On stability for the Ekman boundary layer

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We present a result on well-posedness and stabilitiy for the Ekman boundary layer problem in the space $\mathrm{FM}(\mathbb{R}^2, L^2(\mathbb{R}_+))$, i.e., in the space of $L^2(\mathbb{R}_+)$-valued Fourier transformed Radon measures. We obtain stability in all appearing parameters as time, angle velocity of rotation, viscosity, and layer thickness.

1 Description and main result

The Ekman boundary layder problem is a meteorolocical model for the motion of a rotating fluid (atmosphere) inside a boundary layer, appearing in between a uniform geostrophic flow (wind) and a solid boundary (earth) at which the no slip condition applies. Mathematically this situation is described by the Navier-Stokes equations with Coriolis force

$$\begin{cases} \partial_t u - \nu\Delta u + (u,\nabla)u + 2\Omega e_3 \times u &= -\nabla p \quad \text{in } \mathbb{R}^3_+ \times (0,T), \\ \operatorname{div} u &= 0 \quad \text{in } \mathbb{R}^3_+ \times (0,T), \\ u &= 0 \quad \text{on } \partial\mathbb{R}^3_+ \times (0,T), \\ u|_{t=0} &= u_0 \quad \text{in } \mathbb{R}^3_+. \end{cases} \tag{1}$$

Here the unknowns u and p denote velocity and pressure of the fluid respectively, whereas $e_3 = (0,0,1)$ and the parameters ν and Ω correspond to viscosity and angle velocity of the rotation around the x_3-axis.

There is a famous stationary exact solution to (1) called Ekman spiral and which is given by the vector

$$\mathbf{U}^E(x_3) = U_\infty \left(1 - e^{-x_3/\delta}\cos(x_3/\delta), e^{-x_3/\delta}\sin(x_3/\delta), 0\right)$$

Here $\delta = \sqrt{\nu/\Omega}$ denotes the layer thickness and U_∞ the velocity of the geostrophic flow away from the boundary which is pointing in x_1 direction. The corresponding pressure to $\mathbf{U}^E$ is given by $p^E(x_2) = -\Omega U_\infty x_2$. Remarkable pesistent stability of the Ekman spiral in atmosheric and oceanic boundary layers has been noticed in geophysical literature. We are interested in stability results in the parameters t, Ω, ν, and δ. In particular in the existence of solutions with norms uniformly bounded in Ω in spaces including functions nondecaying at infinity. Results of this type are essential in studies of statistical properties of turbulence, see e.g. [6, 7], and in the analysis of fast oscillating singular limits for system (1), see e.g. [5].

The observation that $\mathbf{U}^E$ depends on the x_3 variable only, i.e., it has infinite energy, and that

$$\lim_{x_3\to\infty} \mathbf{U}^E(x_3) \to (U_\infty, 0, 0)$$

leads to two natural requirements on a potential class $\mathbb{E}$ of initial data:

(i) The class $\mathbb{E}$ should include functions nondecreasing at infinity in tangential direction.

(ii) $u_0 \to (U_\infty, 0, 0)$ if $x_3 \to \infty$ in a certain sense.

A first result on well-posedness for system (1) is obtained in [4] for u_0 in the class

$$\mathbb{E} = \left\{ u \in \dot{B}^0_{\infty,1}(\mathbb{R}^2, L^p(\mathbb{R}_+)^3) + \mathbf{U}^E : \operatorname{div} u = 0,\ u^3|_{\partial\mathbb{R}^3_+} = 0 \right\}, \quad p > 2.$$

Here $\dot{B}^0_{\infty,1}(\mathbb{R}^2, L^p(\mathbb{R}_+)^3)$ stands for the $L^p(\mathbb{R}_+)^3$-valued version of the standard homogeneous Besov space $\dot{B}^0_{\infty,1}(\mathbb{R}^2, \mathbb{C})$. Note that the space $\dot{B}^0_{\infty,1}(\mathbb{R}^2, L^p(\mathbb{R}_+)^3)$ contains $L^p(\mathbb{R}_+)^3$-valued almost periodic functions. Thus, $\mathbb{E}$ satisfies (i) and (ii).

