Seminar on Bass–Serre theory and profinite analogues
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What is this about? Bass–Serre theory is one of the fundamental parts of modern geometric group theory. It was initiated by Serre and further developed by Bass, after attending the former's course at the College de France in 1968/9. Its basic goal is to understand the subgroups and structure of groups (for instance, $\mathrm{SL}(2,\mathbb{Q}_p)$)) which decompose as (generalised) amalgamated products. Serre explains that the initial motivation was a result of Ihara stating that all torsion-free subgroups of $\mathrm{SL}(2,\mathbb{Q}_p)$ are free. The original proof is combinatorial and uses the fact that this group splits as an amalgamated product. One of the achievements of the theory is a characterisation of free groups, amalgamated products and their natural generalisations as groups acting on simplicial trees, with certain restrictions on stabilisers. The group may then be identified with the fundamental group of the quotient graph by the action. From this topological point of view, inspired by the theory of covering spaces, it becomes clear that subgroups of free groups, amalgams, etc are of the same nature, as they also act on the tree corresponding to the overgroup. This generalises many classical theorems in combinatorial group theory proved at the start of the 20th century (e.g. theorems of Nielsen–Schreier, Kurosh).

But how does one obtain this tree on which the group is supposed to act? For the particular case of $\mathrm{SL}(2,\mathbb{Q}_p)$ (or over any local field), it is a special case of a *Bruhat–Tits building*, the p-adic analogue of the symmetric homogeneous space of a real Lie group. We will devote some time to learning about the construction of this tree and understanding how the group acts on it so that the results on amalgams can be applied to prove Ihara's theorem.

We will follow Serre's original book *Trees* [6], which is probably still the best source. If time and energy permit, we will learn about some of the analogues of this theory in the context of pro-p and profinite groups, following [3] and [2].

This is a suggested programme for the Seminar. The aim is to understand well the classical theory, so if you feel that there is too much material in your talk let me know and we can make adjustments to ensure that we achieve the right pace for everybody. Correspondingly, if you feel that you do not have enough things to present, add some more examples, or do some of the exercises in your chapter. There are now many other sources available on this material that you may find helpful to consult. Some are listed in the bibliography.

Talk 1: Amalgams [*[6] Ch. I §1 until Proposition 3*]

These are the main objects of study, generalising free products and free products with amalgamation. Follow §1, giving details of the proof of Proposition 1 and giving an indication for Exercise 2 (an amalgam may be trivial). Prove Theorem 1 and give some of its corollaries. Discuss some familiar examples of amalgams (the first few in §1.5).

Talk 2: More on amalgams [*[6] Ch. I §1, from Proposition 3*] Discuss the examples that did not appear in the previous talk. Do Exercises 1 and 2 of §1.4.

Talk 3: Trees [*[6] Ch. I §2*] This section contains mainly graph-theoretical definitions, the topological realisation of a graph and topological/homotopical considerations (Euler characteristic, each graph has the homotopy type of a bouquet of circles). Do Exercises 1 and 3 of §2.1 and 1 of §2.2.

Talk 4: Trees and free groups [*[6] Ch. I §3*] This talk deals with the special case of groups acting on trees with trivial stabilisers (i.e. freely). Follow §3. This is about the characterisation of free groups as groups that act freely on a regular tree. As a consequence, this yields another proof of the Nielsen–Schreier theorem. The Exercise in §3.1 can be skipped, but it would be nice to see the one in §3.3. If you are topologically inclined, you could give an overview of the appendix (obtaining the same results for groups of homeomorphisms).

Talk 5: Trees & amalgams [*[6] Ch. I §4*] This treats the case of groups acting on trees with non-trivial stabilisers and, in particular, with fundamental domain a segment. This is in fact a characterisation of amalgamated product (Theorems 6 and 7). Discuss examples and (briefly) applications. The fundamental definition in the second part of this section is that of a graph of groups, but only the direct limits of trees of groups are considered. Theorems 9 and 10 are the generalisation of 6 and 7, where "segment" is replaced by "tree of groups".

Talk 6: Graphs of groups [*[6] Ch. I §5.1–5.2*] This is the crux of Bass–Serre theory. In §5 the results of the previous two sections are generalised to groups acting on trees with any fundamental domain. Give both definitions of fundamental group of graph of groups, with examples. Give the proof of Theorem 11 (reduced words are not trivial). This is a bit technical, make sure the general picture/idea is clear.

