GRK 2240 WORKSHOP: p-DIVISIBLE GROUPS

ORGANIZED BY
GEORG LINDEN AND YINGYING WANG

Wintersemester 2019/2020

Introduction

This seminar will serve as an introduction to the theory of p-divisible groups which we will develop around the paper by Tate from 1967 [Tat67] (talks 6–12). As a motivation and to have some examples at our disposal, we begin with a review of some aspects of the arithmetic of elliptic curves (talks 1–5).

Let E be an elliptic curve over a field K (or more generally an abelian variety). One object we can associate to E is the *Tate module*

$$T_\ell(E) \coloneqq \varprojlim_{n\in\mathbb{N}} E[\ell^n](\bar{K}),$$

for any prime ℓ, where $E[\ell^n]$ is the kernel of the multiplication by ℓ^n. Then $T_\ell(E)$ is a $\mathbb{Z}_\ell$-module with

$$T_\ell(E) \cong \begin{cases} \mathbb{Z}_\ell \times \mathbb{Z}_\ell & , \ell \neq \operatorname{char} K \\ 0 \text{ or } \mathbb{Z}_\ell & , \ell = \operatorname{char} K \end{cases} \tag{1}$$

and in fact a continuous $\mathbb{Z}_\ell$-representation of $\operatorname{Gal}(\bar{K}/K)$. The following two theorems demonstrate that $T_\ell(E)$ contains a lot of information about E:

Theorem (Tate's isogeny theorem). *Let E and E' be elliptic curves over a finite field $\mathbb{F}_q$ and let $\ell \neq \operatorname{char} \mathbb{F}_q$ be a prime. Then E and E' are isogeneous if and only if $T_\ell(E)$ and $T_\ell(E')$ are isomorphic as* $\operatorname{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$*-representations.*

Theorem (Néron-Ogg-Shafarevich criterion). *Let E be an elliptic curve over a p-adic local field K, and let $\ell \neq p$ be a prime. Then E has good reduction (i.e. the reduction of E to the residue field of K is non-singular) if and only if $T_\ell(E)$ is unramified as a* $\operatorname{Gal}(\bar{K}/K)$*-representation.*

(The above theorems can be formulated for abelian varieties as well.) However, for some questions, for example concerning deformations of elliptic curves, it does not suffice to only consider $\ell \neq p$: Let $\tilde{E}$ be an elliptic curve over $\mathbb{F}_p$. Let E and E' be two elliptic schemes over $\mathbb{Z}_p$ that lift $\tilde{E}$, i.e. $E_{\mathbb{Q}_p}$ and $E'_{\mathbb{Q}_p}$ are elliptic curves and $E_{\mathbb{F}_p} \cong \tilde{E} \cong E'_{\mathbb{F}_p}$. Then $T_\ell(E_{\mathbb{Q}_p}) \cong T_\ell(\tilde{E}) \cong T_\ell(E'_{\mathbb{Q}_p})$, for $\ell \neq p$. Hence we see that the ℓ-adic Tate module cannot distinguish between different lifts of $\tilde{E}$. On the other hand, it follows from (1) that $T_p(E)$ does not capture enough information either.
To resolve this, one passes to the *p-divisible group* associated to an elliptic curve E (resp. an abelian variety) instead:

$$E[p^\infty] \coloneqq \varinjlim (E[p] \to E[p^2] \to \ldots).$$

Here, we do not only consider the geometric points of the $[p^n]$-torsion but all of the $E[p^n]$ as a formal scheme. Then it is a theorem of Serre and Tate that the deformation theory of elliptic curves (resp. abelian varieties) and the associated p-divisible groups is the same.

In talk 6 to 12, we start with considering finite flat group schemes and we define p-divisible groups in full generality. Then we follow Tate's paper [Tat67], which studies p-divisible groups over a complete noetherian local ring of characteristic 0 with residue field of characteristic p. We work through the proofs of the two main theorems: The faithfulness of the generic fibre functor and the Hodge-Tate decomposition which also implies the Hodge-Tate decomposition of the étale cohomology of abelian varieties:

$$H^1_{\text{ét}}(E_{\mathbb{C}_p}, \mathbb{Q}_p) \otimes \mathbb{C}_p \cong H^1(E_{\mathbb{C}_p}, \mathcal{O}_{E_{\mathbb{C}_p}}) \oplus H^0(E_{\mathbb{C}_p}, \Omega^1_{E_{\mathbb{C}_p}})(-1). \tag{2}$$

Moreover, p-divisible groups also appear in other branches of Arithmetic Geometry. For instance, Tate's paper is one of the foundational papers in p-adic Hodge theory. The Hodge-Tate decomposition (2) is also of interest for studying the p-adic Galois representations coming from the cohomology of these geometric objects. One should also mention the moduli spaces of p-divisible groups. For example, the local Langlands correspondence can be realized via their cohomology (cf. [HT01]).

