / \sim_{\mathcal{F}}$ is the set of equivalence classes of $\sim_{\mathcal{F}}$ in $\prod M_{i}$, denote the eq. cl. by $[g]$ or $[g(i): i \in I]$


## Ultraproduct Construction via Ultrafilters

## Definition

With $\prod M_{i}, \mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,
The ultraproduct $\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure, is defined as follows
$\circledast$ The domain $\mathcal{L}=\prod M_{i} / \sim_{\mathcal{F}}$ is the set of equivalence classes of $\sim_{\mathcal{F}}$ in $\prod M_{i}$, denote the eq. cl. by [ $g$ ] or $[g(i): i \in I]$
$\circledast \forall$ function symbol $f \in \mathcal{L}$, define $f^{\mathcal{M}}$ by

$$
f^{\mathcal{M}}\left(\left[g_{1}\right], \ldots,\left[g_{n}\right]\right)=\left[f^{\mathcal{M}}\left(g_{1}(i), \ldots, g_{n}(i)\right): i \in I\right]
$$

$\forall$ relation symbol $R \in \mathcal{L}$, define $R^{\mathcal{M}}$ by

## Ultraproduct Construction via Ultrafilters

## Definition

With $\prod M_{i}, \mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,
The ultraproduct $\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure, is defined as follows
$\circledast$ The domain $\mathcal{L}=\prod M_{i} / \sim_{\mathcal{F}}$ is the set of equivalence classes of $\sim_{\mathcal{F}}$ in $\prod M_{i}$, denote the eq. cl. by $[g]$ or $[g(i): i \in I]$
$\circledast \forall$ function symbol $f \in \mathcal{L}$, define $f^{\mathcal{M}}$ by

$$
f^{\mathcal{M}}\left(\left[g_{1}\right], \ldots,\left[g_{n}\right]\right)=\left[f^{\mathcal{M}}\left(g_{1}(i), \ldots, g_{n}(i)\right): i \in I\right]
$$

$\circledast \forall$ relation symbol $R \in \mathcal{L}$, define $R^{\mathcal{M}}$ by

$$
\begin{gathered}
\left(\left[g_{1}\right], \ldots,\left[g_{n}\right]\right) \in R^{\mathcal{M}} \\
\left\{i \in I:\left(\left[g_{1}(i)\right], \ldots,\left[g_{n}(i)\right]\right) \in R^{\mathcal{M}_{i}}\right\} \in \mathcal{F}
\end{gathered}
$$

$\forall$ constant symbol $c \in \mathcal{L}$, define $c$

## Ultraproduct Construction via Ultrafilters

## Definition

With $\prod M_{i}, \mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,
The ultraproduct $\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure, is defined as follows
$\circledast$ The domain $\mathcal{L}=\prod M_{i} / \sim_{\mathcal{F}}$ is the set of equivalence classes of $\sim_{\mathcal{F}}$ in $\prod M_{i}$, denote the eq. cl. by $[g]$ or $[g(i): i \in I]$
$\circledast \forall$ function symbol $f \in \mathcal{L}$, define $f^{\mathcal{M}}$ by

$$
f^{\mathcal{M}}\left(\left[g_{1}\right], \ldots,\left[g_{n}\right]\right)=\left[f^{\mathcal{M}}\left(g_{1}(i), \ldots, g_{n}(i)\right): i \in I\right]
$$

$\circledast \forall$ relation symbol $R \in \mathcal{L}$, define $R^{\mathcal{M}}$ by

$$
\begin{gathered}
\left(\left[g_{1}\right], \ldots,\left[g_{n}\right]\right) \in R^{\mathcal{M}} \\
\left\{i \in I:\left(\left[g_{1}(i)\right], \ldots,\left[g_{n}(i)\right]\right) \in R^{\mathcal{M}_{i}}\right\} \in \mathcal{F}
\end{gathered}
$$

$\circledast \forall$ constant symbol $c \in \mathcal{L}$, define $c^{\mathcal{M}}$ by $c^{\mathcal{M}}=\left[c^{\mathcal{M}_{i}}: i \in I\right]$

