GEOMETRIC GROUP THEORY

Matteo Vannacci

Wintersemester 18/19

The goal of this seminar is to understand some of the basic concepts and definitions of "Geometric Group Theory". Historically, one tries to associate a group to a metric space (say a manifold), to streamline information; e.g. one can think about the fundamental group of a manifold. One of the very influencial ideas of Gromov is that this correspondence can somehow be reversed: groups can be thought of as *geometric objects*, for instance via the study of the metric properties of their Cayley graphs.

We will mainly follow the book [2]. There are two talks marked with stars: these are not as accessible as the rest and require more effort. Also if you have suggestions for other related topics, please let me know and I can try to include them in the programme. Finally, if you have any questions about the programme or the preparation of your talk please let me know.

IMPORTANT: Unfortunately we will not have time to delve into the applications to manifolds in the book. In general, you should skip the examples and sections involving too much differential geometry.

PART A: QUASI-ISOMETRIC INVARIANCE

TALK 1: Quasi-Isometry, Quasi-Geodesic Spaces and Geometric Properties of Groups. *Chapter 5 of [2].*

This talk introduces the basic metric notions needed in the seminar with emphasis on Cayley graphs of finitely generated groups.

Follow sections 5.1, 5.2 and 5.3. Skip bilipschitz equivalence. Then continue with Subsections 5.6.1 and 5.6.2.

TALK 2: Amenability is a quasi-isometric invariant. *Chapter 9 of [2].*

In this talk we add a new property to the list of "geometric properties".

Go over the results in Section 9.1. Survey some of the parts of Section 9.2 (at least 9.2.1, but not 9.2.4). Finish with Section 9.3. If there is time you might want to look at some of the relevant exercises.

TALK 3: A Blast from the Past; Word-Growth in Groups. *Chapter 6 of [2].*

This talk surveys the material covered in the previous semester. Note: ideally this talk should be given by one of the attendees of said seminar.

Go throught as much material from Chapter 6 as you can. Skip all the proofs you want, but try to connect the the first talk and last semester as much as you can.

TALK 4: The Svarc-Milnor Lemma. *Chapter 5 of [2].*

The Svarc-Milnor lemma is sometimes called the "fundamental lemma of geometric group theory".

Cover Proposition 5.4.1 and Corollary 5.4.2 in detail. It would be nice to see also some of the content of Subsection 5.4.1 (time permitting). Finally cover the beginning of Section 5.5 and Subsection 5.5.1.

PART B: HYPERBOLIC GROUPS

TALK 5: (Quasi-)Hyperbolic Spaces and Hyperbolic Groups. *Chapter 7 of [2].*

This talk introduces one of the central objects in this seminar: hyperbolic groups.

Skip Section 7.1.1. Cover Section 7.2 in detail. Finish with Definition 7.3.1, Proposition 7.3.2 and Example 7.3.3.

TALK 6: Dehn functions and Dehn presentations. *Chapter 7 of [2].*

This talk deals with the finite presentability of hyperbolic groups. This can be related to certain "isoperimetric inequalities" on the Cayley graphs of hyperbolic groups.

Go through Section 7.4. Introduce Dehn functions (page 200, Definitions 6.E.3, 6.E.4 and 6.E.5) and choose as many exercises as possible between Exercise 6.E.30 and 6.E.36 and 7.E.19 and 7.E.23.[1]

[1] Feel free to come and discuss the choice with me if needed. Also note that one has to work out the solutions to these exercises on their own, but I am more than happy to help.

TALK*: The word problem. *Chapter 12 of [3].*

In this additional talk we look at the word problem in finitely presented groups, relating it to the *halting problem* on Turing machines.

Survey Chapter 12 of [3]. It would be great to arrive at least at the statement of Theorem 12.8, without the proof of Britton's lemma (Lemma 12.7).

TALK 7: Properties of Hyperbolic Groups. *Chapter 7 of [2].*

We continue the study of hyperbolic groups. Here we study elements of infinite order and their centralisers.

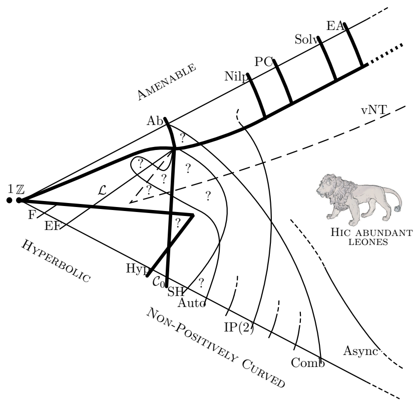
Cover Section 7.5, skip Subsection 7.5.4.

TALK 8: Geometry at infinity. *Chapter 8 of [2].*

The aim of this talk is to introduce "ends" of quasi-geodesic metric spaces and the Gromov boundary of hyperbolic groups.

Go quickly through Section 8.2, if necessary skip the proof of Proposition 8.2.7 and 8.2.8. Concentrate on Section 8.3. Cover the relevant exercises used in the proofs.

PART C



An ounce of action is worth a ton of theory. —F. Engels

The owl of Minerva spreads its wings only with the falling of the dusk. —G. W. F. Hegel

TALKS: Non-Positively Curved Groups.** *Chapter 7 of [2] and Chapter III.Γ.1 of [1].*

The aim of these TWO talks is to define CAT(0)-groups ($\mathcal{C}_0$ in the above picture), these are groups that act "nicely" on a space with "curvature at most 0". These talks could be a seminar on their own and I hope that this could wet the interested attendee appetite for the following semesters.

One can find the basic definitions in Section 7.6 of [2]. The idea is to compare the properties of CAT(0)-groups with those of hyperbolic groups. It turns out that most of the theorems we proved for hyperbolic groups hold for CAT(0)-groups as well! Try to ouline parts of the proof of Theorem 1.1 in Chapter III.Γ.1 of [1].

References

[1] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999.

[2] C. Löh. *Geometric Group Theory. An introduction.* Universitext. Springer, Cham, 2017.

[3] J. Rotman. *An introduction to the theory of groups.* Fourth edition. Graduate Texts in Mathematics, 148. Springer-Verlag, New York, 1995.

All available in the University Catalogue.