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Fluid Flow in Wedge Type Domains

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Maximal regularity for the Stokes equations with perfect slip type boundary conditions on wedge domains is studied. The approach is based on operator theoretical methods, such as the operator sum method and $\mathcal{H}^\infty$-calculus, and the reduced Stokes system.

1 Equations and Main Result

Equations on domains of wedge type naturally arise in the analysis of contact line problems after employing a suitable transformation. Therein the crucial prototype geometry is a wedge. Here we consider the Stokes equations subject to perfect slip boundary conditions given as

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla p &=& f & \text{in} \quad (0,T)\times G, \\ \operatorname{div} \mathrm{u} &=& 0 & \text{in} \quad (0,T)\times G, \\ \nu \times \operatorname{curl} u = 0, \quad u\cdot\nu &=& 0 & \text{on} \quad (0,T)\times \partial G, \\ u(0) &=& u_0 & \text{in} \quad G, \end{array}\right. \tag{1}$$

where

$$G = S_{\varphi_0} \times \mathbb{R} \subset \mathbb{R}^3, \quad S_{\varphi_0} := \{(x_1,x_2)\in\mathbb{R}^2;\ 0<x_1<\infty,\ 0<x_2<x_1\tan\varphi_0\}.$$

We consider the problem stated above in the strong L^p setting for arbitrary angles $\varphi_0 \in (0,\pi)$. It turns out that the perfect slip Stokes operator associated to (1) admits a bounded $\mathcal{H}^\infty$-calculus with angle less than $\pi/2$ in case that a certain spectral condition involving φ_0, γ and p is satsfied. Consequently, system (1) admits maximal regularity in solenoidal Kondrat'ev L^p-spaces of type

$$L^p_{\sigma,\gamma}(G) = \left\{ u \in L^p(G, |(x_1,x_2)|^\gamma d(x_1,x_2,y));\ \int_G u\cdot\nabla\varphi = 0 \ \left(\nabla\varphi \in L^{p'}_{-\gamma p'/p}(G)\right)\right\}.$$

Our main result reads as

Theorem 1.1 *Assume that $1<p<\infty$, $\gamma\in\mathbb{R}$, and $\varphi_0\in(0,\pi)$ satisfy*

$$\min\left\{1, \left(\frac{\pi}{\varphi_0}-1\right)^2\right\} > \left(2-\frac{2+\gamma}{p}\right)^2. \tag{2}$$

Then the perfect slip Stokes operator $\mathcal{A}_S u := -\Delta u$, with $u \in D(\mathcal{A}_S)$ and

$$D(\mathcal{A}_S) := \left\{ u \in L^p_{\sigma,\gamma}(G);\ \nu\times\operatorname{curl} u = 0,\ \nu\cdot u = 0 \text{ on } \partial G, u/|(x_1,x_2)|^2,\ \partial^\alpha u \in L^p_\gamma(G)\ (\alpha\in\mathbb{N}_0^3,\ |\alpha|\le 2)\right\}$$

associated to system (1) admits a bounded $\mathcal{H}^\infty$-calculus on $L^p_{\sigma,\gamma}(G)$ with $\mathcal{H}^\infty$-angle $\phi^\infty_{\mathcal{A}_S} < \pi/2$.

Consequently we obtain maximal regularity of (1). It reads as follows:

Corollary 1.2 *Suppose the assumptions of Theorem 1.1 hold and let $J=(0,T)$ with $T\in(0,\infty)$. Then for each $f \in L^p(J, L^p_{\sigma,\gamma}(G))$ and $u_0 \in (L^p_{\sigma,\gamma}(G), D(\mathcal{A}_S))_{1-1/p,p}$ there exists a unique solution $u \in L^p(J, L^p_{\sigma,\gamma}(G))$ of problem (1) possessing the regularity*

$$u/|(x_1,x_2)|^2, \partial_t u, \partial^\alpha u \in L^p(J, L^p_\gamma(G)) \qquad (|\alpha|\le 2).$$

In particular, the map $[u\mapsto f]$ defines an isomorphism between the corresponding spaces.

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2 Sketch of the Proof

We proceed in several steps.