## An Immediate Example - Ultraproduct of Ordered Fields

The language of ordered fields is $\mathcal{L}_{\text {or }}=\{+,-, \cdot, 0,1,<\}=\mathcal{L}_{\text {Ring }} \cup\{<\}$
Setting:
$\{\mathbb{R}: i \in \mathbb{N}\}$ : a countable collection of copies of $\mathbb{R}$, as $\mathcal{L}_{\text {or }}$-structure $\mathcal{F}$ : an ultrafilter on $\mathbb{N}$

function symbols

$\square$

## An Immediate Example - Ultraproduct of Ordered Fields

The language of ordered fields is $\mathcal{L}_{\text {or }}=\{+,-, \cdot, 0,1,<\}=\mathcal{L}_{\text {Ring }} \cup\{<\}$
Setting:
$\{\mathbb{R}: i \in \mathbb{N}\}$ : a countable collection of copies of $\mathbb{R}$, as $\mathcal{L}_{\text {or }}$-structure $\mathcal{F}$ : an ultrafilter on $\mathbb{N}$

$$
\begin{aligned}
& \rightsquigarrow \mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}} \\
& =\left\{[g]: g(i) \in \mathbb{R}^{\mathbb{N}}\right\}
\end{aligned}
$$

function symbols

$\square$

## An Immediate Example - Ultraproduct of Ordered Fields

The language of ordered fields is $\mathcal{L}_{\text {or }}=\{+,-, \cdot, 0,1,<\}=\mathcal{L}_{\text {Ring }} \cup\{<\}$

## Setting:

$\{\mathbb{R}: i \in \mathbb{N}\}$ : a countable collection of copies of $\mathbb{R}$, as $\mathcal{L}_{\text {or }}$-structure $\mathcal{F}$ : an ultrafilter on $\mathbb{N}$

$$
\begin{aligned}
& \rightsquigarrow \mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}} \\
& =\left\{[g]: g(i) \in \mathbb{R}^{\mathbb{N}}\right\}
\end{aligned}
$$

function symbols

$$
\begin{aligned}
& {[g(i): i \in I]+[h(i): i \in I]=[g(i)+h(i): i \in I]} \\
& {[g(i): i \in I] \cdot[h(i): i \in I]=[g(i) . h(i): i \in I]}
\end{aligned}
$$

## An Immediate Example - Ultraproduct of Ordered Fields

The language of ordered fields is $\mathcal{L}_{\text {or }}=\{+,-, \cdot, 0,1,<\}=\mathcal{L}_{\text {Ring }} \cup\{<\}$

## Setting:

$\{\mathbb{R}: i \in \mathbb{N}\}$ : a countable collection of copies of $\mathbb{R}$, as $\mathcal{L}_{\text {or }}$-structure $\mathcal{F}$ : an ultrafilter on $\mathbb{N}$

$$
\begin{aligned}
& \rightsquigarrow \mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}} \\
& =\left\{[g]: g(i) \in \mathbb{R}^{\mathbb{N}}\right\}
\end{aligned}
$$

function symbols
$[g(i): i \in I]+[h(i): i \in I]=[g(i)+h(i): i \in I]$ $[g(i): i \in I] .[h(i): i \in I]=[g(i) . h(i): i \in I]$
relation symbol

$$
[g]<[h] \Leftrightarrow\{i \in \mathbb{N}: g(i)<h(i)\} \in \mathcal{F}
$$

## An Immediate Example - Ultraproduct of Ordered Fields

The language of ordered fields is $\mathcal{L}_{\text {or }}=\{+,-, \cdot, 0,1,<\}=\mathcal{L}_{\text {Ring }} \cup\{<\}$

## Setting:

$\{\mathbb{R}: i \in \mathbb{N}\}$ : a countable collection of copies of $\mathbb{R}$, as $\mathcal{L}_{\text {or }}$-structure $\mathcal{F}$ : an ultrafilter on $\mathbb{N}$

$$
\begin{aligned}
& \rightsquigarrow \mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}} \\
& =\left\{[g]: g(i) \in \mathbb{R}^{\mathbb{N}}\right\}
\end{aligned}
$$

function symbols

$$
[g(i): i \in I]+[h(i): i \in I]=[g(i)+h(i): i \in I] \quad \text { constant symbols }
$$

$$
[g(i): i \in I] \cdot[h(i): i \in I]=[g(i) \cdot h(i): i \in I]
$$

relation symbol

```
zero [{0,0,0,\ldots}]
unity [{1,1,1,\ldots}]
```

$[g]<[h] \Leftrightarrow\{i \in \mathbb{N}: g(i)<h(i)\} \in \mathcal{F}$

## Łoś' theorem - Fundamental Theorem of Ultraproducts

Setting:
$\left(\mathcal{M}_{i}\right)_{i \in I}$ : a family of $\mathcal{L}$-structures
$\mathcal{F}$ : an ultrafilter $\mathcal{F}$ on I
$\varphi(\bar{x})$ : first order formula in the free variables $\bar{x}$
$\varphi$ true in $\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$

