OBERSEMINAR ALGEBRA AND GEOMETRY
WS 2018/19
THE GROTHENDIECK GROUP OF VARIETIES AND STACKS

Let k be a ground field. The *Grothendieck group of varieties* $K_0(\mathrm{Var}_k)$ is the abelian group generated by isomorphism classes $\{X\}$ of algebraic schemes, modulo the scissor relations $\{X \smallsetminus Y\} = \{X\} - \{Y\}$ for closed subschemes $Y \subset X$. This mysterious group was first introduced by Grothendieck in a letter to Serre dated August 16, 1964 [Correspondance]. It lies at the heart for motivic arguments, in particular for motivic integration. In characteristic zero, Bittner gave a presentation in terms of blowing-ups of smooth schemes with smooth centers.

In March 2009, the late Ekedahl posted two preprints on arXiv, in which he introduced and applied the *Grothendieck group of stacks* $K_0(\mathrm{Stck}_k)$. Among other things, he proved that $K_0(\mathrm{Stck}_k)$ is the localization of the Grothendieck group of varieties $K_0(\mathrm{Var}_k)$, obtained by inverting the Lefschetz class $\mathbb{L}$ and the $\mathbb{L}^n - 1$ for all $n \geq 1$. Here $\mathbb{L} = \{\mathbb{A}^1\}$ denotes the class of the affine line, which is also called the *Lefschetz class*.

For each algebraic group G, in particular for each finite group G, the *classifying stack* BG yields a class $\{BG\}$ in the localized Grothendieck group of varieties. Here BG is the algebro-geometric version of the classifying space from topology, defined as the quotient stack for the trivial G-action on the point $\mathrm{Spec}(k)$. The standard notation for quotient stacks is $[Y/G]$, which is the reason why we should adopt Ekedahl's notation $\{X\}$ for classes in the Grothendieck group of varieties.

The classes $\{BG\} \in S^{-1}K_0(\mathrm{Var}_k)$ yield an algebro-geometric invariant for finite groups G, which is related to rationality problems and invariant theory. For algebraic groups, this invariant can be used to gain more insight into Serre's notion of special groups.

Time and Place: Friday, 12:30-14:00 in 25.22.03.73

Literature:

Bittner 2004: The universal Euler characteristic for varieties of characteristic zero. Compos. Math. 140, 1011–1032.

Ekedahl 2009a: The Grothendieck group of algebraic stacks. Preprint, arXiv:0903.3143.

Ekedahl 2009b: A geometric invariant of a finite group. Preprint, arXiv:0903.3148.

Larsen and Lunts 2003: Motivic measures and stable birational geometry. Mosc. Math. J. 3, 85–95, 259.

Martino 2016: The Ekedahl invariants for finite groups. J. Pure Appl. Algebra 220, 1294–1309.

Martino 2017: Introduction to the Ekedahl invariants. Math. Scand. 120, 211–224.

Poonen 2002: The Grothendieck ring of varieties is not a domain. Math. Res. Lett. 9, 493–497.

Talpo and Vistoli 2017: The motivic class of the classifying stack of the special orthogonal group. Bull. Lond. Math. Soc. 49, 818–823.

Additional sources:

Abramovich, Karu, Matsuki, and Włodarczyk 2002: Torification and factorization of birational maps. J. Amer. Math. Soc. 15, 531–572.

Artin 1971: Algebraic spaces. Yale University Press, New Haven, Conn.-London, 1971.

Artin 1973: Théorèmes de représentabilité pour les espaces algébriques. Les Presses de l'Université de Montréal, Montreal, Que.

Correspondance Grothendieck–Serre. Edited by P. Colmez and J.-P. Serre. Société Mathématique de France, Paris, 2001.

Bogomolov 1988: The Brauer group of quotient spaces of linear representations. Math. USSR-Izv. 30 (1988), 455–485.

Ekedahl 2009c: Approximating classifying spaces by smooth projective varieties. Preprint, arXiv:0905.1538.

Fantechi 2001: Stacks for everybody. In: C. Casacuberta, R. Mir-Roig, J. Verdera and S. Xambó-Descamps (eds.), pp. 349–359. European Congress of Mathematics. Vol. I. Birkhäuser Verlag, Basel, 2001.

Knutson 1971: Algebraic spaces. Springer, Berlin, 1971.

Laumon and L. Moret-Bailly 2000: Champs algebriques. Springer, Berlin.

Looijenga 2002: Motivic measures. Astérisque 276, 267–297.

Olsson 2016: Algebraic spaces and stacks. American Mathematical Society, Providence, RI.

Saltman 1984: Noether's problem over an algebraically closed field. Invent. Math. 77, 71–84.

