GRK 2240 WORKSHOP: C_i-FIELDS

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Introduction

The main subject of this workshop is to study fields in which the following phenomenon arises:

a homogeneous polynomial which has sufficiently more variables than its degree has a non trivial solution.

There are several ways in which one can try to formalize what "sufficiently" should mean. We will focus on a variant introduced by Lang in his thesis under the name of *C_i-fields*.

Definition 1 (Lang)**.** Let i be a non-negative integer. A field k is called C_i if every homogeneous polynomial with coefficients in k of degree d and in strictly more than d^i variables has a non-trivial solution in k.

It is an easy exercise to show that k is C_0 if and only if it is algebraically closed (do it!). Fields which are C_1 were studied by Tsen, and Artin called them *quasi-algebraically closed* before Lang introduced the C_i terminology. Early results imply the following connection to Brauer groups:

Lemma 2. *If k is C_1, then the Brauer group of k is trivial.*

In spite of the previous result and a result of Wedderburn which implies that the Brauer group of a finite field is trivial, Artin conjectured that finite fields are C_1. This was indeed later confirmed by Chevalley [2] and follows by the following stronger result of Warning [8] (nowadays called Chevalley-Warning):

Theorem 3 (Chevalley-Warning)**.** *Let f be a polynomial in n variables with coefficients in a finite field k and let d be its degree. If $n > d$, then the number of solutions of f in k is congruent to 0 modulo p. In particular, finite fields are C_1.*

Lang, Nagata and Tsen studied the behaviour of the C_i condition under field extensions.

Theorem 4 (Lang)**.** *Let k be a C_i field. Then every algebraic extension of k is C_i.*

Theorem 5 (Tsen/Lang-Nagata)**.** *Let k be a C_i-field. If K is an extension of k of transcendence degree n, then K is C_{i+n}.*

Later, Greenberg [3] showed a similar result for fields of power series.

Theorem 6 (Greenberg)**.** *Let k be a C_i field. Then $k(\!(t)\!)$ is C_{i+1}.*

As a Corollary one obtains that the field $\mathbb{F}_p(\!(t)\!)$ is C_2 (and also that $\mathbb{C}(\!(t)\!)$ is C_1 and hence, has trivial Brauer group). Given the similarity between $\mathbb{F}_p(\!(t)\!)$ and the field of p-adic numbers $\mathbb{Q}_p$, Artin conjectured that $\mathbb{Q}_p$ was also C_2. However, this time Artin was wrong. Although his conjecture was confirmed in degree $d \leqslant 3$ (see [6]), Terjanian [7] found counter-examples in degree 4. Surprisingly, despite this negative answer, the following theorem of Ax-Kochen [1] shows that Artin was almost correct in his guess.

Theorem 7 (Ax-Kochen)**.** *Fix $d > 0$. Then, there is a finite set X_d of prime numbers such that for every $p \notin X_d$, every homogenous polynomial with coefficients in $\mathbb{Q}_p$ with strictly more than d^2 variables has a non-trivial solution in $\mathbb{Q}_p$.*

Their proof uses some mild tools from model theory. One of the main tools which we will take as a black box is the following result of Ax and Kochen also proved independently by Ershov (all terms to be later defined during the workshop):

Theorem 8 (Ax-Kochen/Ersov)**.** *Let (K, v) and (L, v) be two Henselian valued fields of equicharacteristic 0. Let vK, vL denote their value groups and Kv, Lv denote their residue fields. If vK and vL, and respectively Kv and Lv are elementarily equivalent, then so are K and L.*

Combining Theorem 8 with an *ultraproduct* construction yields essentially the proof of Theorem 7.

The main objective of workshop is to prove all the above mentioned theorems. Most objects such as Brauer groups, valued fields and ultraproducts will be introduced from scratch. The main result which will be used as a black box is Theorem 8, although all the terms involved in it will be explained.