$\varphi$ true on a "large" subfamily

Łoś' theorem - Fundamental Theorem of Ultraproducts

## Setting:

$\left(\mathcal{M}_{i}\right)_{i \in I}$ : a family of $\mathcal{L}$-structures
$\mathcal{F}$ : an ultrafilter $\mathcal{F}$ on I
$\varphi(\bar{x})$ : first order formula in the free variables $\bar{x}$
$\varphi$ true in $\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$
iff

$\varphi$ true on a "large" subfamily

Theorem (Jerzy Łoś, '55)
For a tuple $\left(\left[g_{1}\right], \ldots,\left[g_{n}\right]\right)$ of elements from $\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$,

$$
\prod \mathcal{M}_{i} / \sim_{\mathcal{F}} \models \varphi\left(\left[g_{1}\right], \ldots,\left[g_{n}\right]\right)
$$

$$
\left\{i \in I: \mathcal{M}_{i} \models \varphi\left(g_{1}(i), \ldots, g_{n}(i)\right)\right\} \in \mathcal{F}
$$

## Łoś' theorem - Applications

## Previously on this talk...

$$
\begin{aligned}
& \mathcal{L}=\mathcal{L}_{\text {Ring }}=\{+,-, \cdot, 0,1\} \\
& \rightsquigarrow\left(\mathcal{M}_{i}\right)_{i \in!} \text { :family of rings } \\
& \mathcal{L}=\mathcal{L}_{\text {ag }}=\{+,-, 0\} \\
& \rightsquigarrow\left(\mathcal{M}_{i}\right)_{i \in!}: \text { family of abelian gps }
\end{aligned}
$$

$\left.\begin{array}{lll|} & \rightsquigarrow \mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}} \\ & =\left\{[g]: g(i) \in \mathbb{R}^{\mathbb{N}}\right\}\end{array}\right] \quad$.

## Łoś' theorem - Applications

Previously on this talk...

```
\mathcal { L } = \mathcal { L } _ { \text { Ring } } = \{ + , - , , , 0 , 1 \}
\rightsquigarrow(\mp@subsup{\mathcal{M}}{i}{}\mp@subsup{)}{i\inl}{}\mathrm{ :family of rings}
\mathcal { L } = \mathcal { L } _ { \text { ag } } = \{ + , - , 0 \}
\rightsquigarrow(\mp@subsup{\mathcal{M}}{i}{\prime}\mp@subsup{)}{i\inl}{}:\mathrm{ :family of abelian gps}
```

Definition
With $\Pi M_{i}, \mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,
The ultraproduct $\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure

$$
\begin{aligned}
& \rightsquigarrow \mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}} \\
& =\left\{[g]: g(i) \in \mathbb{R}^{\mathbb{N}}\right\} \\
& \text { function symbols } \\
& {[g(i): i \in I]+[h(i): i \in I]=[g(i)+h(i): i \in I]}
\end{aligned}
$$

$\left(\mathcal{M}_{i}\right)_{i \in I}$ :family of rings

$$
\rightsquigarrow \mathcal{L}=\mathcal{L}_{\text {Ring }}
$$

$\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}} ;$ an $\mathcal{L}_{\text {Ring }}$-structure

## Łoś' theorem - Applications

Previously on this talk...

```
L}=\mp@subsup{\mathcal{L}}{\mathrm{ Ring }}{}={+,-,\cdot,0,1
\rightsquigarrow(\mp@subsup{\mathcal{M}}{i}{}\mp@subsup{)}{i\inl}{}\mathrm{ :family of rings}
\mathcal { L } = \mathcal { L } _ { \text { ag } } = \{ + , - , 0 \}
\rightsquigarrow(\mp@subsup{\mathcal{M}}{i}{\prime}\mp@subsup{)}{i\inl}{}:\mathrm{ :family of abelian gps}
```