1. Recollection on the geometry of elliptic curves [Sil09, Ch. III. §1,3]

Define Weierstrass equations, the associated discriminant and j-invariant. Discuss Proposition 1.4 but skip the proof if necessary. Give some of the examples in Figures 3.1 and 3.2. Recall the definition of an elliptic curve and prove Proposition 3.1. Use the Riemann-Roch theorem to endow an elliptic curve with a group law (Lemma 3.3, Proposition 3.4 a–d). Show that this group law is a morphism (Theorem 3.6, skip the proof if necessary).

2. Isogenies [Sil09, Ch. III. §4]

Define isogenies of elliptic curves. Introduce the important Example 4.1 of the multiplication by m and prove Proposition 4.2. Define the m-torsion subgroup $E[m]$. Discuss Remark 4.3 about the definition of complex multiplication and present Example 4.4. Consider the translation map on elliptic curves (Example 4.7). State that every isogeny of elliptic curves is a homomorphism (Theorem 4.8), and that the kernel of a non-zero isogeny is a finite subgroup (Corollary 4.9). State Theorem 4.10 and Corollary 4.11. Finally prove Proposition 4.12 and Remarks 4.13.1,2 if time permits.

3. The Tate module of an elliptic curve [Sil09, Ch. III. §6,7]

Introduce the dual isogeny (Theorem 6.1a) and list its properties (Theorem 6.2). Present the Example 4.5. Prove Corollary 6.4 about the structure of $E[m]$. Define the Tate module $T_\ell(E)$ of an elliptic curve E and describe its structure (Proposition 7.1). Introduce the ℓ-adic Galois representation on $T_\ell(E)$ (Remark 7.2) and discuss the related definition of the ℓ-adic cyclotomic character (Remark 7.3). Prove Theorem 7.4, i.e. that the natural map

$$(*) \qquad \mathrm{Hom}(E_1, E_2) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Hom}(T_\ell(E_1), T_\ell(E_2))$$

is injective. Conclude that $\mathrm{Hom}(E_1, E_2)$ is a free $\mathbb{Z}$-module of rank at most 4 (Corollary 7.5). Consider the $\mathrm{Gal}(\bar{K}/K)$-invariant elements in $(*)$ (Remark 7.6) and state Tate's isogeny theorem (Theorem 7.7a): Two elliptic curves over a finite field are isogeneous if and only if their Tate modules are isomorphic Galois representations.

4. Elliptic curves over local fields [Sil09, Ch. VII.]

For an elliptic curve E over a local non-archimedean field define minimal Weierstrass equations and discuss their existence and uniqueness as in Proposition 1.3 a),b) and d). Introduce the reduction $\tilde{E}$ of E and state Proposition 2.1. Prove Proposition 3.1 about the points of finite order in $E(K)$ (You can treat the statement about the formal group associated to E as a black box in this talk). Present some of the Examples 3.3.1–3.3. Recall the notion of the maximally unramified extension of K and the inertia subgroup of $\mathrm{Gal}(\bar{K}/K)$ as necessary and define when an action of $\mathrm{Gal}(\bar{K}/K)$ is unramified. Finally show Proposition 4.1, i.e. that if $\tilde{E}$ is non-singular then the Tate-module $T_\ell(E)$ is unramified, where ℓ is a prime not equal to the residue characteristic of K.

5. The criterion of Néron-Ogg-Shafarevich [Sil09, Ch. VII.]

Introduce the classification of good, multiplicative and additive reduction of an elliptic curve E over a local non-archimedean field K, and explain the connection to the minimal Weierstrass equation of E, Proposition 5.1. Present the Examples 5.2. If time permits, define potentially good reduction and state the semi-stable reduction theorem (Theorem 5.4) and Theorem 5.5. State the Theorem 6.1 by Kodira and Néron and also Corollary 6.2. Prove the criterion of Néron-Ogg-Shafarevich (Theorem 7.1) which states that E has good reduction if and only if the Tate module $T_\ell(E)$ is unramified, for ℓ some (all) primes not equal to the residue characteristic of K. Moreover, draw the Corollaries 7.2 and 7.3.