Most articles so far cited are very well written and interesting to read on their own and speakers are more than welcomed to read the original papers. There is a beautifully written survey book by M. Greenberg [4] on the subject which we can follow for all results except of Theorem 7. The master thesis of A. Kruckman [5] is also a very good survey of Ax-Kochen's theorem which also includes most results we will cover in the workshop. The following is a more detailed schedule for the talks together with other potential references.

Lecture 1: Overview of C_i-fields and motivation

This first lecture should introduce the audience to C_i-fields and provide an overview of the results we will tackle in the workshop. Besides discussing several examples the lecture should contain:

(1) an introduction to Brauer groups and its connexion to C_i-fields (i.e., a proof of Lemma 2);
(2) a proof that finite fields are C_1 via a proof of Theorem 3.

All this material is nicely explained and included in the Chapters 1 and 2 of [4] (they are quite short). Theorem 3 is quite classical and there are different variants, so feel free to show the one you find more interesting.

Lecture 2: Field extensions

This lecture will be devoted to study the behaviour of the C_i condition under field extensions. We will prove Theorems 4 and 5, which are contained in Chapter 3 of [4]. They also appear in [5] (but he also cites [4]). In addition, we will pave the way to show Theorem 6 and therefore some brief recall of discretely valued fields should be given here.

Assuming a result on discretely valued fields (which will be proven in Lecture 3) one can show how to deduce Theorem 6. If time allows, one can even give a short proof of Theorem 6 in the special of $\mathbb{F}_p((t))$, and conclude that such fields are C_2. The special case can be found in Chapter 4 of [4].

Lecture 3: Greenberg's theorem

This lecture will be dedicated to show Theorem 6 in its full generality. This is done by showing a result about solutions of polynomials in discretely valued fields due to Greenberg in [3]. The proof uses some algebraic geometry. A discussion about related problems (which are also discussed in [3]) would be interesting if time permits.

Lecture 4: An introduction to the Ax-Kochen/Ershov principle

In this lecture we will introduce some basic notions of model theory in order to fully understand Theorem 8. Concepts such as first order language, model, formula and elementary equivalence will be introduced. Although this lecture might contain almost no proofs, the speaker should try her/his best to convey the material as simple and appealing as possible! The material is basically contained in [5], but perhaps more background could help here. I will potentially provide notes for this lecture (although they are not written yet).

Lecture 5: Introduction to ultraproducts and Łoś theorem

The purpose of this lecture is to define the ultraproduct construction and discuss a fundamental results about them: Łoś theorem. Independently of C_i-fields, this lecture might be interesting for any algebraist. The speaker should try to provide as many examples and applications as possibles. A nice application to keep in mind is to show a version of a Lefschetz transfer principle for algebraically closed fields, which is in the spirit of the proof of Theorem 7 we will see in Lecture 6.

Ultraproducts and Łoś' theorem are also discussed in [5], but again we might want to say a bit more. These notes (link) also contain nice material (perhaps some parts diverge too much on what we want to cover). As for Lecture 4, I will potentially provide notes for this lecture.

1. Lecture 6: Putting everything together and final remarks

In this lecture we use all results from the previous lectures to give a proof of Theorem 7. The proof is contained in [5]. If time allows (hopefully yes) one could also present the counter-examples of Terjanian [7] to Artin's conjecture (or even indicate more general results which show that $\mathbb{Q}_p$ is actually not C_i for every i!).

References

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[3] Greenberg, M. J. Rational points in Henselian discrete valuation rings. *Inst. Hautes Études Sci. Publ. Math.*, 31 (1966), 59–64. 1, 3

[4] Greenberg, M. J. *Lectures on forms in many variables.* W. A. Benjamin, Inc., New York-Amsterdam, 1969. 2

[5] Kruckman, A. The ax-kochen theorem: an application of model theory to algebra. arXiv:1308.3897 [math.LO], 2013. 2, 3

[6] Lewis, D. J. Cubic homogeneous polynomials over p-adic number fields. *Ann. of Math. (2) 56* (1952), 473–478. 1

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