1. Goals of the program

There are a couple of main goals of this program. Primarily, the goal is to give the PhD students enough familiarity with concepts so that they can get as much as possible out of the Summer school in September.

Aside from this, we have the following goals in chronological order (rather than that of importance):

- understand fundamental examples of Abelian categories and what the definition allows one to do (form the category of chain complexes and compute resolutions),
- understand key constructions in Chain complexes (the mapping cone construction and importation of intuition/techniques from topology),
- understand the derived category of a scheme,
- have some familiarity with the stable Module category,
- understand how the derived category and the stable module category are examples of tt-categories,
- be able to state the definition of the Balmer spectrum along with the main results regarding it.

2. Comments on giving talks

Please follow the outlines and at least state all results mentioned. The individual talks are not self contained and depend on you covering the given material. If you have any questions please contact me directly. As this is introductory, the emphasis should be on concepts as opposed to detailed proofs.

Note: If you need electronic access to an item in the bibliography please contact me.

3. Talk 1: Abelian categories

The goal of this talk is to give the audience enough familiarity with the definition and some familiar examples, so that they can follow the talk on Chain complexes and the various constructions.

Give the definition of an Abelian category in steps (first additive and then abelian) following either Appendix 4 of [6] or Wikipedia, ignoring the extra axioms of Grothendieck. Give The examples of modules over a ring, representations of a finite group, and sheaves of abelian groups.

Make sure to explain kernels and cokernels in the abstract as well as defining the image of a morphism (abstractly/categorically). Specifically, this means defining them as a pullback/pushout. Give the example of sheaves of objects in an Abelian category (using that while kernels of sheaves are still sheaves that you have to re-sheafify cokernels and other colimit constructions such as cokernels). Add that this works for sheaves on a site/Grothendieck topology. Mention the tensor product in each above example. Define exactness. Give the definition of projective and injective object as well as how to construct projective/injective resolutions (provided enough projectives/injectives exist). State Freyd's embedding theorem (as it is a sort of guiding principle), Theorem 1.6.1 of [6].

State definitions of coherent, and quasicoherent sheaves on a Noetherian scheme. State the equivalence of categories between such sheaves on an affine scheme and the relevant module categories. Show that such categories of sheaves are sufficiently closed under the above structures, if time allows.

References for this talk are Wikipedia as well as Weibel's book on homological algebra [6] (or really any book on homological algebra).

Necessary for subsequent talks:

- Freyd's embedding theorem,
- understand cokernels as pushouts,
- formation of resolutions,
- understanding categories of sheaves as abelian categories.

4. Talk 2: Chain complexes

The goal of this talk is to give definitions and explicit constructions. These will be necessary for the construction of the derived category. Be careful about signs in this section, they make a difference.

Give the definition of a chain complex of abelian groups (indicate what a cochain complex is). Explain how the definition extends to an arbitrary abelian category. Define the homology of a chain complex. Give the example of the "interval object" I

$$I := 0 \longleftarrow \mathbb{Z}\{e_0, e_1\} \longleftarrow \mathbb{Z}\{d\} \longleftarrow 0 \cdots$$

where boundary map takes d to $e_1 - e_0$. Also indicate how a projective resolution

$$P_\bullet \longrightarrow M$$

is a morphism from a chain complex $P_\bullet$ to the chain complex which is simply M in degree 0. Explain how the resolution property implies that the map induces an isomorphism in homology.

Define the graded tensor and hom objects (be careful here about signs). Define chain homotopy using I above and the tensor product following the standard topology definition. Give the other definition of chain homotopy and indicate the relationship between the two leaving the proof that they are equivalent as an exercise (by other definition I mean the one given on wikipedia). Show that chain homotopic maps, using the second definition, induce the same map in homology. Define $\mathbb{Z}[n]$, the complex with only one nonzero entry, and use it to define the shift of a a chain complex. Show how this shift behaves with respect to homology. Define bounded, and finite type complexes. Indicate how all of the above generalizes to chain complexes of objects in an abelian category. Then define Ext and Tor, very briefly if time allows.

References for this Wikipedia, [6] (in particular Sections 1.1, 1.2, 1.4, and 1.5) as well as section 1.3.2 of [3]. Necessary for subsequent talks:

- Resolutions as an q.i. object,
- the interval object and notion of homotopy,

5. Talk 3: Derived categories

The goal of this talk is to introduce various definitions and constructions and convince the audience that the notion is not so unfamiliar.

Define homotopy equivalence of chain complexes as well as quasi-isomorphism (q.i.). Recall that a homotopy equivalence is necessarily as q.i. and that a q.i. is a homotopy equivalence if the domain complex is bounded and comprised of projective module (see 10.4.7 of [6]).

See Weibel section 1.5 for the following. Construct the mapping cylinder, Mf, of a map

$$f : X \longrightarrow Y$$

of chain complexes as the pushout $Y \cup_f I \otimes X$ and demonstrate that it is homotopic to Y as is f homotopic to the inclusion of X as a subcomplex of Mf. Consider $Cf := Mf/X$ and write down the formula for the differential (see Weibel. Explain that there is a long exact sequence in homology associated to the short exact sequence in chain complexes

$$X \longrightarrow Mf \longrightarrow Cf.$$

Show that $C1 = 0$ and that $C0 = x[1]$, provided one has time to do so.

Give the definition of a derived category of a ring as well as the homotopy category following wikipedia. Also give the definitions of the bounded varieties of these categories. Spell out the universal properties of $D(A)$ and $K(A)$ following the universal properties of a localized category. Define the localization of a category, see Definition 10.3.1 of [6], and remark on its similarity to the localization of a ring. When defining the localization give the definition but not all of the conditions on S. (Have them in your notes in case someone insists). Give Weibel's example 10.3.2. Mention that objects in the derived category can be represented by preferred objects in the homotopy category as in Theorem 10.4.8 of Weibel.

