Seminar/Reading group on Invariant Random Subgroups Wintersemester 2016/17

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Version 1: September 12, 2016

What is this about? Invariant random subgroups (IRSs) are a new area of research at the crossroads of group theory, ergodic theory, and random graph theory. An IRS of a group is a random variable on the space of all subgroups of the group, which is invariant under conjugation. As such it can be viewed as a probabilistic generalization of a normal subgroup and of a lattice in a locally compact group. One can then naturally ask how do properties of normal subgroups and of lattices compare to IRSs. This has been done for several classes of groups ([1],[4],[7],[8],[9],[11]).

Since IRSs correspond to stabilizers of probability measure preserving actions, their study is intimately linked to ergodic theory [14]. They also appear as probabilistic limits of manifolds of increasing volume ([1]) and have connections to sofic groups. IRSs also have connections to characters and representation theory. In this setting, they were introduced under a different name by Vershik, who classified those of the infinite symmetric group and called for more classification results for other groups.

Our aim is to understand some aspects of the theory developed so far and its applications. Along the way we will cover other notions such as amenability, random walks, and Kazhdan's property (T), which are well-established of interest in other areas of mathematics.

Some videos of lectures about the subject are included in the reference list ([2], [11])

This is a suggested programme for the Seminar. It may vary according to the audience's needs and wishes (keep track of the version number!).

Talk 1: Introduction

What are Invariant Random Subgroups (IRS)? Why are they studied? Why should you come to this seminar?

Talk 2: Basics

Explain Section 3 of [4]. Namely, define the Chabauty topology on $Sub(G)$ and why this space is compact; correspondence between $Sub(G)$ and space of rooted Schreier graphs of G (reminder of what a Schreier graph is); definition of IRS and examples; Propositions 12 and 13 with proof (every IRS corresponds to a stabilizer of a measure preserving action); [unimodularity can be skipped if needed]; definition of Benjamini–Schramm convergence (see also Section 1 of[6]), recalling what weak convergence of measures means.

Talks 3 & 4: IRSs of groups of intermediate growth

The goal of these two talks is to understand [9], where the authors find uncountably many ergodic IRSs of a group of intermediate growth. I suggest the following split (speakers to coordinate with each other in any case).

Talk 3 Motivation. Why do we want to find continuous and ergodic IRSs? Recall notion of word growth; since groups of polynomial growth have countably many subgroups (using Gromov's theorem and that finitely generated virtually nilpotent groups are finitely presented), groups of intermediate growth are "smallest" ones that can support continuous IRSs. Define Grigorchuk groups G_ω and show main properties used in the paper (just infinite, branch action). Define universal group U_Λ and show it has intermediate growth (Section 5).

Talk 4 Recall necessary statements from previous talk. Find uncountably many continuous ergodic IRSs of U_Λ (Section 6 in detail).

Talk 5: Amenability, IRSs and random walks

The goal of this talk is to understand the original and the IRS version of Kesten's criterion for amenability of a group. The notion of amenability should be recalled/explained with examples and basic properties (there are lots of resources for this, e.g., [12, Chapter 0], [16, Chapter 10], [5, Appendix G]). Introduce random walks on (Cayley graphs of) groups and spectral radius ([4, Section 2], [10, Section 2]). Prove Kesten's criterion for amenability in the IRS setting (i.e., go through [4, Section 4] – the original [10, Section 3] might help).

[Depending on interest, we could have an extra talk on amenability and/or random walks].

Talk 6: a crash course on Kazhdan's property (T) Kazhdan defined property (T) for locally compact groups in the mid 1960s to show that a large class of lattices are finitely generated. This notion, at first sight about representations, has become basic in areas such as group theory, ergodic theory, differential geometry, operator algebras, combinatorics and computer science (the latter through the use of groups with (T) to build expander graphs). Property (T) is important in our setting because of its role in rigidity results for lattices in (semi)simple Lie groups of higher rank (e.g. Margulis' Normal Subgroup Theorem and the more general Stuck–Zimmer theorem, which state in some sense that these groups have few normal subgroups).

This talk is intended as an introduction to Property (T) and its main properties, following Chapter 1 of [5]: Section 1.1 (Definition, compact groups have (T), amenable+(T)=compact), Section 1.3 (Theorems 1.3.1 and 1.3.4), Section 1.4 (sketch of why $SL(n, K)$ has (T)), Theorem 1.7.1 (a lattice has (T) if and only if the ambient group has it.)

Talk 7: IRSs of higher rank Lie groups Sufficiently nice higher-rank semisimple Lie groups have very few IRSs (this is a generalization of the Stuck–Zimmer theorem which is in turn one of Margulis' Normal Subgroup Theorem). Proof of Theorems 4.1 and 4.2 of [1], talking the Stuck-Zimmer Theorem (4.3) as a black box.

Talks 8 & 9: IRSs of free groups The aim of these talks is to understand the first two main results of [7] (i.e., Corollary 3.4 and Theorem 4.1). Speakers to coordinate with each other. I suggest working through Section 3 in the first talk and through Section 4 in the second.

Talk 10 Several options available:

- Talk on Stuck–Zimmer theorem used in Talk 7 ([14, Section 2]).
- Classification of IRSs of the infinite symmetric group ([15]).
- IRSs of simple groups:
  - $PSL(n, F)$ for $n \geq 3$ and F countable ([3], [13]).
  - Higman–Thompson groups ([8]).

  In both cases, there are no non-trivial IRSs.

References

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