Definition
With $\Pi M_{i}, \mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,
The ultraproduct $\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure

$$
\begin{array}{ll}
\qquad \begin{array}{ll} 
& \rightsquigarrow \mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}} \\
& =\left\{[g]: g(i) \in \mathbb{R}^{\mathbb{N}}\right\}
\end{array} & \\
\text { function symbols } & \\
\hline[g(i): i \in I]+[h(i): i \in I]=[g(i)+h(i): i \in I] & \text { constant symbols } \\
{[g(i): i \in I] \cdot[h(i): i \in I]=[g(i) \cdot h(i): i \in I]} & \\
& \begin{array}{l}
\text { zero }[\{0,0,0, \ldots\}] \\
\text { unity }[\{1,1,1, \ldots\}]
\end{array} \\
\hline
\end{array}
$$

$\left(\mathcal{M}_{i}\right)_{i \in I}$ :family of rings

$$
\rightsquigarrow \mathcal{L}=\mathcal{L}_{\text {Ring }}
$$

$$
\text { Is } \mathcal{M} \text { a ring? }
$$

$$
\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}} ; \text { an } \mathcal{L}_{\text {Ring }} \text {-structure }
$$

## Łoś' theorem - Applications

Previously on this talk...

```
\mathcal { L } = \mathcal { L } _ { \text { Ring } } = \{ + , - , , , 0 , 1 \}
\rightsquigarrow(\mp@subsup{\mathcal{M}}{i}{}\mp@subsup{)}{i\inl}{}\mathrm{ :family of rings}
\mathcal { L } = \mathcal { L } _ { \text { ag } } = \{ + , - , 0 \}
\rightsquigarrow(\mp@subsup{\mathcal{M}}{i}{}\mp@subsup{)}{i\inI}{\prime}:\mathrm{ :family of abelian gps}
```

Definition
With $\Pi M_{i}, \mathcal{F}$ and $\sim_{\mathcal{F}}$ as above,
The ultraproduct $\mathcal{M}=\Pi \mathcal{M}_{i} / \sim_{\mathcal{F}}$, an $\mathcal{L}$-structure

|  |  $\sim \mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$ <br>  $=\left\{[g]: g(i) \in \mathbb{R}^{\mathbb{N}}\right\}$ |
| :--- | :--- |
|  |  |
| function symbols |  |
| $[g(i): i \in I]+[h(i): i \in I]=[g(i)+h(i): i \in I]$ |  |
| $[g(i): i \in I] \cdot[h(i): i \in I]=[g(i) \cdot h(i): i \in I]$ | constant symbols |
| relation symbol | zero $[\{0,0,0, \ldots\}]$ |
| $[g]<[h] \Leftrightarrow\{i \in \mathbb{N}: g(i)<h(i)\} \in \mathcal{F}$ |  |
| unity $[\{1,1,1, \ldots\}]$ |  |

$\left(\mathcal{M}_{i}\right)_{i \in I}$ :family of rings

$$
\rightsquigarrow \mathcal{L}=\mathcal{L}_{\text {Ring }}
$$

Is $\mathcal{M}$ a ring? Answer: YES

$$
\mathcal{M}=\prod \mathcal{M}_{i} / \sim_{\mathcal{F}} ; \text { an } \mathcal{L}_{\text {Ring }} \text {-structure }
$$

## Łoś' theorem - Applications

Corollary
The ultraproduct of groups/rings/fields is again a group/ring/field.

Proposition
If almost all of the $K_{i}$ are algebraically closed fields, then so is $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$

$$
\left(\forall a_{0}, a_{1}, \ldots, a_{n}\right)(\exists x)\left(a_{n}=0 \vee a_{0}+a_{1} x+\ldots+a_{n} x^{n}=0\right)
$$

holds for almost all of $K_{i}$ $\xrightarrow{\text { Los }}$ holds for $\prod_{i \in}, K_{i} / \sim_{\mathcal{F}}$

## Łoś' theorem - Applications

## Corollary

The ultraproduct of groups/rings/fields is again a group/ring/field.
Proposition
If almost all of the $K_{i}$ are algebraically closed fields, then so is $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$.
$\left(\forall a_{0}, a_{1}, \ldots, a_{n}\right)(\exists x)\left(a_{n}=0 \vee a_{0}+a_{1} x+\ldots+a_{n} x^{n}=0\right)$
holds for almost all of $K_{\text {; }}$ $\xrightarrow{\text { Los }}$ holds for $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$

## Łoś' theorem - Applications

## Corollary

The ultraproduct of groups/rings/fields is again a group/ring/field.
Proposition
If almost all of the $K_{i}$ are algebraically closed fields, then so is $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$.

$$
\left(\forall a_{0}, a_{1}, \ldots, a_{n}\right)(\exists x)\left(a_{n}=0 \vee a_{0}+a_{1} x+\ldots+a_{n} x^{n}=0\right)
$$

holds for almost all of $K_{i}$ $\xrightarrow{\text { Los }}$ holds for $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$

## Łoś' theorem - Applications

## Corollary

The ultraproduct of groups/rings/fields is again a group/ring/field.