1. We consider the parabolic resolvent problem

$$(\lambda - \Delta)u = f \;\text{ in } G, \quad \nu \times \operatorname{curl} u = 0 \;\text{ on } \partial G, \quad u \cdot \nu = 0 \;\text{ on } \partial G \tag{3}$$

in $L^p_\gamma(G) := L^p(G, |x|^\gamma d(x_1, x_2, y))$ for suitable $\gamma \in \mathbb{R}$. By introducing polar coordinates and Euler transformation we transform (3) isomorphically onto a layer. Moreover, by a suitable scaling we transform in a way that we may work in unweighted L^p-spaces. For $\Omega := \mathbb{R} \times (0, \varphi_0) \times \mathbb{R}$ the resulting system is given through

$$e^{2x}\lambda v_x - e^{2x}\partial_y^2 v_x + P(\partial_x)v_x - \partial_\varphi^2 v_x + v_x + 2\partial_\varphi v_\varphi = g_x \quad \text{in } \Omega, \tag{4}$$

$$e^{2x}\lambda v_\varphi - e^{2x}\partial_y^2 v_\varphi + P(\partial_x)v_\varphi - \partial_\varphi^2 v_\varphi + v_\varphi - 2\partial_\varphi v_x = g_\varphi \quad \text{in } \Omega, \tag{5}$$

$$e^{2x}\lambda v_y - e^{2x}\partial_y^2 v_y + P(\partial_x)v_y - \partial_\varphi^2 v_y = g_y \quad \text{in } \Omega, \tag{6}$$

$$\partial_\varphi v_x = 0, \;\; v_\varphi = 0, \;\; \partial_\varphi v_y = 0 \quad \text{on } \partial\Omega, \tag{7}$$

where $P(\cdot)$ is a second order polynomial, i.e.

$$P(\partial_x) = -\left(\partial_x^2 + 2\left(2 - \frac{\gamma+2}{p}\right)\partial_x + \left(2 - \frac{\gamma+2}{p}\right)^2\right).$$

2. We treat (4)-(7) by operator sum methods similar to [2]. To this end, we build up (4)-(7) step by step by operators given in unweighted L^p-spaces. Note that here a special challenge arises when dealing with non-commuting operators as e^{2x} and $P(\partial_x)$, where the full strength of [2, Theorem 3.1] is needed. To this end, the Labbas-Terreni commutator condition has to be verified, which indeed holds true in our case. Therefore we may show that the transformed Laplacian given in system (4)-(7) plus a positive shift admits a bounded $\mathcal{H}^\infty$-calculus with angle less than $\pi/2$. In order to prove that the perfect slip Laplacian $\mathcal{A}$ admits a bounded $\mathcal{H}^\infty$-calculus with $\phi^\infty_{\mathcal{A}} < \frac{\pi}{2}$ it remains to get rid of the shift. Essential for this purpose is the spectrum of the compounded ∂_φ operator L acting on $(0, \varphi_0)$. It consists of a pure positive point spectrum with smallest eigenvalue

$$\lambda_1 = \min\left\{1, \left(\frac{\pi}{\varphi_0} - 1\right)^2\right\}.$$

An application of well-known results to the sum of operators admitting $\mathcal{BIP}$ yields the invertibility of the complete resulting operator sum in (4)-(7) if $\sigma(-L)$ and the spectrum of the remaining operator sum are disjoint. This, in turn, is the case if the spectral condition (2) is satisfied. The invertibility allows us to remove the shift and we obtain the $\mathcal{H}^\infty$-calculus to be valid for the transformed Laplacian. Transforming back to the wedge and to weighted L^p-spaces the assertion follows for the perfect slip Laplacian.

3. The $\mathcal{H}^\infty$-calculus result carries over to the Stokes operator: perfect slip boundary conditions allow for a convenient treatment of $\mathcal{A}_S$ as the part of the Laplacian $\mathcal{A}$ in $L^p_{\sigma,\gamma}(G)$, i.e. the solenoidal subspace of $L^p_\gamma(G)$. Thus we may employ

$$(\lambda - \mathcal{A}_S)^{-1} = (\lambda - \mathcal{A})^{-1}|_{L^p_{\sigma,\gamma}(G)}$$

for all $\lambda \in \rho(\mathcal{A}) = \rho(\mathcal{A}_S)$, which readily yields that the results obtained for $\mathcal{A}$ carry over to $\mathcal{A}_S$. This property of perfect slip boundary conditions is already utilized in [1]. Consequently, Theorem 1.1 is proved. □

Remark 2.1 Note that in particular we obtain a well-posedness result in the usual L^p-setting without weight, i.e., we can cover the case $\gamma = 0$. Theorem 1.1 implies this for any $\varphi_0 \in (0, \pi)$ and $p = p(\varphi_0) \in (1, \infty)$ close to 1. Then duality and interpolation yield a bounded $\mathcal{H}^\infty$-calculus on $L^p_\sigma(G)$ for $1 < p < \infty$.

References

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