6. Affine group schemes and Finite flat group schemes

Let R be a commutative ring. Define (commutative) affine group schemes over R as group objects in the category of affine R-schemes [Tat97, §1.1-1.6] (cf. [Dem72, Ch. I], [Sha86, §2]). If time permits, also mention that an affine group scheme over R is a representable functor from the category of R-algebras to groups [Tat97, §1.6]. Construct kernels in the category of affine group schemes and mention the difficulty of constructing cokernels [Tat97, §1.7-1.8]. Present the Hopf algebra interpretation for affine group schemes Spec(A) where A is an R-algebra [Tat97, §2.2] (cf. [Sha86, §2]). Give the examples $\mathbb{G}_a, \mathbb{G}_m, \mu_n$, [Tat97, §2.4-2.7] and constant group schemes and diagonalizable group schemes [Tat97, §2.6, §2.10]. Finally, define finite flat group schemes and present examples [Tat67, §1.1 (a), (b)].

7. Duality and structure of Finite flat group schemes

Define and explain the character group scheme for an affine commutative group scheme [Tat97, §2.9]. Present Cartier duality of finite flat group schemes and illustrate with examples [Tat67, §1.2] (cf. [Sha86, §4]). Define étale and connected finite flat group schemes, and give the example that for an elliptive curve E/k, where $\operatorname{char} k = p$, the finite group schemes $E[\ell^n]$ are étale ($\ell \neq p$) but the group schemes $E[p^n]$ are never étale.
Briefly discuss quotients of finite flat group schemes [Tat97, §3.1-3.6] in order to introduce the connected-étale exact sequence for a finite flat group scheme over a complete noetherian local ring [Tat97, §3.7] (cf. [Tat67, §1.3-1.4]).

8. Formal groups and p-divisible groups

Define p-divisible groups and morphism of p-divisible groups following [Tat67, §2.1]. Work out the examples in [Tat67, §2.1] (cf. [Sha86, §6]).
Let R be a complete local ring. First, give the classical definition of n-dimensional formal Lie groups over R [Tat67, §2.2]. Give the example of the formal group associated to an elliptic curve [Sil09, Ch IV]. Show that one can also define a formal group as a representable functor from the category of profinite R-algebras to group [Sha86, §5] (cf. Talk 6).
Define étale and connectedness for formal group schemes [Dem72, Ch. II.7] and present the connected-étale exact sequence for formal groups [Sha86, §5].
Show that a p-divisible group over R is a p-torsion commutative formal group for which the multiplication by p map is an isogeny [Tat67, §2.2] (cf. [Sha86, §6]).

9. Formal Lie groups and Duality for p-divisible groups

State that there is a categorical equivalence between the category of connected p-divisible groups and the category of divisible commutative formal Lie groups over R and outline the constructions [Tat67, §2.2 Prop 1] (cf. [Sha86, §6 Theorem]). Show that this theorem implies every connected p-divisible group is smooth [Dem72, Ch. II] (cf. [Sha86, §6]).
Define the Cartier dual for a p-divisible group, give examples and present Cartier duality of p-divisible groups [Tat67, §2.3] (cf. [Sha86, pp. 63-64]).

10. Galois modules $\Phi(G)$ and $T(G)$ [Tat67, §2.4]

Let G be a p-divisible group over a complete local ring R with residue field of characteristic p and fraction field K. Define the group of points of G with values in an extension of R. State and justify the connected-étale short exact sequence for group of points of G when R has a perfect residue field as in Proposition 4. State and prove Corollary 1 and 2. Introduce the tangent space of G at origin and the logarithm. Give examples of the logarithm in the case of the multiplicative group $\mathbb{G}_m$ and elliptic curves.
Define the Tate module $T(G)$ and Tate comodule $\Phi(G)$ for a p-divisible group G. Show that they each determines the generic fibre of G over R, and present the cannonical isomorphisms of $T(G)$ and $\Phi(G)$ (c.f. [Sha86, §6]). Give examples of $T(G)$ and $\Phi(G)$, e.g. in the case of $\mathbb{G}_m$ and abelian varieties.