## Proposition

If almost all of the $K_{i}$ are algebraically closed fields, then so is $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$.

$$
\left(\forall a_{0}, a_{1}, \ldots, a_{n}\right)(\exists x)\left(a_{n}=0 \vee a_{0}+a_{1} x+\ldots+a_{n} x^{n}=0\right)
$$

holds for almost all of $K_{i}$
$\xrightarrow{\text { Los }}$ holds for $\prod_{i \in I} K_{i} / \sim J$

## Łoś' theorem - Applications

## Corollary

The ultraproduct of groups/rings/fields is again a group/ring/field.

## Proposition

If almost all of the $K_{i}$ are algebraically closed fields, then so is $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$.

$$
\left(\forall a_{0}, a_{1}, \ldots, a_{n}\right)(\exists x)\left(a_{n}=0 \vee a_{0}+a_{1} x+\ldots+a_{n} x^{n}=0\right)
$$

holds for almost all of $K_{i}$
$\xrightarrow{\text { Los }}$ holds for $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$

## Łoś' theorem - Applications

Proposition
$\left\{K_{i}\right\}_{i \in I}$ : a collection of fields such that for each prime p , only finitely many $K_{i}$ have characteristic $p$.

Then $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$, has characteristic zero.

Consider, for a fixed prime $p,(\exists a)(p a-1=0)$
$\left\{i \in I\right.$ : the statement holds in $\left.K_{i}\right\} \in \mathcal{F}$
$\xrightarrow{\text { Los }}$ the statement holds over $\prod_{i \in 1} K_{i} / \sim_{\mathcal{F}}$

## Łoś' theorem - Applications

## Proposition

$\left\{K_{i}\right\}_{i \in I}$ : a collection of fields such that for each prime p , only finitely many $K_{i}$ have characteristic $p$.

Then $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$, has characteristic zero.

Consider, for a fixed prime $p,(\exists a)(p a-1=0)$
$\left\{i \in I\right.$ : the statement holds in $\left.K_{i}\right\} \in \mathcal{F}$
$\xrightarrow{\text { Los }}$ the statement holds over $\prod_{i \in 1} K_{i} / \sim_{\mathcal{F}}$

## Łoś' theorem - Applications

## Proposition

$\left\{K_{i}\right\}_{i \in I}$ : a collection of fields such that for each prime p , only finitely many $K_{i}$ have characteristic $p$.

Then $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$, has characteristic zero.

Consider, for a fixed prime $p,(\exists a)(p a-1=0)$

$$
\left\{i \in I: \text { the statement holds in } K_{i}\right\} \in \mathcal{F}
$$

$\xrightarrow{\text { Los }}$ the statement holds over $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$

## Łoś' theorem - Applications

## Proposition

$\left\{K_{i}\right\}_{i \in I}$ : a collection of fields such that for each prime p , only finitely many $K_{i}$ have characteristic $p$.

Then $\prod_{i \in I} K_{i} / \sim_{\mathcal{F}}$, for any ultrafilter $\mathcal{F}$, has characteristic zero.

Consider, for a fixed prime $p,(\exists a)(p a-1=0)$

$$
\begin{aligned}
& \quad\left\{i \in I \text { : the statement holds in } K_{i}\right\} \in \mathcal{F} \\
& \stackrel{\text { Los }}{\Longrightarrow} \text { the statement holds over } \prod_{i \in I} K_{i} / \sim_{\mathcal{F}}
\end{aligned}
$$

## Field of Complex Numbers - External Point of View

Setting:

$$
\begin{aligned}
& \mathbb{P}=\{p \in \mathbb{N}: p \text { prime }\} \\
& \left\{\mathbb{F}_{p}^{\text {alg }}\right\}_{p \in \mathbb{P}}, \text { as } \mathcal{L}_{\text {Ring }} \text {-structure }
\end{aligned}
$$