Maybe state remark 10.3.3 of Weibel out loud, but do not spend more than 1 minute on it (Also, the various Grothendieck versions of abelian categories are meant to deal with these issues. While they are legitimate mathematical issues, they are not our focus.). Mention the category of (left) fractions construction on page 381 and 382 of [6] if time permits. This is messy, try not to spend too much time on it.

References for this talk are Wikipedia, and Sections 10.3 and 10.4 of [6] (Or any homological algebra book). Necessary for subsequent talks:

- Cofiber/cone construction as well as shift,
- definition of perfect complexes.

6. Talk 4: The Stable Module category

The main goal of this talk is to state the definition and describe some objects in the category in certain examples.

Introduce the stable module category of a ring. Define the operation Ω as well as Ω^{-1} using surjections from projective modules and injections into injective modules. Note that these constructions are not well defined on the module category, but are on the stable module category. Show that both of these are functors on the module category using the universal property of projective/injective modules as well as kernels and cokernels. Show that $k[G]$ is a self injective algebra (that projective modules are injective and vice versa) (if you feel you will be short on time just state this). Deduce from this that Ω and Ω^{-1} are inverse to one another. Relate Hom in the stable module category to cohomology (see the Wikipedia article on the stable module category for this).

Mention the fact that the trivial module being projective depends on the ground ring. Recall Maschke's theorem and demonstrate its failure with respect to $\mathbb{F}_p[C_p]$-modules. Then explain how this demonstrates that the stable module category measures how far $k[G]$ is from being a semi-simple algebra.

Define also the cone of a map $f : M \to N$ via the pushout of

$$PM \xrightarrow{f} N.$$

Compute $\Omega(M)$ for M various modules over $\mathbb{F}_p[x]/(x^p)$ (recall that $\mathbb{F}_p[x]/(x^p) \cong \mathbb{F}_p[C_p]$). (especially the trivial module $\mathbb{F}_p$ for $p \leq 5$).

State the cohomology of C_p with coefficients in $\mathbb{F}_p$ (If you feel there is lots of time give the computation of the abelian group structure). Define elementary abelian p-groups and give their cohomology as described in [3]. State Theorem 2.21 of [3], due to Quillen. Mention how this identifies $Spec^h(H^*(G;k))$ as a quotient of affine spaces, see discussion after 2.21.

Time permitting: Define cohomological varieties (V_G and $V_G(M)$) as in Definition 2.29 of [3]. State Theorem 2.30 of [3]. Relate these to how closed subsets of a topology behave.

References are Wikipedia, Section 1.2 of [3], Sections 1.2 and 1.3 of [4].
Necessary for subsequent talks:

- Construction of Ω^{-1} as a functor on $stmod_{k[G]}$,
- Construction of cone/cofiber of a map
- Identification of cohomology as an object in the stable module category.

7. Talk 5: Tensor triangulated categories

The goal of this talk is to give rudimentary definitions and highlight how the objects of talks 3 and 4 give us examples of tt categories.

Define a triangulated category. Do not state the octahedral axiom as listed in various references. Instead use the version given as T3 of [5]. Define a tensor triangulated category. Define an exact/tensor functor. Define compact objects in a triangulated category. Define thick subcategories. Show that the Kernel of an exact functor is thick.

Explain the equivalence

$$T/ker(L) \cong im(L).$$

Also mention Verdier quotients following [2] Remark 19.

Explain how the derived category of a scheme is tensor triangulated. Explain how the stable module category is a tensor triangulated category. Identify the compact objects in each example. See [2] Sections 1.2 and 1.4 for this. (By the above I mean highlight the relevant structures from previous talks that establish these facts.) Point out that the cofiber construction is not a functor as there is not a canonical map between cofibers. Such a canonical map only exists on the category of chain complexes, it depends on the model one takes for the cofiber and not just its equivalence class.

Indicate that restriction to a subscheme induces an exact tensor functor on the derived category. Similarly, restriction to a subgroup does as well.

References for this are Wikipedia, [5], and [2].

Necessary for subsequent talks:

- Definition of thick subcategories.
- Explanation of Verdier quotients,
- Demonstrating that examples from talks 3 and 4 are tensor triangulated categories.

8. Talk 6: tt-Geometry

The goal of this talk is to give the main example applications of tt-Geometry as well as the main constructions. Specifically, we want to understand all of the objects involved in the correspondences as well as some of the maps.

All of the following refer to items in Balmer's survey [2]. Give Theorem 6 and follow with Definition 7 and construction 8. State Proposition 11. Define Thomason subsets as in Remark 12 as well as specialization closed subsets. Then state Theorems 14 and 16 as well as Remark 17.

Briefly summarize Definition 20, Remarks 22 and 23, and give construction 24. State, as briefly as possible, Theorem 25 and summarize Remark 26. Then give Construction 29.

State Theorems 42 and 43. State Theorem 54 and go through Remark 56. State Theorem 57. In doing so recall $spec^h$ and $Proj$ of graded rings.

The key reference for this is [2]. However, there are other works of Balmer, such as [1], that may be helpful.

References

[1] Paul Balmer, *Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings*, Algebr. Geom. Topol. **10** (2010), no. 3, 1521–1563. MR 2661535

[2] ———, *Tensor triangular geometry*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 85–112. MR 2827786

[3] David J. Benson, Srikanth Iyengar, and Henning Krause, *Representations of finite groups: local cohomology and support*, Oberwolfach Seminars, vol. 43, Birkhäuser/Springer Basel AG, Basel, 2012. MR 2951763

[4] Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988. MR 935124

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[6] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324 (95f:18001)

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