Welcome to the

Ringvorlesung

Motivic integration (gentle intro)

§1 What is Mot. Int?

- Can integrate in $\mathbb{Q}_p$, $\mathbb{F}_p((t))$ ("p-adic int.")
- Mot. Int is abstract analogue of this
    - in other valued fields K like $K = k((t))$
    - Mot. Integrals take values in some modified Grothendieck ring
    - works uniformly in K
- Applications:
    - can use properties of integration (like change of var fmla) to obtain geometric information
    - uniform p-adic integration
    - transfer results between $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ for $p \gg 0$
- Ex 3 approaches to mot. int.
    - Kontsevich
    - Cluckers-Loeser
    - Hrushovski-Kazhdan

  In the end, not so different

§2 p-adic integration

- Let K be a non-arch. loc. fld, i.e. a finite ext of $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$

$$\begin{array}{ccc} \mathbb{Z}\cup\{\infty\} \xleftarrow{v} & K & \\ & \cup & \\ & O_K & \leftarrow \text{val. ring} \\ & \downarrow{\scriptstyle \text{res}} & \\ & k & \leftarrow \text{res. fld} \end{array}$$

Example: $K = \mathbb{F}_q((t)) = \{ \sum_{i\geq N} a_i t^i \mid N\in\mathbb{Z},\ a_i \in \mathbb{F}_q \}$

$\cup$

$O_K = \mathbb{F}_q[[t]] \xrightarrow{\text{res}} \mathbb{F}_q$

$p := \operatorname{char} \mathbb{F}_q$ $(q = p^s)$

$\underset{a}{\overset{!!}{}}$ If $a_N \neq 0$: $v(a) = N$

$ac(a) = a_N \in \mathbb{F}_q^\times$

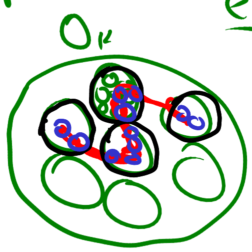
$(ac(0) := 0)$

- $(K,+)$ is a loc. cpct gp $\Rightarrow$ has Haar measure μ normalized s.t.

$$\mu(O_K) = 1$$

- Obtain the product measure on K^n

$p=7$



Ex: $\bullet\ \mu(O_K) = 1$

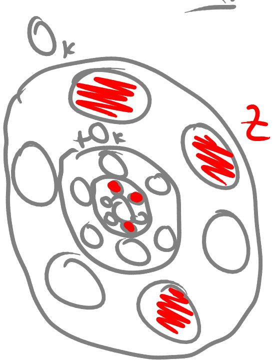
- $O_K = \dot{\bigcup\limits_{b\in\mathbb{F}_q}} \underbrace{(b + tO_K)}_{\{b + \sum_{i\geq 1} a_i t^i \mid a_i \in \mathbb{F}_q\}}$ all these have the same measure as tO_K

$\Rightarrow \overset{1}{\overset{||}{\mu(O_K)}} = q\cdot \mu(tO_K) \Rightarrow \mu(tO_K) = q^{-1}$

- Similarly: $\mu((t^r\cdot O_K)^n) = q^{-rn}$ $\forall r\in\mathbb{Z}$
- Can determine $\mu(Z)$ for $Z \subset K$ measurable by approximating by balls:
  - Let's suppose $Z \subset O_K^n$
  - Let $\pi_r : O_K \longrightarrow O_K / t^r O_K$ induces $\pi_r : O_K^n \to (O_K/t^r O_K)^n$

$$\sum_{i\geq 0} a_i t^i \longmapsto (a_0, \dots, a_{r-1})$$

  - $\#\pi_r(Z) = \#$ translates of $(t^r O_K)^n$ in O_K meeting Z
  - $\mu(Z) = \lim\limits_{r\to\infty} \#\pi_r(Z)\cdot q^{-rn}$



Ex: $Z = \{x^2 \mid x \in O_K\}$ Suppose $p \neq 2$

$z \in Z \Leftrightarrow 2 \mid v(z) \wedge ac(z) \in \bar{Z} := \{x^2 \mid x \in \mathbb{F}_q^\times\}$

