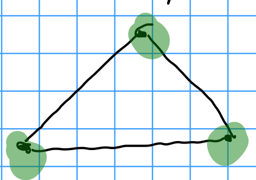
Group homology

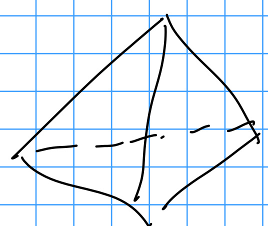
§ 0 introduction

what is it?

(Co)homology

top spaces: spaces are built from simple pieces (like cells/discs for CW-cplxs or simplices for simplicial sets)





homology measures how nontrivial topology arises from gluing these simple pieces

derived functors

e.g. Tor measures failure of $- \otimes_R N$ to be exact

R ring, N module

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

$M_1 \otimes N \to M_2 \otimes N$ may fail to be inj.

analog: modules are built from simpler pieces (from free modules)

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

F_i free modules

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more specifically: for gp homology

to a group G (discrete) and a G-representation M

we associate gp homology $H_*(G, M)$

gp cohomology $H^*(G, M)$

describes how G-representations are built from somehow "free" representations where does it appear / what is it used for?

- study gps: e.g. classification of (central) extensions of gps
- study gp actions on topological spaces: (e.g. existence of fixed pt free actions, classification of smooth actions on mfds)
- representation theory (e.g. of finite gps) (support varieties, gp cohomology "controls" derived cat of G-rep[?] extensions)
- number theory: Galois cohomology (cohomology of Galois gp acting on stuff, Hilbert 90, Brauer gp)
- algebraic K-theory: related to homology of $GL_\infty(R)$ R commutative ring
- Hilbert's third problem: (scissors congruence classification of polytopes in classical geometries E^n, S^n, H^n) relates to group homology of isometry gps made discrete

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plan: §1 definition

§2 easy computations

§3 formal properties

§4 an example $SL_2(\mathbb{Z})$

§1 Definition topological & algebraic view

1.1 top. view G gp

let EG be contractible space w/ proper free G-action $G \times EG \to EG$

quotient $BG = EG/G$ classifying space for G

def: gp homology ⓐ $H_*(G, \mathbb{Z}) := H_*(BG, \mathbb{Z})$

rmk: alternative for discrete gps:

$$\pi_i(BG) \cong \begin{cases} G & i=1 \\ 0 & \text{otherwise} \end{cases}$$

$\rightsquigarrow$ up to htpy equiv independent of choice of EG & G-action

expl:

- $T^n = B\mathbb{Z}^n$ $\qquad H^*(\mathbb{Z}^n, \mathbb{Z}) = \Lambda^*_{\mathbb{Z}} \mathbb{Z}^n$

  $T^n = \mathbb{R}^n / \mathbb{Z}^n$

- $\mathbb{RP}^\infty = BC_2$ $\qquad H^*(C_2, \mathbb{Z}) = \mathbb{Z}[x]/(2x)$, $\deg x = 2$

  $\mathbb{RP}^\infty = S^\infty / \text{antipodal}$

- $Gr_n(\mathbb{R}^\infty) = BGL_n(\mathbb{R})$ (topological gp)

ⓐ $H^*(G, \mathbb{Z}) := H^*(BG, \mathbb{Z})$

general construction:

as simplicial set:

G gp: simpl. set EG

$$EG_n = G^{\times(n+1)}$$

w/ facemaps $d_i : (g_0, \dots, g_n) \longmapsto (g_0, \dots, \hat{g}_i, \dots, g_n)$

contractible, free G-action

$$g \cdot (g_0, \dots, g_n) = (g g_0, \dots, g g_n)$$

$BG = EG/G$

more explicitly $BG_n = G^{\times n}$

rewriting elements: bar notation

$$EG_n/G \xrightarrow{\simeq} BG_n$$

$$G(1, g_1, g_1 g_2, \dots, g_1 \cdots g_n) \longmapsto [g_1 | \dots | g_n]$$

face maps:

$$d_i : [g_1 | \dots | g_n] \longmapsto \begin{cases} [g_2 | \dots | g_n] & i = 0 \\ [g_1 | \dots | g_i g_{i+1} | \dots g_n] & 0 < i < n \\ [g_1 | \dots | g_{n-1}] & i = n \end{cases}$$

then get BG as geometric realization of this simplicial set.

(1.2) algebraic view

- consider gp ring $\mathbb{Z}[G]$
  as ab gp $\mathbb{Z}[G] = \left\{ \sum n_i [g_i] \;\middle|\; \begin{array}{l} n_i \in \mathbb{Z} \\ g_i \in G \end{array} \right\}$ finite formal sums

  ring structure
$$[g]\cdot[h] = [gh]$$

- G-module : modules over $\mathbb{Z}[G]$

rmk: can also do this for other coeff rings, e.g. $\mathbb{C}[G]$ grp. G-representations

def: G gp, M G-mod.
$$H_n(G, M) := \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$
$$H^n(G, M) := \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

can compute these via free resolutions of $\mathbb{Z}$ as $\mathbb{Z}[G]$-mod.

