RESEARCH SEMINAR ON GROUP THEORY – SOSE 2017

ALGEBRAIC CHARACTERISATIONS OF p-ADIC ANALYTIC GROUPS

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The first objective of the seminar is to establish the following **characterization of p-adic analytic groups**; see [2, Theorem 8.1].

Theorem. *A topological group G has the structure of a p-adic analytic group if and only if G has an open subgroup which is a powerful finitely generated pro-p group.*

Moreover, one shows that whenever G admits such an analytic structure then it is already unique. Indeed, there are several independent group-theoretic characterizations of p-adic analytic pro-p groups (at least 15 of them, according to Interlude A in [3]). Each one of them can be regarded as an answer to the p-adic version of **Hilbert's fifth problem**. We plan to include a talk with an overview of the solution to the problem in the setting of real Lie groups, for comparison to the developments in the p-adic case.

To a large degree the theory of p-adic analytic groups was developed by M. Lazard in the 1960s. The group-theoretic aspects of his work were taken up and reinterpreted in the 1980s by A. Lubotzky and A. Mann. Starting with their concept of powerful pro-p groups one can reconstruct most of the group-theoretic consequences of Lazard's theory with a minimum of analytic methods; this is the approach taken in [2].

The second goal of the seminar is to understand the **Lie correspondence** between (uniformly powerful) p-adic analytic groups and (powerful) $\mathbb{Z}_p$-Lie algebras; see [2, Theorem 9.10]. This correspondence lies at the heart of typical applications of the theory of p-adic analytic groups. In spirit, it is somewhat similar to the Lazard Correspondence that applies to finite p-groups G of nilpotency class less than p and (finite) nilpotent Lie rings L of nilpotency class less than p; see [7] or [5, Chapters 9,10]. In this context, subgroups of a finite p-group G correspond bijectively to Lie subrings of L and similarly, normal subgroups $N \trianglelefteq G$ correspond bijectively to ideals of L. Also, the nilpotency class and the order of the objects involved remain invariant under this correspondence.

In the p-adic setting, a subgroup of a uniformly powerful pro-p group is typically not (uniformly) powerful again. In [3] it is explained how one can nevertheless set up a Lie correspondence between torsion-free p-adic analytic pro-p groups of dimension less than p and residually-nilpotent $\mathbb{Z}_p$-Lie lattices of dimension less than p.

To streamline things, we will assume throughout that p is an odd prime. (The theory for $p = 2$ has little extra twists.) We shall primarily follow the book *Analytic pro-p groups* by J. D. Dixon, M. P. F. du Sautoy, A. Mann and D. Segal. The first chapter of [6] might be a helpful resource too, providing a short overview of the subject and a wider perspective. Later talks will also be based on research papers listed below.

We currently plan to have 9 talks, but this can be adjusted during the course of the seminar. Depending on the participants interests, it would be possible to add further individual talks on topics such as: Hausdorff dimension on p-adic analytic groups ([1]), cohomology of p-adic analytic groups (Chapter 11 of [9]) or Lubotzky's Linearity Criterion (Interlude B of [2]). We will discuss such options at a later stage. Here is a short outline of the envisioned talks.

Talk 1: Introduction and basic definitions. Follow Chapters 1 and 2 in [2] but you may skip Section 1.3. It is recommended to skip the separate treatment of finite p-groups in Section 2.1 and to define and treat powerful pro-p groups directly without any detours (see Section 3.1).

Talk 2: Powerful pro-p groups and Lie rings. Continue with Chapters 3 and 4 in [2]. Sections 4.4 and 4.6 can probably be skipped, except for Theorem 4.20 (it is enough to state the result). As an example, one can mention the group $\mathrm{GL}_d(\mathbb{Z}_p)$ in parallel with developing the theory, inspired by Chapter 5.

Talk 3: The Campbell-Hausdorff formula. Follow Chapter 6 in [2], skipping details where necessary, in particular (most of) Sections 6.2 and 6.6 can probably be skipped.

Talk 4: The Lie algebra. Follow Chapter 7 in [2]. Go quickly through Section 7.1 and treat in more detail Sections 7.2 and 7.3.

Talk 5: p-Adic analytic groups and uniform pro-p groups. Follow Chapter 8 in [2]. Highlight the main results: Theorem 8.1, Theorem 8.18 and Theorem 8.31.

Talk 6: Hilbert's fifth problem. This is an essentially independent talk from the others, and is intended as an overview of the real fifth problem in Hilbert's list (as opposed to the p-adic analogue, which is the characterization of p-adic analytic groups). The aim is therefore to explain what the problem is and the main steps in its solution, comparing it to the p-adic case. Explain the problem and general philosophy, following Section 1.1 of [8], and present how the main ingredients of the solution are put together, as in Section 1.6 (using the necessary theorems in between these Sections).

Talk 7: Lie Theory for uniform pro-p groups and powerful $\mathbb{Z}_p$-Lie lattices. Follow Sections 9.2, 9.3 and 9.4 in [2]. Put some emphasis on the following major results: Theorem 9.4 and Theorem 9.10.

Talk 8: Analytic pro-p groups of dimension smaller than p. Follow the article [3]. Focus on the Lie correspondence between torsion-free p-adic analytic pro-p groups of dimension less than p and residually-nilpotent $\mathbb{Z}_p$-lattices of dimension less than p.

Talk 9: Analytic groups over general pro-p rings. Select from Chapter 13 in [2] and [4]. The aim of this talk is to introduce analytic pro-p groups over more general pro-p rings, to recover some of the familiar results from the p-adic setting and to highlight open problems.

Talks 10+: Extra topics See above suggestions.

References

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[8] T. Tao, *Hilbert's fifth problem and related topics*, Graduate Studies in Mathematics Volume 153. American Mathematical Society (2014). `https://terrytao.files.wordpress.com/2012/03/hilbert-book.pdf`

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