$\vee\ z = 0$

use Hensel's Lemma

$$Z = \dot{\bigcup_{b\in\bar{Z}}} (b + tO_K) \cup \dot{\bigcup_{b\in\bar{Z}}} t^2(b + tO_K) \cup \dot{\bigcup_{\dots}} t^4 \dots$$

$$\mu(Z) = \#\bar{Z}\cdot q^{-1} + \#\bar{Z}\cdot q^{-3} + \#\bar{Z}\cdot q^{-5} + \dots$$
$$= \#\bar{Z}\cdot(q^{-1}+q^{-3}+q^{-5}+\dots) = \#\bar{Z}\cdot q^{-1}\cdot\frac{1}{1-q^{-2}}$$
$$= \#\bar{Z}\cdot(\#\mathbb{F}_q)^{-1}\cdot\frac{1}{1-(\#\mathbb{F}_q)^{-2}}$$

(*) • To integrate a fctn $f: \mathcal{O}_K^n \to \mathbb{Z}$:
$$\int_{\mathcal{O}_K^n} f = \sum_{m\in\mathbb{Z}} \mu(\{x\in\mathcal{O}_K^n \mid f(x)=m\})\cdot m$$

• Observation: $Z\subset K^n$ ⇝ obtained expression of $\mu(Z)$ in terms of cardinalities of subsets of k^m

§3 Kontsevich mot. int.: ideas/plan

$\mathcal{O}_K = k[[t]]$

• Goal: Measure subsets of $\mathcal{O}_K^n$ for $K=k((t))$ with k infinite

• $\mu(\mathcal{O}_K)=1$, $\mu(t\mathcal{O}_K) = \frac{1}{\#k} = 0$?? Not interesting.
Instead: Let μ take values in some ring $\hat{M}$ containg a formal symbol $\mathbb{L}$ for "$\#k$" ... and also $\mathbb{L}^{-1}$ $(\mathbb{L}=[\mathbb{A}^1])$

• Then "$\mu(t^r\mathcal{O}_K^n) = \mathbb{L}^{-rn}$" makes sense.

• We will do this for k alg. closed, char $k=0$

$$K_0(\mathrm{Var}_k) \xrightarrow{\text{not inj}} \underbrace{K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]}_{M}$$

§4 The codomain of the measure

(Copy $\mathbb{F}_p[[t]]$-measure: $X\subset\mathbb{F}_p^n$)

• Fix k alg. closed field of char. 0. $K:=k((t))$ $\mathcal{O}_K:=k[[t]]$

• Grothendieck ring of var. over k:

$$K_0(\mathrm{Var}_k) := \langle [X] \mid X \text{ affine variety over } k\rangle \Big/ \begin{array}{l}[X]=[Y] \text{ if } X\cong Y\\ [Y]=[X]+[Y\setminus X]\quad X\subset Y \text{ closed subvariety}\\ Y\setminus X \text{ is affine}\end{array}$$

free ab. gp ~~generated by the [X]~~

subset of k^n defined by (finitely many) polynomial eqns

Turn this into a ring by setting $[X]\cdot[Y] := [X\times Y]$

• Given X variety over k and $Z\subset X$ constructible, we have a well-defined $[Z]\in K_0(\mathrm{Var}_k)$

constructible: finite boolean combination of subvarieties

- Set $\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathrm{Var}_k)$
- Set $\mathcal{M}_k := K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$

  (Ex X, Y non-iso s.t. $\mathbb{A}^1 \times X \cong \mathbb{A}^1 \times Y$
  $\Downarrow$
  $\mathbb{L}\cdot[X] = \mathbb{L}\cdot[Y]$ in $\mathcal{M}_k$
  $\Downarrow$
  $[X] = [Y]$ in $\mathcal{M}_k$ (but not in $K_0(\mathrm{Var}_k)$))
- $\mathcal{M}_{k,\le d} := \left\{ \frac{[X]-[Y]}{\mathbb{L}^r} \;\middle|\; \max\{\dim X, \dim Y\} - r \le d \right\}$ $(d \in \mathbb{Z})$
- $\hat{\mathcal{M}}_k :=$ completion of $\mathcal{M}_k$ with respect to this filtration

  ($(a_i)_{i\in\mathbb{N}}$ is a "cauchy-sequence" if $\forall d\, \exists N: \forall i,j > N: a_i - a_j \in \mathcal{M}_{k,\le d}$)

(Check that $\hat{\mathcal{M}}_k$ is a ring)