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

F_n = free $\mathbb{Z}[G]$ module gen^d by $[g_1|\dots|g_n]$

ε: $F_0 \cong \mathbb{Z}[G]$ gen^d by $[\,]$
$$\mathbb{Z}[G] \to \mathbb{Z} : [\,] \longmapsto 1$$

$$d_n : [g_1|\dots|g_n] \longmapsto g_1[g_2|\dots|g_n] + \sum_{j=1}^{n-1} (-1)^j [g_1|\dots|g_j g_{j+1}|\dots|g_n] + [g_2|\dots|g_n]$$

$$H_*(G, M) = H_*\left(F_* \otimes_{\mathbb{Z}[G]} M\right)$$

$$H^*(G, M) = H^*\left(\mathrm{Hom}_{\mathbb{Z}[G]}(F_*, M)\right)$$

<u>rmk:</u> top construction of BG provides a free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$

$\rightsquigarrow$ topological & algebraic views / definition yield the same result.

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next talk: - easy computations $H_1(G, \mathbb{Z}) = G^{ab}$

- formal properties
- example computation $SL_2\mathbb{Z}$

$BG = EG/G$ in stacks notation

$\underset{\sim}{*//G}$ classifying stack of G

§2 easy computations

(2.1) $H_0(G,M) = \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$ (from def as Tor)

G gp, M $\mathbb{Z}[G]$-mod $= M_G := M / \langle gm - m \mid g \in G, m \in M \rangle$

coinvariants

biggest quotient of M w/ triv. G-action.

$H^0(G,M) = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ (from def as Ext)

$= M^G = \{ m \in M \mid gm = m \quad \forall g \in G \}$

invariants biggest submodule of M w/ triv. G-action.

rmk: sometimes even if we only care about invariants may be useful to translate question into higher homology gps (via long exact seq, homological alg)

expl: $H_0(\mathrm{Isom}(\mathbb{H}^3), \text{polytopes})$

$\longleftrightarrow \quad H_3(SL_2\mathbb{C}, \mathbb{Q})$

exhibits a link to K theory & regulators

2.2 prop: $H_1(G,\mathbb{Z}) = G/[G,G] = G^{ab}$

abelianization

pf:
$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z}$$
$$[\,] \longmapsto 1$$

gen as $\mathbb{Z}[G]$ mod

$$[g] \longmapsto g[\,] - [\,]$$
$$[g_1|g_2] \longmapsto g_1[g_2] - [g_1g_2] + [g_1]$$

$$F_2 \otimes \mathbb{Z} \xrightarrow{d_2 \otimes \mathbb{Z}} F_1 \otimes \mathbb{Z} \xrightarrow{d_1 \otimes \mathbb{Z}} F_0 \otimes \mathbb{Z}$$
$$[g] \longmapsto g[\,] - [\,]$$

$g[\,] = [\,]$

$[\,] - [\,] = 0$

$d_1 = 0$

$$[g_1|g_2] \longmapsto g_1[g_2] - [g_1g_2] + [g_1]$$
$$= [g_1] + [g_2] - [g_1g_2]$$

$$\rightsquigarrow \quad H_1 = \frac{\ker d_1 \otimes \mathbb{Z}}{\operatorname{Im} d_2 \otimes \mathbb{Z}} = \frac{F_1 \otimes \mathbb{Z}}{\operatorname{Im} d_2 \otimes \mathbb{Z}} = \frac{\mathbb{Z}[G]}{[g_1] + [g_2] - [g_1g_2]}$$

$[g_1g_2] = [g_2g_1]$

$\rightsquigarrow$ isom to $G/[G,G] = G^{ab}$

(23) G gp X CW cplx, w/ cellular free G-action homeo to S^{2n-1}

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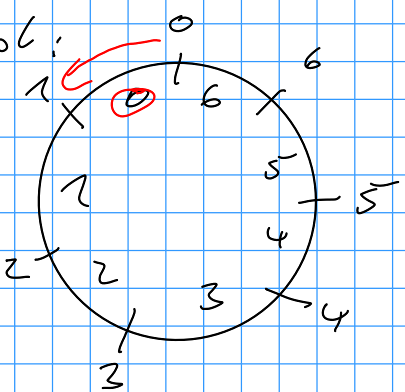
expl.:



C_7-action by rotation

generator σ

$\partial[0] = [1] - [0] = \sigma - 1$

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cellular cplx ⇝ exact seq of ab gps

$$0 \to \underset{H_{2n-1}(X)}{\mathbb{Z}} \xrightarrow{\eta} C_{2n-1}(X) \xrightarrow{d} \cdots \to C_1(X) \to C_0(X) \xrightarrow{\varepsilon} \underset{H_0(X)}{\mathbb{Z}} \to 0$$

G-action

patch together ∞ many copies along $C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\eta} C_{2n-1}(X)$

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expl.: C_n $C_1(S^1)$ $C_0(S^1)$

$$0 \to \mathbb{Z} \to \mathbb{Z}[C_n] \xrightarrow{\sigma - 1} \mathbb{Z}[C_n] \to \mathbb{Z} \to 0$$

$$\eta \circ \varepsilon = N := 1 + [\sigma] + \cdots + [\sigma^{n-1}]$$

⇝ periodic resolution

$$\cdots \to \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{\sigma - 1} \mathbb{Z}[C_n] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