However, it seems that this class is inappropriate for stability investigations. This relies essentially on the fact that the Poincaré-Riesz semigroup $(e^{-tB_\Omega})_{t\geq 0}$ generated by the Coriolis operator $B_\Omega u := 2\Omega e_3 \times u$ proved to be unbounded in t and

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Ω in the space $L^p(\mathbb{R}^3)$, unless $p = 2$. The uniform boundedness in L^2 is a consequence of the fact that here a multiplier result as

$$\|\mathcal{F}^{-1} m \mathcal{F}\|_{\mathcal{L}(L^2(\mathbb{R}^3)^3)} \le \|m\|_{L^\infty(\mathbb{R}^3)^{3\times 3}} \tag{2}$$

for bounded $m : \mathbb{R}^3 \to \mathbb{C}^3$ is available by virtue of Placherel's theorem and since the symbol of the Poincaré-Riesz is bounded (and uniformly bounded in t and Ω). Here $\mathcal{L}(X)$ denotes the class of all bounded operators on a Banach space X, and $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform and its inverse respectively. Hence, in order to obtain results on stability, besides the natural requirements (i) and (ii), a potential class of initial data should also admit a multiplier result as (2). A suitable class satisfying these requirements and which is still large enough is the class

$$\mathbb{E} = \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(\mathbb{R}_+)) + \mathbf{U}^E,$$

where

$$\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(\mathbb{R}_+)) = \Big\{ \mathcal{F}u : \ u \in \mathrm{M}(\mathbb{R}^2, L^2(\mathbb{R}_+)^3), \ |u|(\{0\}) = 0, \ \mathrm{div}\, \mathcal{F}u = 0, \ \mathcal{F}u^3|_{\partial\mathbb{R}^3_+} = 0 \Big\}$$

equipped with the norm $\|\mathcal{F}u\|_{\mathrm{FM}(X)} := \|u\|_{\mathrm{M}(X)} := |u|(\mathbb{R}^3)$. Here $\mathrm{M}(\mathbb{R}^3, X)$ denotes the space of finite X-valued Radon measures and $|u|(\mathcal{O}) := \sup \left\{ \sum_{A\in\Pi} \|u(A)\|_X : \ \Pi \text{ finite partition of } \mathcal{O} \right\}$ the total variation measure of an open set $\mathcal{O} \subseteq \mathbb{R}^2$. Typical examples important for our purposes and included in the space $\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(\mathbb{R}_+))$ are given by almost periodic functions as

$$u_0(x) := \sum_{j=1}^{\infty} a_j e^{i\lambda_j \cdot x}, \quad x \in \mathbb{R}^2, \ \lambda_j \neq 0, \ a_j \in L^2(\mathbb{R}_+)^3, \ \sum_{j=1}^{\infty} \|a_j\|_2 < \infty.$$

The significance of the space $\mathrm{FM}_{0,\sigma}(\mathbb{R}^3, \mathbb{C})$ for the Navier-Stokes equations with rotation in the whole space $\mathbb{R}^3$ was already pointed out in [2] and [3]. In [2] a local-in-time existence result is proved with an existence interval independent of Ω. In [3] global-in-time solvability and exponential stability is derived for initial data $u_0 \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^3, \mathbb{C})$ such that $\mathrm{supp}\, \mathcal{F}u_0$ is contained in a sum-closed frequency set. Moreover, the required smallness condition is explicitly given in terms of the Reynolds number and all the results are independent of Ω. Consequently, we have stability in all appearing parameters.

Adopting the ideas from [2] and [3] we can prove a corresponding result for the Ekman boundary layer problem, i.e., for system (1). Our main result reads as follows.

Theorem 1.1 *Let* $\nu, \delta > 0$ *and* $\Omega \in \mathbb{R}$. *For each* $u_0 \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(\mathbb{R}_+)) + \mathbf{U}^E$ *there exists a* $T_0 \ge C\|u_0\|^2_{\mathrm{FM}(L^2)}$ *with* $C > 0$ *independent of* Ω *and a unique classical solution* $u \in C([0, T_0], \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(\mathbb{R}_+))) + \mathbf{U}^E$ *of system (1).*

The proof is based on a generator result independent of Ω for the linearized Stokes equations (system (1) without the term $(u, \nabla)u$). The essential ingredient in deriving the generator result, in turn, is a multiplier result on the space of Fourier transformed Radon measures $\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(\mathbb{R}_+))$ corresponding to (2). More precisely, we can prove the estimate

$$\|\mathcal{F}^{-1} m \mathcal{F}\|_{\mathcal{L}(\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(\mathbb{R}_+)))} \le \|m\|_{L^\infty(\mathbb{R}^2, \mathcal{L}(L^2(\mathbb{R}_+)))^{3\times 3}}$$

for $m \in \left[C(\mathbb{R}^2 \setminus \{0\}, \mathcal{L}(L^2(\mathbb{R}_+))) \cap L^\infty(\mathbb{R}^2, \mathcal{L}(L^2(\mathbb{R}_+))) \right]^{3\times 3}$. The contraction mapping principle applied on the mild formulation of (1) then yields Theorem 1.1.

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