11. Hodge-Tate decomposition for p-divisible groups

Let G be a p-divisible group over R as in Talk 10 and assume that R is of characteristic 0. Explain the connection between the Tate module of G and its dual G' [Sha86, p. 65]. Construct the pairings in [Tat67, §4] (cf. [Sha86, p. 66]). State and prove Theorem 3 of [Tat67], which gives descriptions of points of G and the tangent space of G in terms of the Tate module of G' (c.f. [Sha86, §6]). Skip the proof of [Tat67, Prop 11].

The Hodge-Tate decomposition follows from Theorem 3 [Tat67, Cor 2]. Sketch the proof here and show how this can be applied to the case of abelian varieties over R (c.f. [Sha86, §6]). One may also mention that this result is later generalized by Faltings [Fal88, Ch III, Theorem 4.1].

12. Main Theorem [Tat67, §4.2]

Let R be as in Talk 11. State and prove Theorem 4, which tells us that morphism of general fibres of p-divisible groups over R of mixed characteristc can be uniquely lifted to a morphism of the p-divisible groups. In other words, the functor that sends p-divisible groups G defined over R to $G \otimes_R K$ is fully faithful. State and prove the corollaries.

If time permits, state and explain this theorem in the case when R is of positive characteristic [Ber80] [Jon98].

References

[Ber80] Pierre Berthelot. "Théorie de Dieudonné sur un anneau de valuation parfait". fr. In: *Annales scientifiques de l'École Normale Supérieure* 4e série, 13.2 (1980), pp. 225–268.

[Dem72] Michel Demazure. *Lectures on p-divisible groups*. Lecture Notes in Mathematics, Vol. 302. Springer-Verlag, Berlin-New York, 1972.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*. Vol. 151. Annals of Mathematics Studies. With an appendix by Vladimir G. Berkovich. Princeton University Press, Princeton, NJ, 2001, pp. viii+276. ISBN: 0-691-09090-4.

[Jon98] A. J. de Jong. "Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic". In: *Invent. Math.* 134.2 (1998), pp. 301–333. ISSN: 0020-9910.

[Sha86] Stephen S. Shatz. "Group schemes, formal groups, and p-divisible groups". In: *Arithmetic geometry (Storrs, Conn., 1984)*. Springer, New York, 1986, pp. 29–78.

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*. Second. Vol. 106. Graduate Texts in Mathematics. Springer, Dordrecht, 2009, pp. xx+513. ISBN: 978-0-387-09493-9.

[Tat67] J. T. Tate. "p-divisible groups". In: *Proc. Conf. Local Fields (Driebergen, 1966)*. Springer, Berlin, 1967, pp. 158–183.

[Tat97] John Tate. "Finite flat group schemes". In: *Modular forms and Fermat's last theorem (Boston, MA, 1995)*. Springer, New York, 1997, pp. 121–154.

[Voi07] Claire Voisin. *Hodge theory and complex algebraic geometry. I*. English. Vol. 76. Cambridge Studies in Advanced Mathematics. Translated from the French by Leila Schneps. Cambridge University Press, Cambridge, 2007. ISBN: 978-0-521-71801-1.

Additional references

[Far04] Laurent Fargues. "Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales". In: *Astérisque* 291 (2004). Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, pp. 1–199.

[Lev10] Brandon Levin. *p-Divisible Groups: Part II*. Lecture notes. 2010.

[Lie00] Max David Lieblich. "Galois Representations arising from p-divisible groups". Diploma thesis. Harvard University, 2000.

[Mor15] Matthew Morrow. *p-Divisible Groups*. Lecture notes. 2015.

[Ser95] Jean-Pierre Serre. "Groupes p-divisibles (d'après J. Tate)". In: *Séminaire Bourbaki, Vol. 10*. Soc. Math. France, Paris, 1995, Exp. No. 318, 73–86.

[Sti12] Jakob Stix. *A course on finite flat group schemes and p-divisible groups*. Lecture notes. 2012.

[You16] Alex Youcis. *p-divisible groups, formal groups, and the Serre-Tate theorem*. Aug. 2016. URL: https://ayoucis.wordpress.com/2015/08/10/p-divisible-groups-formal-groups-and-the-serre-tate-theorem/.

[Zha17] Luming Zhao. "Comparison theorems for p-divisible groups". MA thesis. Universite Bordeaux 1, 2017.