⇝ explicit computation of hlgy of C_n

prop: $H_i(C_n, \mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & i \text{ odd } \geq 0 \\ \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$

prop: G acts freely & cellularly on S^{2n-1}
then G has periodic cohomology

$$H^{i+2n}(G, M) \cong H^i(G, M) \qquad i > 0$$

§3 formal properties

(a) functoriality $H_n(G,-)$, $H^n(G,-) : \mathbb{Z}[G]\mathrm{Mod} \to Ab$

(b) long exact sequences

(c) $g : H \to G$ gp hom

exact functor $\mathrm{Res}^H_G : \mathbb{Z}[G]\text{-}\mathrm{Mod} \to \mathbb{Z}[H]\mathrm{Mod}$

& induced maps $H_*(H, \mathrm{Res}^H_G(M)) \to H_*(G, M)$

(d) induction functors

$$\mathrm{Ind}^G_H \dashv \mathrm{Res}^H_G \dashv \mathrm{Coind}^G_H$$

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} - \qquad\qquad \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], -)$$

<u>Shapiro lemma:</u> $H \subseteq G$ subgp, M H-mod

$$H_*(H, M) \cong H_*(G, \mathrm{Ind}^G_H(M))$$

(e) transfer maps $H \subseteq G$ subgp of finite index

$$tr : H_*(G, M) \longrightarrow H_*(H, \mathrm{Res}^H_G(M))$$

(f) product structures

(4) <u>example $SL_2\mathbb{Z}$</u>

$SL_2\mathbb{Z}$ = 2×2 det 1 integer matrices acts on hyperbolic plane by Möbius transformations

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

$$SL_2\mathbb{Z} \times \mathbb{H} \longrightarrow \mathbb{H}$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) \longmapsto \frac{az+b}{cz+d}$$

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from $\mathbb{H}$ or the tree T
get $SL_2\mathbb{Z}$-action on CW-complex

(contractible, from metric of negative curvature)

but not classifying space! (S, ST have fixed pts)

$\rightsquigarrow$ get complex $C_*(X)$ $\qquad X = \mathbb{H}$ or T

of $SL_2\mathbb{Z}$-mod s.th. $C_*(X) \simeq \mathbb{Z}$
quasi iso.

$\rightsquigarrow$ $\underline{H_*(G, C_*(X))} \cong H_*(G, \mathbb{Z})$

homology of total cplx of $F_* \otimes C_*(X)$

$$C_*(\tilde{T}) : \quad C_1(\tilde{T}) \xrightarrow{\partial} C_0(\tilde{T})$$

edges $\qquad\qquad$ vertices

$SL_2\mathbb{Z} \cdot$ edge connecting i and ω $\qquad$ $SL_2\mathbb{Z} \cdot i$, $\quad SL_2\mathbb{Z} \cdot \omega$

$$\begin{array}{ccc} H_*(SL_2\mathbb{Z}, C_1(\tilde{T})) & & H_*(SL_2\mathbb{Z}, C_0(\tilde{T})) \\ \text{Shapiro } \wr\| \quad \cong \operatorname{Ind}_{Stab}^{SL_2\mathbb{Z}}(\mathbb{Z}) & & \wr\| \text{ Shapiro} \\ H_*(C_2, \mathbb{Z}) & \xrightarrow{\partial} & H_*(C_4, \mathbb{Z}) \oplus H_*(C_6, \mathbb{Z}) \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & & \langle S \rangle \qquad \langle ST \rangle \end{array}$$

Prop: long exact seq.

$$H_{i+1}(SL_2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\partial} H_i(C_2, \mathbb{Z}) \to H_i(C_4, \mathbb{Z}) \oplus H_i(C_6, \mathbb{Z})$$
$$\to H_i(SL_2\mathbb{Z}, \mathbb{Z}) \to H_{i-1}(C_2, \mathbb{Z})$$

explicit formulas,
easier $\quad H_*(PSL_2\mathbb{Z}, \mathbb{Z}) \cong H_*(\underset{\langle S \rangle}{C_2}, \mathbb{Z}) \oplus H_*(\underset{\langle ST \rangle}{C_3}, \mathbb{Z})$

Scholium:
- use geometry of nice actions to compute gp homology
- technical tool: spectral seq.
- for gps acting on trees get general formulas for homology (applies to amalgams, HNN-extensions, ...)

- for arithmetic gps (such as $SL_n \mathbb{Z}$) have nice actions on assoc. symmetric spaces

$$SL_n \mathbb{Z} \curvearrowright \underbrace{SL_n \mathbb{R} / SO(n)}_{\text{contractible}}$$

action has finite stabilizers & can find cell structures.

still difficult to work out exactly

- $SL_3 \mathbb{Z}$ computed by Soulé 1979
- $SL_4 \mathbb{Z}$ computed up to 2-torsion
- rational computation up to $SL_8 \mathbb{Z}$ (computer)