Choose an ultrafilter $\mathcal{F}$ on $\mathbb{P}$
$\rightsquigarrow \mathbb{F}^{*}=\prod_{p \in \mathbb{P}} \mathbb{F}_{p}^{\text {alg }} / \sim_{\mathcal{F}}$ is a field

## Moreover $\mathbb{F}^{*}$

has characteristic 0 . is algebraically closed has the cardinality of continuum

## Field of Complex Numbers - External Point of View

Setting:

$$
\begin{aligned}
& \mathbb{P}=\{p \in \mathbb{N}: p \text { prime }\} \\
& \left\{\mathbb{F}_{p}^{\text {alg }}\right\}_{p \in \mathbb{P}}, \text { as } \mathcal{L}_{\text {Ring }} \text {-structure }
\end{aligned}
$$

Choose an ultrafilter $\mathcal{F}$ on $\mathbb{P}$
$\rightsquigarrow \mathbb{F}^{*}=\prod_{p \in \mathbb{P}} \mathbb{F}_{p}^{\text {alg }} / \sim_{\mathcal{F}}$ is a field

Moreover $\mathbb{F}^{*}$

* has characteristic 0 .
is algebraically closed.
has the cardinality of
continuum.


## Field of Complex Numbers - External Point of View

Setting:

$$
\begin{aligned}
& \mathbb{P}=\{p \in \mathbb{N}: p \text { prime }\} \\
& \left\{\mathbb{F}_{p}^{\text {alg }}\right\}_{p \in \mathbb{P}}, \text { as } \mathcal{L}_{\text {Ring }} \text {-structure }
\end{aligned}
$$

Choose an ultrafilter $\mathcal{F}$ on $\mathbb{P}$
$\rightsquigarrow \mathbb{F}^{*}=\prod_{p \in \mathbb{P}} \mathbb{F}_{p}^{\text {alg }} / \sim_{\mathcal{F}}$ is a field

Moreover $\mathbb{F}^{*}$

* has characteristic 0 .
$\circledast$ is algebraically closed.
has the cardinality of


## Field of Complex Numbers - External Point of View

Setting:

$$
\begin{aligned}
& \mathbb{P}=\{p \in \mathbb{N}: p \text { prime }\} \\
& \left\{\mathbb{F}_{p}^{\text {alg }}\right\}_{p \in \mathbb{P}}, \text { as } \mathcal{L}_{\text {Ring }} \text {-structure }
\end{aligned}
$$

Choose an ultrafilter $\mathcal{F}$ on $\mathbb{P}$
$\rightsquigarrow \mathbb{F}^{*}=\prod_{p \in \mathbb{P}} \mathbb{F}_{p}^{\text {alg }} / \sim_{\mathcal{F}}$ is a field

Moreover $\mathbb{F}^{*}$
$\circledast$ has characteristic 0 .
$\circledast$ is algebraically closed.
$\circledast$ has the cardinality of continuum.

## Field of Complex Numbers - External Point of View

Setting:

$$
\begin{aligned}
& \mathbb{P}=\{p \in \mathbb{N}: p \text { prime }\} \\
& \left\{\mathbb{F}_{p}^{\text {alg }}\right\}_{p \in \mathbb{P}}, \text { as } \mathcal{L}_{\text {Ring }} \text {-structure }
\end{aligned}
$$

Choose an ultrafilter $\mathcal{F}$ on $\mathbb{P}$
$\rightsquigarrow \mathbb{F}^{*}=\prod_{p \in \mathbb{P}} \mathbb{F}_{p}^{\text {alg }} / \sim_{\mathcal{F}}$ is a field

Moreover $\mathbb{F}^{*}$
$\circledast$ has characteristic 0 .
$\circledast$ is algebraically closed.
$\circledast$ has the cardinality of continuum.

$$
\mathbb{F}^{*} \simeq \mathbb{C}
$$

## Non-standard Reals - Internal Point of View

## Setting:

$\{\mathbb{R}: i \in \mathbb{N}\}$ : a collection of copies of $\mathbb{R}$, as an
$\mathcal{L}_{\text {or }}$-structure
$\mathcal{F}$ : an ultrafilter on $\mathbb{N}$

Consider $\mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$
Observations:

* This structure $\mathcal{R}$ contains infinitesimal numbers.



## Non-standard Reals - Internal Point of View

## Setting:

$\{\mathbb{R}: i \in \mathbb{N}\}$ : a collection of copies of $\mathbb{R}$, as an
$\mathcal{L}_{\text {or }}$-structure
$\mathcal{F}$ : an ultrafilter on $\mathbb{N}$

Consider $\mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$
Observations:

* This structure $\mathcal{R}$ contains infinitesimal numbers.