Talk 7: Structure of a group acting on a tree [*[6] Ch. I. §5.3–5.5*] This is the culmination of the last 3 talks, generalising the characterisations of free groups and of amalgams to groups acting on trees with arbitrary fundamental domain. The group is then the fundamental group of the graph of groups obtained as fundamental domain. Give the proof of Kurosh's theorem as an application.

Talk 8: Amalgams and fixed points [*[6] Ch. I §6*] Serre's property (FA) (acting on every tree fixed-point-freely) almost characterises groups which are *not* amalgams. Skip Proposition 22; it might be discussed in another talk. Give examples of groups with and without (FA). The second part of the section is devoted to showing that $\mathrm{SL}(3, \mathbb{Z})$ has (FA). Skip details in the subsections leading up to this if necessary (many proofs can be given by pictures).

Talk 9: The tree of SL_2 [*[6] Ch. II, §1.1–1.3*] Define the tree X of lattices of a two-dimensional vector space over a local field K. Go through Exercises 1 and 2 of §1.1. Define $\mathrm{GL}(V)$ and the subgroups that will be relevant to us. Explain Exercise 2. Present the types of stabilisers of the action of $\mathrm{GL}(V)$ on X. The notes [1] might be helpful.

Talk 10: Decomposing $\mathrm{SL}(V)$ **as an amalgam** [*[6] Ch. II, §1.4–1.6*] Recall the necessary notions from the previous talk. Prove that $\mathrm{SL}(V)$ is an amalgam, which follows from the characterisation of amalgams from §4 and the fact that $\mathrm{SL}(V)$ acts on X with an edge as fundamental domain. If time permits, go through the presentations of $\mathrm{SL}(\mathbb{Z}[1/2])$ and $\mathrm{SL}(\mathbb{Z}[1/3])$. Prove Ihara's theorem on subgroups of $\mathrm{GL}(V)$. Then either go through the locally compact case, or through the proof of Nagao's theorem on the decomposition of $\mathrm{GL}_2(k[t])$ as an amalgamated product.

Talk 11: (optional topic) Connection with Buildings and Tits systems [*[6] Ch. II, §1.7*] This is an interesting excursion giving a glimpse into the theory of buildings. The section concerns groups with a Tits system (or BN pair) of a certain type, which, by the previous theory, also decompose as amalgams.

Talk 12: Free pro-p groups, amalgams and graphs [[3] §4, 1, 2; [2] §2.1] Much of the classical Bass–Serre theory that was covered previously carries through to the setting of profinite groups, but we will concentrate on pro-p groups, following the self-contained chapter [3]. Extra, more up-to-date material can be found in [2] and is worth consulting. Free groups and amalgams are defined analogously to the abstract setting, by universal properties ([3] §4). The relevant sections in [4] might also be useful. Then follow §1, 2 of [3] to define profinite graphs and pro-p trees (§2.1 of [2] may be helpful here).

Talk 13: Pro-p groups acting on pro-p trees [[3] §3–5] Define the action of a pro-p group on a profinite graph and show that the Cayley graph of a free pro-p group is a pro-p tree (Theorem 3.3). (A primer on homological methods might be needed. Consult the references given and skip technical details if necessary.) Go through some of the main results in §3 and §4 on subgroups of amalgams and HNN extensions of pro-p groups. Some of the proofs and results appear in greater generality in [2].

References

[1] A. Raghuram and B. Sury. Groups acting on trees. Notes of a course given at the Indian Institute of Technology in Guwahati, India, December 2002. `http://www.isibang.ac.in/~sury/tree.pdf`

[2] L. Ribes. Profinite graphs and groups. *Springer*, 2017.

[3] L. Ribes and P. Zalesskii. Pro-p trees and applications. In New Horizons in pro-p groups. *Springer*, 2000.

[4] L. Ribes and P. Zalesskii. Profinite groups. *Springer*, 2010.

[5] P. Scott and T. Wall, Topological methods in group theory. In Homological Group Theory, *London Math. Soc. Lecture Notes*, vol. 36, 1979, pp. 137–204.

[6] J.-P. Serre. Trees. *Springer Verlag*, 1980.