MODEL THEORY OBERSEMINAR: TOPOS THEORY

Introduction

The plan of this semester's Oberseminar is to cover the following topics on topos theory

(1) (April 13) Fast track to categorical notions
(2) (April 20) Elementary toposes / Grothendieck toposes
(3) (April 27) Semantics I
(4) (May 04) Semantics II
(5) (May 11) Sequents, syntactic categories and completeness I
(6) (May 18) Sequents, syntactic categories and completeness II
(7) (May 25) Applications: Deligne theorem and classical completeness
(8) (June 01) Classifying toposes I
(9) (June 11) Classifying toposes II
(10) (June 22) Topos-theoretic spectra I
(11) (June 29) Topos-theoretic spectra II

The seminar has essentially 3 parts. During the first part, after doing a recap of the needed categorical background, we introduce elementary and Grothendieck toposes. Then, we introduce categorical logic, giving both its semantics, a syntactic deduction system, and we sketch a proof of a completeness theorem for such a logic. As an application we will obtain a theorem of Deligne and recover classical completeness of first order logic. The second and third parts deal, respectively, with classifying toposes (which already appeared in the proof of completeness) and topos-theoretic spectra.

Most of the talks (except for 1, 2, 7) are paired and thought to be worked out either in pairs (at least to decide in pairs how to divide the material). In what follows we give the literature for each session. The descriptions are short and the sources sometimes a bit on the formal side – ask Peter for an overview of your talk (on Rocketchat or peter.arndt@hhu.de)!

Lecture 1: Fast track to categorical notions

The following notions should be recalled/defined: limit, colimit, (co)equalizers, pullbacks, pushouts, mono and epimorphisms, exponential objects. Show their construction in Set. References are [2, Chapter 3] and [6, Chapter 1]. A result which should be worth mentioning is the following: "A pullback of a monomorphism is a monomorphism". If time allows, introduce subobjects.

Lecture 2: Elementary and Grothendieck toposes

Introduce elementary and Grothendieck toposes. The first 10 pages of the following (Notes), or these notes, should be a good resource for talk 2. State the following theorem (without proof) "elementary toposes have all finite colimits". Several examples should be given: functor category Set^C for any small category C (perhaps prove this if times allows) see Section 3, Pages 23-26 of (Notes) and [4, Prop 1.12]. Another example: sheaves on a topological space (maybe in their incarnation as local homeomorphisms). Finally, we should show that the subobjects of an object form a Heyting algebra.

Lectures 3 and 4: Semantics I and II

The first lecture will introduce the semantics and cover the material in [5, p-818-828], with first examples (e.g. show that a morphism is an epi iff the formal statement saying that it is surjective is satisfied). The second lecture will focus on examples, and in particular on Rings, covering some material in [7]. Another interesting source of examples is Blechschmidt's PhDthesis (some slides available on this website).

Lecture 5 and 6: sequents, sintactic cateories and completness I and II

These two lectures will cover the material in [5, D1.3-D1.4]. Here the syntactic calculi of sequents will be introduced and the completeness theorem will be sketched.

Lecture 7: Applications: Deligne theorem and classical completeness

The material of this section corresponds to [5, D1.5].

Lecture 8 and 9: Classifying toposes

The material of this section corresponds to [6, Chapter 8, sections 3-6]. Another source is Chapter 3 of []Buckley's PhDThesis.

Lecture 10 and 11: Topos-theoretic spectra

The material of this section corresponds to [1] and [3].

References

[1] Coste, M. Localisation, spectra and sheaf representation. In *Applications of sheaves (Proc. Res. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., Univ. Durham, Durham, 1977)* (1979), vol. 753 of *Lecture Notes in Math.*, Springer, Berlin-New York, pp. 212–238. 2

[2] Goldblatt, R. *Topoi*, second ed., vol. 98 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1984. The categorial analysis of logic. 1

[3] Johnstone, P. T. Rings, fields, and spectra. *J. Algebra 49*, 1 (1977), 238–260. 2

[4] Johnstone, P. T. *Topos theory*. London Mathematical Society Monographs, Vol. 10. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1977. 1

[5] Johnstone, P. T. *Sketches of an elephant: a topos theory compendium. Vol. 1*, vol. 43 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 2002. 2

[6] Mac Lane, S., and Moerdijk, I. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition. 1, 2

[7] Mulvey, C. Intuitionistic algebra and representations of rings. In *Recent advances in the representation theory of rings and C^*-algebras by continuous sections (Sem., Tulane Univ., New Orleans, La., 1973)*. 1974, pp. 3–57. Mem. Amer. Math. Soc., No. 148. 2