- Ex: $a := \sum_{i=0}^{\infty} \mathbb{L}^{-i} \in \hat{\mathcal{M}}_k$

  $a\cdot(1-\mathbb{L}^{-1}) = 1 \Rightarrow a = \frac{1}{1-\mathbb{L}^{-1}}$

To measure $Z \subset k[[t]]^n$ with "$\dim Z = d$", want a d-dim measure

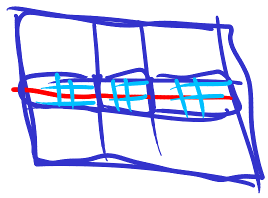
§5 Motivic measure (Kontsevich)

(Motivation: think of $k = \mathbb{F}_p$)

- Set $\pi_r : k[[t]] \longrightarrow k[[t]]/t^r \xrightarrow{\sim} k^r$

  $\sum_{i\ge 0} a_i t^i \longmapsto (a_0, \dots, a_{r-1})$

$(\mathbb{F}_3[[t]])^2$



- Given $Z \subset k[[t]]^n$
- Call Z measurable if
  - $\pi_r(Z)$ is constructible for every r
  - the limit exists
- $\mu_d(Z) = \lim_{r\to\infty} [\underbrace{\pi_r(Z)}_{\subset k^{r\cdot n}}]\cdot \mathbb{L}^{-r\cdot d}$

$\mu(t^s \cdot \mathbb{F}_p[[t]]) = p^{-s}$

- Ex: $\mu_1(\underbrace{t^s\cdot k[[t]]}_{=}) = \mathbb{L}^{-s}$

  $Z = \{ \sum_{i\ge 0} a_i t^i \mid a_0, \dots, a_{s-1} = 0 \}$

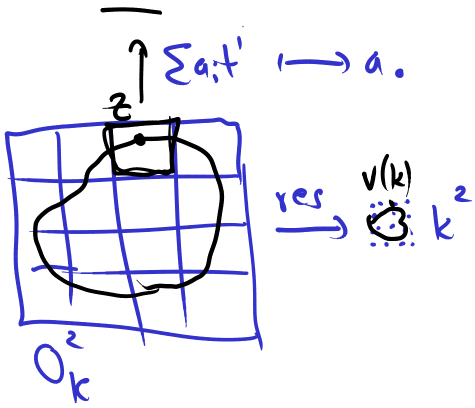
  $$\pi_r(Z) = \begin{cases} \{(0,\dots,0)\} & r \le s \quad \Rightarrow [\pi_r(Z)] = 1 \\ \underbrace{(0,\dots,0)}_{r} \times \underbrace{k \times \dots \times k}_{r-s} & r \ge s \quad \Rightarrow [\pi_r(Z)] = \mathbb{L}^{r-s} \end{cases}$$

  $$\lim_{r\to\infty} \underbrace{[\pi_r(Z)]\cdot\mathbb{L}^{-r}}_{\overset{r\ge s}{=}\ \mathbb{L}^{r-s}\cdot\mathbb{L}^{-r}} = \mathbb{L}^{-s}$$

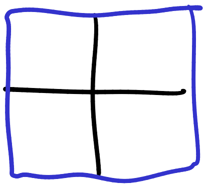
- let V be an affine variety over k $\rightsquigarrow$ $V(k) \subset k^n$, $Z := V(k[[t]])$
- Thm 1: Such Z are measurable.

- Thm 2: If V is smooth of dim. d, then

$$\mu_1(Z) = [V(k)] \cdot \mathbb{L}^{-d}$$

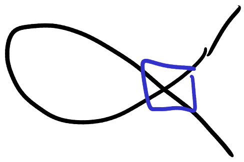


- Ex: V defined by $x \cdot y = 0$



$\mu_1(Z) = 2$

but $[V(k)] \cdot \mathbb{L}^{-1} = (2 \cdot \mathbb{L} - 1) \cdot \mathbb{L}^{-1} = 2 - \mathbb{L}^{-1}$



(*) also works motivically. Want to integrate fctns like: $\mathcal{O}_k^n \longrightarrow \mathbb{Z}$

$f \in k[x_1, \ldots, x_n]$ $\qquad \underline{x} \longmapsto v(f(\underline{x}))$

Better versions of mot. int use, instead of $\hat{\mathcal{M}}$, $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}, (1-\mathbb{L}^{-k})^{-1}]_k$