Włodarczyk 2003: Toroidal varieties and the weak factorization theorem. Invent. Math. 154, 223–331.

Program:

October 12, Talk 1, N.N.: [Bittner 2004], Section 2 and 3.

Introduce the Grothendieck group of varieties $K_0(\mathrm{Var}_k)$ and the Lefschetz class $\mathbb{L} = \{\mathbb{A}^1\}$. Explain the Bittner presentation in detail [2004]. This relies on resolution of singularities, together with the Weak Factorization Theorem [Włodarczyk 2003; Abramovich et al. 2003]. Discuss the statement of the latter.

October 19, Talk 2, N.N.: [Bittner 2004], Section 4.

Briefly explain the category of Chow motives $\mathcal{M}_k$ following [Scholl 1994], and show that there is a homomorphism of rings

$$K_0(\mathrm{Var}_k) \longrightarrow K_0(\mathcal{M}_k),$$

as explained in [Bittner 2004], Section 4.

October 26, TBA.

November 2, Talk 3, N.N.: [Poonen 2002].

Show that the ring $K_0(\mathrm{Var}_k)$ must contains zero-divisors, according to Poonen [2002]. This relies on some facts about abelian varieties, which should discussed.

November 9, Talk 4, N.N.: [Larsen and Lunts 2003], Section 2.

Show that the residue class ring

$$K_0(\mathrm{Var}_k)/(\mathbb{L}) = \mathbb{Z}[\mathrm{SB}]$$

is the monoid ring on the monoid SB of stable birational equivalence classes of varieties, following Larsen and Lunts [2003], Section 2.

November 16, Talk 5, N.N.

Discuss the notion of algebraic spaces, a generalization of schemes where the Zariski topology is replaced by the étale topology. Explain the advantages of algebraic space via fundamental examples: Denormalizations of schemes, quotients by finite group actions, contractions of negative-definite curves on surfaces and higher-dimensional generalizations. Sources: for example [Olsson 2016; Artin 1971, 1973; Knutson 1971].

November 23, Talk 6, N.N.

Discuss the notion of algebraic stacks, a further generalization of schemes where the set of A-valued points $X(A)$ from the Yoneda functor is replaced by a fiber category $\mathscr{X}_A$. The objects of these categories are typically schemes or sheaves comprising a moduli problem, but in general are rather unrestricted. Stress examples: the stack $\mathscr{M}_g$ of curves of genus g, moduli stacks $\mathscr{B}un_{G,C}$ of principal bundles, and quotient stacks $[X/G]$. Also discuss inertia stacks $I_{\mathscr{X}}$. Sources: for example [Olsson 2016; Fantechi 2001; Laumon and Moret-Bailly 2000].

November 30, Talk 7, N.N.: [Ekedahl 2009a], Section 1.

Introduce the Grothendieck group of stacks $K_0(\mathrm{Stck}_k)$, following Ekedahl [2009a], Section 1, and show that

$$K_0(\mathrm{Stck}_k) = S^{-1}K_0(\mathrm{Var}_k),$$

where S is the multiplicative system in $K_0(\mathrm{Spc}_k) = K_0(\mathrm{Var}_k)$ generated by the Lefschetz class $\mathbb{L}$ and the differences $\mathbb{L}^n - 1$, $n \geq 2$. See also [Martino 2016, 2017].

December 7, TBA.

December 14, Talk 8, N.N.: [Ekedahl 2009b], Section 3 and 4.
Introduce the algebro-geometric invariant $\{BG\} \in K_0(\mathrm{Stck}_k)$, where G is a finite or algebraic group. Discuss the examples of finite groups G with $\{BG\} = 1$ given in [Ekedahl 2009b], Section 3 and 4. See also [Martino 2016, 2017].

December 21: no Oberseminar.

January 11, Talk 9, N.N.: [Ekedahl 2009b], Section 5.
Show that there are finite groups G with $\{BG\} \neq 1$, and discuss the relation to the group cohomology $H^2(G, \mathbb{Q}/\mathbb{Z})$, after Ekedahl [2009b], Section 5. Explain the necessary results from Saltman [1984] and Bogomolov [1987] about invariant theory. See also [Martino 2016, 2017].

January 18, Talk 10, N.N.: [Talpo and Vistoli 2017].
Discuss Serre's notion of special group and show that they satisfy the relation $\{BG\} = \{G\}^{-1}$ in the localized Grothendieck group of varieties. Show that the orthogonal groups $G = \mathrm{SO}$ also satisfy this relation, following Talpo and Vistoli [2017].

January 25, February 3: TBA.

February 1: Program discussion for next semester.