Consider the eq. cl. $\varepsilon=\left[\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}\right]$


## Non-standard Reals - Internal Point of View

## Setting:

$\{\mathbb{R}: i \in \mathbb{N}\}$ : a collection of copies of $\mathbb{R}$, as an
$\mathcal{L}_{\text {or }}$-structure
$\mathcal{F}$ : an ultrafilter on $\mathbb{N}$

Consider $\mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$
Observations:

* This structure $\mathcal{R}$ contains infinitesimal numbers.

Consider the eq. cl. $\varepsilon=\left[\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}\right]$

$$
\rightsquigarrow \mathcal{R} \models 0<\varepsilon
$$

$$
\text { as }\left\{n \in \mathbb{N}: 0<\frac{1}{n}\right\}=\mathbb{N} \in \mathcal{F}
$$

## Non-standard Reals - Internal Point of View

## Setting:

$\{\mathbb{R}: i \in \mathbb{N}\}$ : a collection of copies of $\mathbb{R}$, as an
$\mathcal{L}_{\text {or }}$-structure
$\mathcal{F}$ : an ultrafilter on $\mathbb{N}$

Consider $\mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$
Observations:

* This structure $\mathcal{R}$ contains infinitesimal numbers.

Consider the eq. cl. $\varepsilon=\left[\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}\right]$

$$
\begin{gathered}
\rightsquigarrow \mathcal{R} \models 0<\varepsilon \\
\text { as }\left\{n \in \mathbb{N}: 0<\frac{1}{n}\right\}=\mathbb{N} \in \mathcal{F}
\end{gathered}
$$

$$
\begin{gathered}
\frac{\text { Moreover }}{r \in \mathbb{R}^{>0}} \mathcal{R} \models \varepsilon<[\{r, r, r, \ldots\}] \text {, where } \\
\text { as }\left\{n \in \mathbb{N}: \frac{1}{n}<r\right\} \in \mathcal{F}
\end{gathered}
$$

## Non-standard Reals - Internal Point of View

The ultraring $\mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$
Observations:
$* \mathcal{R}$ contains elements larger than any real number
$\mathcal{R}$ does not contain a largest element

Consider the eq. cl. $\omega=[\{1,2,3, \ldots\}]$
$\leadsto \mathcal{R} \neq[\{r, r, r, \ldots\}]<\omega$, for any real
number $r$

Consider $(\exists x)(\forall y) y<x$

It does not hold in $\mathbb{R}$, so must be false in $\mathcal{R}$.

## Non-standard Reals - Internal Point of View

The ultraring $\mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$
Observations:
$\circledast \mathcal{R}$ contains elements larger than any real number

Consider the eq. cl. $\omega=[\{1,2,3, \ldots\}]$
$\rightsquigarrow \mathcal{R} \models[\{r, r, r, \ldots\}]<\omega$, for any real number $r$.

$$
\text { as }\{n \in \mathbb{N}: r<n\} \in \mathcal{F}
$$

## Non-standard Reals - Internal Point of View

The ultraring $\mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$
Observations:
$\circledast \mathcal{R}$ contains elements larger than any real number

* $\mathcal{R}$ does not contain a largest element

Consider the eq. cl. $\omega=[\{1,2,3, \ldots\}]$
$\rightsquigarrow \mathcal{R} \models[\{r, r, r, \ldots\}]<\omega$, for any real number $r$.

$$
\text { as }\{n \in \mathbb{N}: r<n\} \in \mathcal{F}
$$

## Non-standard Reals - Internal Point of View

The ultraring $\mathcal{R}=\prod_{i \in \mathbb{N}} \mathbb{R} / \sim_{\mathcal{F}}$
Observations:
$\circledast \mathcal{R}$ contains elements larger than any real number

* $\mathcal{R}$ does not contain a largest element

Consider $(\exists x)(\forall y) y<x$
Consider the eq. cl. $\omega=[\{1,2,3, \ldots\}]$
It does not hold in $\mathbb{R}$, so must be false in $\mathcal{R}$.
$\rightsquigarrow \mathcal{R} \models[\{r, r, r, \ldots\}]<\omega$, for any real number $r$.

$$
\text { as }\{n \in \mathbb{N}: r<n\} \in \mathcal{F}
$$

